VARIATIONAL APPROACH TO THE BÄCKLUND TRANSFORMATIONS

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Variational approach to the Bäcklund transformations is derived on the basis of strong or semi-strong necessary condition for extremum of a functional. The obtained method is applied to the sine-Gordon and Korteweg–de Vries equations.

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1. Introduction

There are different ways in which the Bäcklund transformations may be achieved [1–5]. One of them, a variational approach [4] seems to be very promising. A fair connection between the Bäcklund transformations for some soliton equations and the calculus of variations has been shown to exist and applied to a class of second-order partial differential equations in m independent variables. This class contains the sine-Gordon equation as a special case for $m = 2$. Also the variational approach after certain adaptations of the technique was successfully applied to the Korteweg–de Vries equation together with its modifications. However, the Rund's approach leads to a particular form of the Bäcklund transformation. He assumes that a pair of the partial differential equations

$$\begin{aligned} E(u) &= 0\,, \\ D(v) &= 0\,, \end{aligned} \tag{1}$$

corresponds to the Euler–Lagrange equations resulting from the variational principles given by the Lagrangians $L(u)$ and $L(v)$, respectively. It is important to say that Euler–Lagrange equations are the necessary conditions for the extremum of action functional to exist (necessary conditions). If there exists a relation between the functions u and v and their derivatives which are such, as to imply that the difference $L(u) - L(v)$ is a divergence, then $E(v) = 0$, whenever (1) is satisfied, and conversely. Thus the relations: $E(u) = 0$ and $E(v) = 0$ possess the desired property for the Bäcklund transformation. They are of a very particular form because there is no coupling between u and v like in the original pair (1). It would be more fundamental to start from the general form of the least action principle leading directly to a general form of the Bäcklund transformations of (1). This can be achieved by extending the Rund's constraint:

$$L(u) - L(v) = \text{divergence} \tag{2}$$

to a more general one: $\delta\Phi^* = 0$, where

$$\Phi^* = \int_{t_1}^{t_2} \int_X (L(u) + \lambda_0 L(v)) dt dx + I \,, \tag{3}$$

and where I is a topological invariant.

Such approach is more universal and may have a structure which enables one to reduce the derivation of the Bäcklund transformation to an algorithm. There is also a particular question how to apply the calculus of variations in order to derive the partial differential equations of the lower order then the order of the corresponding Euler–Lagrange equations? Moreover, the set of solutions of the looked for equations must satisfy the Euler–Lagrange equation. It is obvious that (1) is invariant with respect to the scaling of its action functional Φ^* by the topological invariants. One can consider the topological invariant in (3) as a constraint for the extremum of the action functional: $I = \text{const}$. This constraint is trivial and does not contribute to the field equation (1) being the necessary conditions. But if one formulates variational problem which breaks invariance of necessary conditions with respect to the gauge scaling by I, than one derives new field equations. If this new variational problem guarantees that the set of solutions of new equations is included in the solution set of (1) then one derive new method for the non-linear field theory. In order to realize this idea we introduce and apply the concept of strong necessary condition. After some calculations the new equations of motion appear to be the Bäcklund transformations. The paper is organized in the following way: In Chapter 2 we introduce the strong necessary condition concept and we derive equations of motion.

Chapter 3 presents derivation of the Bäcklund transformation for the sine-Gordon equation. In Chapter 4 we introduce concept of the semi-strong necessary condition and we present its application to the KdV equation. In conclusions we discuss how to generalize the results of the present paper and suggest new possible applications of the strong and semi-strong necessary conditions concepts.

2. Equations of motion resulting from the strong necessary condition concept

We shall consider a partial differential equation of motion resulting from the least action principle: $\delta\Phi[u] = 0$, where:

$$\Phi[u] = \int_{E^2} F(u, u_{,x}, u_{,t}) dx dt \tag{4}$$

and

$$u(\cdot, t) \in C^1 ,$$
$$u(x, \cdot) \in C^1 .$$

Let us analyse a variation

$$\delta\Phi[u] = \int_{E^2} (F_{,u}\delta u + F_{,u_{,x}}\delta u_{,x} + F_{,u_{,t}}\delta u_{,t}) dx dt , \tag{5}$$

where δu is the increment of $u(x)$. The necessary condition and the assumption for the fixed boundaries lead to the Euler–Lagrange equation:

$$F_{,u} - D_x F_{,u_{,x}} - D_t F_{,u_{,t}} = 0 . \tag{6}$$

The order of (6) is always higher then the highest order of derivative of $u(x)$ appearing in (4). The reason for increasing the orders are operators D_x and D_t in (6). Therefore, in order to derive the Bäcklund transformation from the least action principle we should not apply the Euler–Lagrange's equation as the necessary condition. However, the only way to satisfy $\delta\Phi = 0$ without (6) is to set up the following conditions:

$$F_{,u_{,x}} = 0 , \tag{7}$$
$$F_{,u_{,t}} = 0 , \tag{8}$$
$$F_{,u} = 0 . \tag{9}$$

All solutions of (7)–(9) satisfy the Euler–Lagrange equation (6) but in most cases the set of solutions of (7)–(9) is trivial ($u = \text{const}$) or empty. In order to extend this set we introduce integral constraints for the extremum of $\Phi[u]$: $I_1 = c_1, I_2 = c_2$, where

$$I_1[u] = \int\limits_{E^2} W_1(u, u_{,x}) dxdt \,, \tag{10}$$

$$I_2[u] = \int\limits_{E^2} W_2(u, u_{,t}) dxdt \,, \tag{11}$$

and where c_i are constants. These constraints should not change the Euler-Lagrange equation, therefore I_i-s must be topological invariants. Topological invariant means that I_i remains constant while its argument u varies locally: $\delta I_i \equiv 0$. Applying the necessary condition for a conditional extremum of functional to exist we obtain:

$$\begin{aligned}
&\lambda_0 \int\limits_{E^2} (F_{,u}\delta u + F_{,u_{,x}}\delta u_{,x} + F_{,u_{,t}}\delta u_{,t}) dxdt \\
&\lambda_1 \int\limits_{E^2} (W_{1,u}\delta u + W_{1,u_{,x}}\delta u_{,x} + W_{1,u_{,t}}\delta u_{,t}) dxdt \\
&\lambda_2 \int\limits_{E^2} (W_{2,u}\delta u + W_{2,u_{,x}}\delta u_{,x} + W_{2,u_{,t}}\delta u_{,t}) dxdt = 0 \,.
\end{aligned} \tag{12}$$

The concept of strong necessary condition applied to (12) leads to the following equations of motion:

$$\lambda_0 F_{,u} + \lambda_1 W_{1,u} + \lambda_2 W_{2,u} = 0 \,, \tag{13}$$

$$\lambda_0 F_{,u_x} + \lambda_1 W_{1,u_x} + \lambda_2 W_{2,u_x} = 0 \,, \tag{14}$$

$$\lambda_0 F_{,u_t} + \lambda_1 W_{1,u_t} + \lambda_2 W_{2,u_t} = 0 \,. \tag{15}$$

Contrary to the Euler–Lagrange equation now the topological invariants contribute to the field equation. For a scalar field $u(x,t)$ the W_i can be chosen in the following way:

$$\begin{aligned}
W_1 &= D_x G_1(u) \,, \\
W_2 &= D_t G_2(u) \,.
\end{aligned} \tag{16}$$

Expressions (13)–(15) establish a simultaneous set of equations for $u(x,t), G_1$, and G_2. It must be stressed again that any solution of (13),(15) for $u(x)$ satisfies the Euler–Lagrange equation. Relation between the Euler's theorem and the strong necessary conditions is presented on Fig.1 and Fig.2.

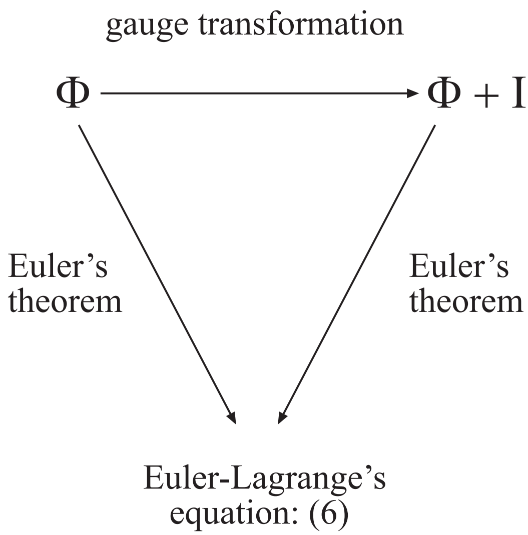

Fig. 1. The invariance of the Euler–Lagrange's equation with respect to the gauge transformation.

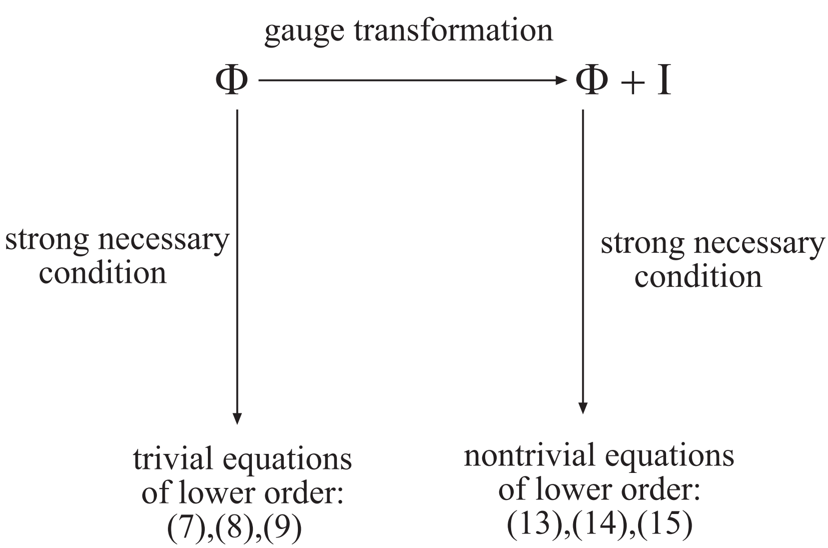

Fig. 2. Influence of the gauge transformation on the field equations resulting from the strong necessary condition concept.

3. Bäcklund transformations for the sine-Gordon equation

Let us consider the sine-Gordon equation in the light-cone coordinate system:

$$u_{,xt} = \sin(u)\,. \tag{17}$$

This model possesses poor topology, *e.g.* all configurations of the field $u(x,t)$ satisfying the boundary conditions $u(\partial E^2) = \mathrm{const}$ are classified by the elements of homotopy group $\pi_2(S^1)$ which is trivial. Nondivergence topological invariants appear to be degrees of elements of $\pi_n(S^n)$ or Hopf invariants of $\pi_{2n-1}(S^n)$ homotopy groups. Therefore, in order to construct them one has to combine an appropriate number of fields $u, v, ...$, accordingly to the dimension of the independent variables space. It is also necessary to assume the boundary conditions. In the sine-Gordon case the dimension of the independent variables space is equal two, therefore we have to consider a model corresponding to the homotopy group $\pi_2(S^2)$. Thus, we combine two independent sine-Gordon equations:

$$\begin{aligned} u_{,xt} &= \sin(u)\,, \\ v_{,xt} &= \sin(v)\,. \end{aligned} \tag{18}$$

An action functional for the two component model (18) is of the following form:

$$\Phi[u,v] = \int\limits_{E^2} \left[\frac{1}{2}u_{,x}u_{,t} - \cos(u) + \lambda_0\left(\frac{1}{2}v_{,x}v_{,t} - \cos(v)\right)\right] dxdt\,. \tag{19}$$

For the purpose of the strong extremum concept we suppose the constraints defined by the topological invariants:

$$I_1 = \int\limits_{E^2} G_1(u,v)(u_{,x}v_{,t} - u_{,t}v,x)dxdt\,, \tag{20}$$

$$I_2 = \int\limits_{E^2} D_x G_2(u,v)dxdt\,, \tag{21}$$

$$I_3 = \int\limits_{E^2} D_t G_3(u,v)dxdt\,. \tag{22}$$

Expanding (21), (22) we get:

$$\lambda_2 I_2 = \int\limits_{E^2} (f_{,u}u_{,x} + f_{,v}v_{,x})dxdt\,, \tag{23}$$

$$\lambda_3 I_3 = \int\limits_{E^2} (g_{,u}u_{,t} + g_{,v}v_{,t})dxdt\,, \tag{24}$$

where we introduce a new notation: $\lambda_1 G_1 = G, \lambda_2 G_2 = f$ and $\lambda_3 G_3 = g$. Minimizing $\Phi^* = \Phi + \lambda_1 I_1 + \lambda_2 I_2 + \lambda_3 I_3$ we derive the following set of equations:

$$\begin{aligned} \sin(u) &+ G_{,u}(u_{,x}v_{,t} - u_{,t}v_{,x}) + (f_{,uu}u_{,x} + f_{,uv}v_{,x}) \\ &+ (g_{,uu}u_{,t} + g_{,uv}v_{,t}) = 0\,, \end{aligned} \tag{25}$$

$$\begin{aligned} \lambda_0 \sin(v) &+ G_{,v}(u_{,x}v_{,t} - u_{,t}v_{,x}) + (f_{,uv}u_{,x} + f_{,vv}v_{,x}) \\ &+ (g_{,uv}u_{,t} + g_{,vv}v_{,t}) = 0\,, \end{aligned} \tag{26}$$

$$\frac{1}{2}u_{,t} + Gv_{,t} + f_{,u} = 0\,, \tag{27}$$

$$\frac{1}{2}u_{,x} - Gv_{,x} + g_{,u} = 0\,, \tag{28}$$

$$\frac{\lambda_0}{2}v_{,t} - Gu_{,t} + f_{,v} = 0\,, \tag{29}$$

$$\frac{\lambda_0}{2}v_{,x} + Gu_{,x} + g_{,v} = 0\,. \tag{30}$$

It is easy to prove that any solution of (25)–(30) satisfies Euler's equations (18). Equations (25) and (26) can be expressed by divergences:

$$\sin(u) + D_x(f_{,u} + Gv_{,t}) + D_t(g_{,u} - Gv_{,x}) = 0\,, \tag{31}$$

$$\sin(u) + D_x(f_{,v} + Gu_{,t}) + D_t(g_{,v} - Gu_{,x}) = 0\,. \tag{32}$$

Following (27)–(30) we express the arguments of D_x and D_t in (31) and (32) by $u_{,x}, u_{,t}, v_{,x}$ and $v_{,t}$. Results are equations (18)□.

All equations (25)–(30) must be self-consistent. Formally we have six simultaneous equations for the five unknown functions: u, v, G, f, g. We reduce the number of equations from six to four by making them linearly dependent. We achieve this by the following Ansatz:

$$\frac{\lambda_0}{4} + G^2 = 0\,, \tag{33}$$

$$2Gf_{,u} + f_{,v} = 0\,, \tag{34}$$

$$2Gg_{,u} - g_{,v} = 0\,. \tag{35}$$

Condition (33) implies that G must be a constant. For further calculations we choose $\lambda_0 = -1$ and $G = \pm \frac{1}{2}$. Now we have to satisfy four equations (25)–(27) and (29). This we realize by identifying ((25),(26)) with ((27),(29)). The last one we solve using the following Ansatz for f and g:

$$f = \mu \cos\frac{u+v}{2}\,,$$

$$g = \nu \cos\frac{u-v}{2}\,.$$

We derive from (25) and (26):

$$\sin\frac{u-v}{2}=\frac{\mu}{2}\frac{u_{,x}+v_{,x}}{2}\,, \tag{36}$$

$$\sin\frac{u+v}{2}=\frac{\nu}{2}\frac{u_{,t}-v_{,t}}{2}\,, \tag{37}$$

and from (27)–(30):

$$\frac{u_{,t}-v_{,t}}{2}-\frac{\mu}{2}\sin\frac{u+v}{2}=0\,, \tag{38}$$

$$\frac{u_{,x}+v_{,x}}{2}-\frac{\nu}{2}\sin\frac{u-v}{2}=0\,. \tag{39}$$

(40)

We get consistency of (36)–(39) for $\mu\nu=4$ or letting $\frac{\mu}{2}=\alpha$ and $\frac{\nu}{2}=\frac{1}{\alpha}$. Finally, (36),(37) is equivalent to (38),(39) and forms the Bäcklund transformation for the sine-Gordon system.

4. Semi-strong necessary condition concept and KdV equation

In order to make our formalism more universal we have to extend the strong necessary condition concept to a semi-strong one. Let Φ be a functional on a set of differentiable functions. These functions can be regarded as elements of the space C^2. Let Φ depend on the higher derivatives of $u(x,t)$:

$$\Phi[u]=\int\limits_{t_1}^{t_2}\int\limits_{X}F(u,u_{,t},u_{,x},u_{,xx})dxdt\,. \tag{41}$$

Accordingly, one can investigate the necessary condition for the extremum of (41) to exist:

$$\int\limits_{t_1}^{t_2}\int\limits_{X}(\frac{\partial F}{\partial u}\delta u+\frac{\partial F}{\partial u_{,t}}\delta u_{,t}+\frac{\partial F}{\partial u_{,x}}\delta u_{,x}+\frac{\partial F}{\partial u_{,xx}}\delta u_{,xx})dxdt=0\,. \tag{42}$$

Now, there are more then one possibilities to satisfy (42) without Euler–Lagrange's equation. One of them is a semi-strong necessary condition:

$$F_{,u} = 0\,, \tag{43}$$

$$F_{,u_{,t}} = 0\,, \tag{44}$$

$$F_{,u_{,x}}-D_xF_{,u_{,xx}} = 0\,. \tag{45}$$

If (41) depends on the higher derivatives of $u(x,t)$ up to $u(x,t)_{,kx}$ then (45) takes the following extended form:

$$F_{,u_{,x}} - D_x F_{,u_{,xx}} + D_x^2 F_{,u_{,xxx}} - D_x^3 F_{,u_{,xxxx}} + ... + (-1)^{k-1} D_x^{k-1} F_{,u_{,kx}} = 0\,, \quad (46)$$

where $u_{,kx}$ means the derivative of the order k.

Semi-strong necessary condition concept supplies a helpful tool for the theory of non-linear partial differential equations. We present its application to the Korteweg–de Vries equation.
The topology associated with the KdV equation is similar to that associated with the sine-Gordon equation. Therefore, we have to consider two independent fields $\bar{u}$ and $\bar{v}$ governed by KdV equations:

$$\begin{aligned} \bar{u}_{,t} - 6\bar{u}\bar{u}_{,x} + \bar{u}_{,xxx} &= 0\,, \\ \bar{v}_{,t} - 6\bar{v}\bar{v}_{,x} + \bar{v}_{,xxx} &= 0\,. \end{aligned} \quad (47)$$

It is easy to see that the Lagrangian density

$$\begin{aligned} &F(u, v, u_{,x}, v_{,x}, u_{,t}, v_{,t}, u_{,xx}, v_{,xx}) \\ &= \tfrac{1}{2}u_{,x}u_{,t} - u_{,x}^3 - \tfrac{1}{2}u_{,xx}^2 + \lambda(\tfrac{1}{2}v_{,x}v_{,t} - v_{,x}^3 - \frac{1}{2}v_{,xx}^2) \end{aligned} \quad (48)$$

generates (47) as the corresponding Euler–Lagrange equations, where $\bar{u} = u_{,x}$ and $\bar{v} = v_{,x}$, [6]. We attach three topological invariants with the following functional densities:

$$\begin{aligned} W_1 &= G_1(u,v)(u_{,x}v_{,t} - u_{,t}v_{,x})\,, \\ W_2 &= D_t G_2(u,v)\,, \\ W_3 &= D_x G_3(u, v, u_{,x}, v_{,x}, u_{,xx}, v_{,xx}, u_{,xxx}, v_{,xxx})\,. \end{aligned} \quad (49)$$

Effective action functional including the topological constraints defined by (49) takes the following form:

$$\begin{aligned} \Phi^*[u,v] = \int_{E^2} &(F(u, v, u_{,x}, v_{,x}, u_{,t}, v_{,t}, u_{,xx}, v_{,xx}) + W_1(u, v, u_{,x}, v_{,x}, u_{,t}, v_{,t}) \\ &+ W_2(u, v, u_{,t}, v_{,t}) + W_3(u, v, u_{,x}, v_{,x}, u_{,xx}, v_{,xx}, u_{,xxx}, v_{,xxx}, u_{,xxxx}, v_{,xxxx}))dxdt\,. \end{aligned} \quad (50)$$

Applying to (50) the concept of the semi-strong condition we derive the following set equations of motion:

$$\begin{aligned} G_{1,u}(u_{,x}v_{,t} - u_{,t}v_{,x}) + (G_{2,u})_{,t} + (G_{3,u})_{,x} = 0\,, \\ G_{1,v}(u_{,x}v_{,t} - u_{,t}v_{,x}) + (G_{2,v})_{,t} + (G_{3,v})_{,x} = 0\,, \end{aligned} \quad (51)$$

$$\frac{1}{2}u_{,x} - G_1 v_{,x} + G_{2,u} = 0\,,$$
$$\frac{\lambda}{2}v_{,x} + G_1 u_{,x} + G_{2,v} = 0\,, \tag{52}$$

$$\frac{1}{2}u_{,t} - 3u_{,x}^2 + u_{,xxx} + G_1 v_{,t} + G_{3,u} = 0\,,$$
$$\lambda(\frac{1}{2}v_{,t} - 3v_{,x}^2 + v_{,xxx}) - G_1 u_{,t} + G_{3,v} = 0\,. \tag{53}$$

Performance of partial derivatives with respect to x (1,2 and 3 degrees) generates a lot of complicated terms. (51),(52) and (53) have the final reduced form obtained with the aid of the following theorem.
If W_3 is a divergence of an arbitrary order n:

$$W_3 = D_x G_3(u, v, u_{,x}, v_{,x}, u_{,xx}, v_{,xx}, ..., u_{,nx}, v_{,nx})\,, \tag{54}$$

then the semi-strong necessary condition:

$$F^*_{,u,x} - D_x F^*_{,u,xx} + D_x^2 F^*_{,u,xxx} + ... + (-1)^{n-1} D_x^{(n-1)} F^*_{,u,nx} = 0\,,$$
$$F^*_{,v,x} - D_x F^*_{,v,xx} + D_x^2 F^*_{,v,xxx} + ... + (-1)^{n-1} D_x^{(n-1)} F^*_{,v,nx} = 0\,, \tag{55}$$

can be reduced to (53), where $F^* = F + W_1 + W_2 + W_3$. The proof can be done by induction starting from $n = 2$: $G_3(u, v, u_{,x}, v_{,x}, u_{,xx}, v_{,xx})$. (51)–(53) establish a system of six simultaneous equations for the five unknown functions: u, v, G_1, G_2, G_3. In order to make them consistent we have to substitute $G_1 = \frac{1}{2}$, $\lambda = -1$ and

$$G_2 = \frac{1}{12}(u - v)^3 + \gamma(u - v)\,,$$
$$G_3(u, v, ..., u_{,xxx}, v_{,xxx}) = \left[-\frac{3}{2}(u_{,x}^2 + v_{,x}^2) + \frac{1}{2}(u_{,xxx} + v_{,xxx})\right](u - v)\,, \tag{56}$$

where γ is an arbitrary real constant. In consequence two equations (52) reduce to one equation:

$$(u + v)_{,x} = 2\gamma + \frac{1}{2}(u - v)^2 \tag{57}$$

and (53) to the following one:

$$(u - v)_{,t} - 3(u_{,x}^2 - v_{,x}^2) + (u - v)_{,xxx} = 0\,. \tag{58}$$

Finally, using (57) we reduce (51) to

$$(u - v)\left[(u - v)_{,t} - 3(u_{,x}^2 - v_{,x}^2) + (u - v)_{,xxx}\right] = 0\,. \tag{59}$$

(57)–(59) establish the Bäcklund transformation for the Korteweg–de Vries equation.

5. Concluding remarks

Application of minimization procedure with the topological constraints has been developed since twenty years [7–10] and applied to non-linear models associated with non-trivial homotopy groups $\pi_n(S^n)$ and $\pi_{2n-1}(S^n)$. In all these cases the derived equations were equivalent to the Bogomolny decomposition [11, 12]. However, the models connected to trivial homotopy groups generate a new quality. For instance, the sine-Gordon or Korteweg–de Vries equations are associated with the trivial homotopy group $\pi_2(S^1)$. In order to generate a topological invariant different than a divergence we had to attach a second independent field model associated with identical homotopy group (18), (47). In this way we obtained the models connected to $\pi_2(S^2)$. Then, applying the strong necessary condition concept to such complex models we derived the Bäcklund transformations. One can imagine a lot of new possibilities in application of this procedure. First, (18) or (47) need not to consist of identical equations. For instance, instead of two KdV equations one can consider two independent equations from the KdV-hierarchy, or a combination of KdV and modified KdV ones. Second, one can combine two or more different topologies leading to the nontrivial homotopy group. Both ways lead to a new class of associated equations of the lower degree than the original equation of motion. The Bogomolny decomposition and the Bäcklund transformations establish the first two classes.

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