A TWISTORIAL DESCRIPTION OF THE DYNAMICS OF COMPLEXIFIED ELECTROMAGNETIC FIELDS

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A method which enables one to build up explicit least-action principles in the non-projective twistor spaces is applied to the context of the theory of complexified Maxwell fields. The freedom in the choices of spinor kernels for the integrands of the universal contour integrals for interacting fields gives rise to the possibility of constructing several Lagrangian densities for the system being considered. It appears that the Lorenz-gauge condition is intrinsically tied in with the inner structure of the twistor dynamics. The configurations involving the kernels for the potential and current density turn out to suggest a natural variational prescription for deriving the equations of motion for the potential. It is shown that the equations for the fields can be derived directly from coupled statements which carry only field quantities.

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1. Introduction

A method has recently been proposed [1,2] which makes feasible the construction of twistorial least-action principles for complexified spinless and spinning fields. Loosely speaking, the twistor Lagrangian densities of interest are built up by simply inserting into the expressions for the standard densities the integrands of the universal contour integrals for interacting fields [3] along with the usual two-twistor expression for the (holomorphic) partial-derivative operator on Complex Minkowski space CM. In the spinning case, the usefulness of the structures which result out of the completion of this procedure rests essentially upon the freedom in the selection of spinor kernels for the contour integrals that produce the fields. The dynamical statements are all set upon bounded eight-real-dimensional domains contained in the topological product of the non-projective twistor space T with its dual T^*. Upon actually working out the variations of the actions in either

case, one holds fixed the arguments of the respective twistor functions and assumes that the variation itself commutes with the derivatives which eventually occur in the expressions for the densities involved. Normally, the sets of variations are constituted by arbitrary non-singular scalar functions that are defined on the (compact) closures of the domains and likewise required to vanish on the boundaries, the twistor fields as well as their derivatives being supposedly continuous in the closures.

Indeed, the method was initially utilized [1] to describe the dynamics of classical Klein–Gordon fields, the corresponding twistor equations of motion having appeared as statements carrying the well-known Penrose wave operators [4]. The adaptation of the techniques to spinning systems was basically accomplished in connection with the presentation of a description for Dirac fields [2]. A remarkable result arising in the latter work, which apparently brings out a typical feature of spinning systems, is associated to the dynamical admissibility of spinor variations. The arbitrariness borne by the choice of spinor kernels thus produced the generation of many densities that are related to one another through a simple set of interchange rules. It was shown, in effect, that the relevant equations of motion not only coincide with the statements which are obtained by transvecting the Dirac equations with suitable kernels, but are also independent of which density is effectively selected out.

In the present work, we apply the method to complexified Maxwell fields. It will be seen that both the multiplicity of Lagrangian densities and the admissibility of spinor variations will take place once again. We will show that the Lorentz-gauge condition is intimately tied in with the structure of the twistor dynamics. The traditional variational role of the electromagnetic potential will be carried over to the situation allowed for at this stage. Accordingly, the configurations involving the potential and current-density kernels provide a naive prescription for carrying out the derivation of the equations of motion for the potential in a natural way. A particular result that emerges from the implementation of this prescription is the occurrence of three equivalent wave equations which carry the Penrose operators together with appropriate twistor functions. It appears that the equations of motion for the fields can be derived in a straightforward way from coupled field statements which involve scalar variations. It turns out that the twistorial representation of the theory agrees with the relations which arise from trivially translating the CM-version of Maxwell's equations. Notwithstanding the fact that the key procedures can be carried through in much the same way as for the spinning case referred to before, we will build up a new set of integral devices which considerably facilitate working out the pertinent least-action principle. For the sake of completeness, we will also show the details of the product-space construction provided in Ref. [1].

The only reason lying behind the elaboration of our work is related to the belief that it would be worthwhile to look into the inner structure of the electromagnetic system, thereby verifying whether such an investigation might bring out dynamical features other than the ones exhibited hitherto. We hope that the description to be presented here will fully justify our motivation. The paper has been outlined as follows. Section 2 briefly reviews the formulation of the theory as given in Ref. [5]. In Section 3, we recall the contour integrals which are of immediate interest to us and exhibit the main kernel structures. A symbolic form of the action is constructed in Section 4, the equations of motion being derived afterwards in Section 5. We make a few remarks on our representation in Section 6. The fields will be considered as wave functions, and no attempt will be made herein to provide a cohomological interpretation of the field dynamics. All the spinor and twistor conventions adopted by Penrose and Rindler [3] will be taken for granted from the beginning. The standard coordinates x^a of an arbitrary point of CM will be split up according to the prescription $x^a = \xi^a + i\eta^a$, with ξ^a and η^a being both components of real vectors. The covariant alternating tensor for the canonical basis of CM will be denoted by e_{abcd}.

2. Formulation of the theory in CM

The Maxwell system is defined at x^a as the set

$$\mathrm{MS} = \{\phi_{AB}(x), \psi_{A'B'}(x), \Phi_{AA'}(x), J_{AA'}(x)\}\,, \tag{2.1}$$

where, in particular, the quantities $\phi_{AB}(x)$ and $\psi_{A'B'}(x)$ are the electromagnetic fields. Such quantities are symmetric spinors and show up as independent states of photons, being additionally looked upon as uncharged massless fields of spin ± 1. Either of them thus describes locally the six degrees of freedom of the system. The quantity $\Phi_{AA'}(x)$ stands for the (vector) electromagnetic potential whereas $J_{AA'}(x)$ denotes a (divergenceless) current density which effectively plays the role of a source for the fields and potential. Each entry of (2.1) can be regarded as a mapping on the tensor product between adequate $\mathrm{SL}(2,C) \otimes \mathrm{SL}(2,C)$-fibers over x^a. We have the explicit field-potential relationships

$$\phi_{AB}(x) \;=\; \nabla_{A'(A}\,\Phi^{A'}_{B)}(x) \tag{2.2a}$$

and

$$\psi_{A'B'}(x) \;=\; \nabla_{A(A'}\,\Phi^{A}_{B')}(x)\,, \tag{2.2b}$$

which come directly from the defining expression for the Maxwell bivector

$$F_{AA'BB'}(x) = 2\nabla_{[AA'}\,\Phi_{BB']}(x)\,, \tag{2.3}$$

where $\nabla_{AA'} = \partial/\partial x^{AA'}$ and the square brackets denote skew-symmetrization over the index pairs. We thus have the unambiguous splitting relation

$$F_{AA'BB'}(x) = \varepsilon_{A'B'}\phi_{AB}(x) + \varepsilon_{AB}\psi_{A'B'}(x)\,, \tag{2.4}$$

with the ε's being the canonical "metric" spinors.

The skewness of $F_{ab}(x)$ implies that the fields satisfy the Bianchi identity

$$\nabla^{B}_{A'}\phi_{AB}(x) = \nabla^{B'}_{A}\psi_{A'B'}(x)\,. \tag{2.5}$$

We often consider this identity as the first half of Maxwell's equations, which is deemed to be equivalent to Eqs. (2.2). The second half is the essentially dynamical part of the theory, and arises out of the variational principle [5]

$$\delta \int\limits_{\Omega} \mathcal{L}_{\mathbf{M}}\, \underset{\sim}{\Omega} = 0\,. \tag{2.6}$$

In (2.6), $\underset{\sim}{\Omega}$ defines an elementary eight-real-dimensional volume of a bounded region $\Omega \subset$ CM through

$$\underset{\sim}{\Omega} \;=\; d^4\xi \wedge d^4\eta\,, \tag{2.7a}$$

where

$$d^4\lambda \;=\; \frac{1}{4!}e_{abcd}d\lambda^a \wedge d\lambda^b \wedge d\lambda^c \wedge d\lambda^d\,, \tag{2.7b}$$

with $x^a \in \Omega$ and λ denoting either ξ or η. The quantity $\mathcal{L}_{\mathrm{M}}$ is the complete CM-Lagrangian density for the system, which is written out explicitly as

$$\mathcal{L}_{\mathrm{M}} = \frac{1}{8\pi}\left[\phi_{AB}(x)\phi^{AB}(x) + \psi_{A'B'}(x)\psi^{A'B'}(x)\right] + \Phi_{AA'}(x)J^{AA'}(x)\,. \tag{2.8}$$

Usually, $\delta\,\Phi_{AA'}(x)$ is taken as the variation that vanishes on the boundary $\partial\Omega$ of Ω. Combining (2.5) with the field equations that result from (2.6) then yields the gauge-invariant statements

$$\nabla^{B}_{A'}\phi_{AB}(x) = 2\pi J_{AA'}(x) = \nabla^{B'}_{A}\psi_{A'B'}(x)\,, \tag{2.9}$$

which amount to the entire theory in Ω.

The wave equations that control the propagation of the fields and potential in Ω read [5]

$$\Box\phi_{AB}(x) \;=\; 4\pi\nabla_{A'(A}J^{A'}_{B)}(x)\,, \tag{2.10a}$$

$$\Box\psi_{A'B'}(x) \;=\; 4\pi\nabla_{A(A'}J^{A}_{B')}(x)\,, \tag{2.10b}$$

$$\Box \Phi_{AA'}(x) = 4\pi J_{AA'}(x) + \nabla_{AA'} \Lambda(x)\,, \tag{2.11}$$

with

$$\Box \;=\; \nabla_{CC'}\nabla^{CC'} \;=\; \nabla_c \nabla^c \tag{2.12a}$$

and

$$\Lambda(x) \;=\; \nabla^{CC'}\, \Phi_{CC'}(x) \tag{2.12b}$$

being the D'Alembertian operator and the Lorentz scalar. Evidently, the divergencelessness of $J_{AA'}(x)$ can be stated as either of the relations

$$\nabla_{A'[A} J^{A'}_{B]}(x) = 0, \nabla_{A[A'} J^{A}_{B']}(x) = 0\,, \tag{2.13}$$

and, consequently, we can drop the symmetrization round brackets from the right-hand sides of Eqs. (2.10). In case the fields bear a specific energy character, we may extend the domain of definition of the system by carrying out an analytic continuation into CM.

3. Contour integrals and spinor-kernel structures

Let $\{Z^\alpha, W_\beta\}$ be a pair of non-null twistors through x^a, which are prescribed by

$$Z^\alpha \;=\; (\omega^A(x), \pi_{A'}) = (ix^{AA'}\pi_{A'}, \pi_{A'}) \in \mathrm{T}_\pm\,, \tag{3.1a}$$

$$W_\beta \;=\; (\lambda_A, \mu^{A'}(x)) = (\lambda_A, -ix^{AA'}\lambda_A) \in \mathrm{T}^*_\pm\,, \tag{3.1b}$$

with $\mathrm{T}_\pm$ and $\mathrm{T}^*_\pm$ standing for the non-null slices of T and T^*, respectively. Such twistors effectively satisfy the incidence condition $Z^\mu W_\mu = 0$, and thence can be viewed as two-complex-dimensional planes $\alpha_{\boldsymbol{Z}}$ and $\beta_{\boldsymbol{W}}$ lying in CM whose intersection is a complex null geodesic $\mathcal{N}_{\boldsymbol{ZW}}$ that contains x^a (see, for instance, Ref. [4]). In a formal way, we have

$$\alpha_{\boldsymbol{Z}} \cap \beta_{\boldsymbol{W}} = \mathcal{N}_{\boldsymbol{ZW}} \ni x^a\,. \tag{3.2}$$

It is useful to introduce the (Poincaré-invariant) non-projective differential forms

$$d^2 Z \;\doteqdot\; \tfrac{1}{2}(I_{\mu\nu} dZ^\mu \wedge dZ^\nu) = (I_{\mu\nu} A^\mu B^\nu) d\alpha \wedge d\beta\,, \tag{3.3a}$$

and

$$d^2 W \;\doteqdot\; \tfrac{1}{2}(I^{\rho\sigma} dW_\rho \wedge dW_\sigma) = (I^{\rho\sigma} E_\rho F_\sigma) d\tau \wedge d\theta\,, \tag{3.3b}$$

where $I_{\mu\nu}$ and $I^{\rho\sigma}$ denote the ordinary infinity twistors [3,4], and

$$Z^\mu \;=\; \alpha A^\mu + \beta B^\mu, W_\rho = \tau E_\rho + \theta F_\rho\,, \tag{3.3c}$$

with α, β, τ, θ being independent nowhere-vanishing complex parameters and A^μ, B^μ, E_ρ, F_ρ standing for fixed auxiliary twistors through x^a. We notice that

$$d^2Z = (I_{\mu\nu}A^\mu B^\nu)(\alpha d\alpha)\wedge d\kappa\,, \tag{3.4a}$$

$$d^2W = (I^{\rho\sigma}E_\rho F_\sigma)(\tau d\tau)\wedge d\zeta\,, \tag{3.4b}$$

where $\kappa=\beta/\alpha$ and $\zeta=\theta/\tau$. We thus write the symbolic contour integrals for the fields

$$\phi_{AB}(x)=\frac{1}{(2\pi i)^4}\int\limits_{\Gamma_{f_{(\mathcal{F})}}}\mathcal{F}_{AB}f_{(\mathcal{F})}(Z^\alpha,W_\beta)d^2Z\wedge d^2W \tag{3.5}$$

and

$$\psi_{A'B'}(x)=\frac{1}{(2\pi i)^4}\int\limits_{\Gamma_{g_{(\mathcal{G})}}}\mathcal{G}_{A'B'}g_{(\mathcal{G})}(Z^\alpha,W_\beta)d^2Z\wedge d^2W\,, \tag{3.6}$$

along with the formal expression for the potential

$$\Phi_{AA'}(x)=\frac{1}{(2\pi i)^4}\int\limits_{\Gamma_{P_{(\mathcal{P})}}}\mathcal{P}_{AA'}P_{(\mathcal{P})}(Z^\alpha,W_\beta)d^2Z\wedge d^2W\,, \tag{3.7}$$

where $\{\mathcal{F}_{AB},\,\mathcal{G}_{A'B'},\,\mathcal{P}_{AA'}\}$ carries decomposable kernels to be determined.

In accordance with the conventional approach [3], the (two-variable) twistor fields and potential $\{\Theta(Z^\alpha,W_\beta)\}$ that enter into the integrands of Eqs. (3.5)–(3.7) are independent meromorphic functions on $\mathrm{T}_\pm\times\mathrm{T}^*_\pm$, each of which being ultimately taken to be also homogeneous in both variables. The contours $\{\Gamma_\Theta\}$ are four-real-dimensional compact contours having the toroidal topology $(S^1\times S^1)\times(S^1\times S^1)$. All of them have to be specified in such a way that non-vanishing outcomes are produced when one actually performs the integrals. We will make another point regarding this specification in a moment.

The projective version of (3.5)–(3.7) can be immediately obtained by performing integrals that carry simple one-forms [2]. We have the explicit formulae

$$\phi_{AB}(x)=\frac{1}{(2\pi i)^2}\int\limits_{\gamma_{f_{(\mathcal{F})}}}\mathcal{F}_{AB}f_{(\mathcal{F})}(Z^\alpha,W_\beta)\delta Z\wedge\delta W\,, \tag{3.8}$$

$$\psi_{A'B'}(x)=\frac{1}{(2\pi i)^2}\int\limits_{\gamma_{g_{(\mathcal{G})}}}\mathcal{G}_{A'B'}g_{(\mathcal{G})}(Z^\alpha,W_\beta)\delta Z\wedge\delta W \tag{3.9}$$

and

$$\Phi_{AA'}(x) = \frac{1}{(2\pi i)^2} \int\limits_{\gamma_{P_{(\mathcal{P})}}} \mathcal{P}_{AA'} P_{(\mathcal{P})}(Z^\alpha, W_\beta) \delta Z \wedge \delta W \,, \tag{3.10}$$

with

$$\delta Z \doteq I_{\mu\nu} Z^\mu dZ^\nu = (I_{\mu\nu} A^\mu B^\nu)(\alpha^2 d\kappa) \,, \tag{3.11a}$$

$$\delta W \doteq I^{\rho\sigma} W_\rho dW_\sigma = (I^{\rho\sigma} E_\rho F_\sigma)(\tau^2 d\zeta) \tag{3.11b}$$

and $\gamma_\Theta \simeq S^1 \times S^1$.

Each S^1-piece is thought here of as an oriented circled loop which suitably surrounds the singularities lying in the respective complex plane. It follows that we can sort out a single overall contour for all the integrals of either type. In the projective case, in particular, it is frequently convenient to take each of the pieces borne by the common $(S^1 \times S^1)$-contour as a circle which intersects the interior of the other, with a similar construction certainly applying to the non-projective case as well. We should mention that projective structures of this kind were utilized earlier in a work dealing with general techniques for explicitly evaluating twistor diagrams in real Minkowski space [7]. Therefore, using Eqs. (2.2) and the scaling-invariant relation at x^a

$$\nabla_{AA'} = i(\pi_{A'} \hat{\pi}_A - \lambda_A \hat{\lambda}_{A'}) \,, \tag{3.12a}$$

where

$$\hat{\pi}_A \doteq \frac{\partial}{\partial \omega^A}, \hat{\lambda}_{A'} \doteq \frac{\partial}{\partial \mu^{A'}} \,, \tag{3.12b}$$

we accordingly obtain the kernel prescriptions

$$\begin{aligned} \mathcal{F}_{AB} &: \lambda_A \lambda_B, 2\lambda_{(A} \hat{\pi}_{B)}, \hat{\pi}_A \hat{\pi}_B \,, \\ \mathcal{G}_{A'B'} &: \pi_{A'} \pi_{B'}, 2\pi_{(A'} \hat{\lambda}_{B')}, \hat{\lambda}_{A'} \hat{\lambda}_{B'} \,, \\ \mathcal{P}_{AA'} &: \lambda_A \pi_{A'}, \pi_{A'} \hat{\pi}_A + \lambda_A \hat{\lambda}_{A'}, \hat{\pi}_A \hat{\lambda}_{A'} \,, \end{aligned}$$

up to a Cauchy-theorem "gauge-freedom". Hence, the relevant homogeneity degrees are supplied by the schemes

			Z	W
$f_1(Z^\alpha, W_\beta) \doteq f_{(\mathcal{F})}$	for	$\mathcal{F}_{AB} = \lambda_A \lambda_B$:	-2	-4
$f_2(Z^\alpha, W_\beta) \doteq f_{(\mathcal{F})}$	for	$\mathcal{F}_{AB} = 2\lambda_{(A} \hat{\pi}_{B)}$:	-1	-3
$f_3(Z^\alpha, W_\beta) \doteq f_{(\mathcal{F})}$	for	$\mathcal{F}_{AB} = \hat{\pi}_A \hat{\pi}_B$:	0	-2

			Z	W
$g_1(Z^\alpha, W_\beta) \doteq g_{(\mathcal{G})}$	for	$\mathcal{G}_{A'B'} = \pi_{A'} \pi_{B'}$:	-4	-2
$g_2(Z^\alpha, W_\beta) \doteq g_{(\mathcal{G})}$	for	$\mathcal{G}_{A'B'} = 2\pi_{(A'} \hat{\lambda}_{B')}$:	-3	-1
$g_3(Z^\alpha, W_\beta) \doteq g_{(\mathcal{G})}$	for	$\mathcal{G}_{A'B'} = \hat{\lambda}_{A'} \hat{\lambda}_{B'}$:	-2	0

and

$$\begin{array}{lll} & & Z \quad W \\ P_1(Z^\alpha, W_\beta) \doteqdot P_{(\mathcal{P})} & \text{for} & \mathcal{P}_{AA'} = \lambda_A \pi_{A'} : \quad -3 \quad -3 \\ P_2(Z^\alpha, W_\beta) \doteqdot P_{(\mathcal{P})} & \text{for} & \mathcal{P}_{AA'} = \pi_{A'} \hat{\pi}_A + \lambda_A \hat{\lambda}_{A'} : \quad -2 \quad -2 \\ P_3(Z^\alpha, W_\beta) \doteqdot P_{(\mathcal{P})} & \text{for} & \mathcal{P}_{AA'} = \hat{\pi}_A \hat{\lambda}_{A'} : \quad -1 \quad -1 \end{array}$$

whence each integral bears a scaling-invariant integrand. We observe that the potential kernels are invariant under the simultaneous interchanges

$$\pi_{A'} \rightleftharpoons \lambda_A \,, \quad \hat{\pi}_A \rightleftharpoons \hat{\lambda}_{A'} \,, \tag{3.13}$$

with the kernel for $P_2(Z^\alpha, W_\beta)$ being also scaling invariant. The former invariance property exhibits the "reality" of $\Phi_{AA'}(x)$. For simplicity, we will no longer write the arguments of the twistor functions.

It is evident that the functional structures $\{\mathcal{S}_{AA'} S_{(\mathcal{S})}\}$ for $J_{AA'}(x)$ may be specified by invoking (2.9) and operating with (3.12a) on either set of field integrals. Equivalently, this specification can also be achieved by utilizing a procedure which takes into account Eq. (2.11) along with the operator compositions

$$\blacksquare_{\boldsymbol{ZW}} = (-2)\hat{\boldsymbol{Z}} \circ \hat{\boldsymbol{W}} = (-2)\hat{\boldsymbol{W}} \circ \hat{\boldsymbol{Z}} = \blacksquare_{\boldsymbol{WZ}} \tag{3.14}$$

and the definitions

$$\hat{\boldsymbol{Z}} \doteqdot \lambda_A \hat{\pi}^A = I^{\rho\sigma} W_\rho \frac{\partial}{\partial Z^\sigma} \,, \tag{3.15a}$$

$$\hat{\boldsymbol{W}} \doteqdot \pi_{A'} \hat{\lambda}^{A'} = I_{\mu\nu} Z^\mu \frac{\partial}{\partial W_\nu} \,. \tag{3.15b}$$

What results is, in effect, that the kernels $\{\mathcal{S}_{AA'}\}$ for the current density coincide with the kernels for the potential. Because of the commutativity property of the (scaling-invariant) compositions (3.14), we can unambiguously denote as $\blacksquare$ either of the second-order derivative operators involved. Such objects are the Penrose wave operators mentioned in Section 1.

The pattern of the operator (3.12a), which is clearly real in the sense of (3.13), enables us to see easily that all the potential kernels yield $\Lambda(x) = 0$. Hence, the Lorentz-gauge condition is deeply rooted into the spinor-kernel structure of the system. The coincidence between the potential and current-density kernels essentially enhances this striking feature of the system, as can be seen at once by calling upon Eq. (2.13). Consequently, the vanishing of the Lorentz scalar effectively entails the divergencelessness of $J^{AA'}(x)$.

Furthermore, the right-hand sides of (2.2) can now be rewritten in a somewhat simpler way as $\nabla_{A'A}\Phi_B^{A'}(x)$ and $\nabla_{AA'}\Phi_{B'}^{A}(x)$. We thus have the field-potential relations

$$if_k \;=\; (-1)\hat{\boldsymbol{W}}P_k \tag{3.16a}$$

and

$$(-ig_k) \;=\; (-1)\hat{\boldsymbol{Z}}P_k\,, \tag{3.16b}$$

with $k = 1, 2, 3$. The identity (2.5) turns out to be expressed as

$$\hat{\boldsymbol{Z}}(if_k) = \hat{\boldsymbol{W}}(-ig_k)\,. \tag{3.17}$$

We end this section by observing that, on the basis of the field and wave equations of Section 2, one can derive a complete set of relationships involving adequately the spinor–kernel schemes given above. However, some of these structures will not play any crucial role in the development of Sections 4 and 5, whence they will not be spelt out here (see Ref. [6]).

4. The dynamical principle

Let $\tau^{\pm}$ be a bounded open subset of $\mathrm{T}_{\pm} \times \mathrm{T}_{\pm}^{*}$ whose entries are locally defined by the real and imaginary parts of $\omega^A(x)$ and $\mu^{A'}(x)$ according to the elementary prescription

$$(Z^0, Z^1) \;\leftrightarrow\; \omega^A(x) = B^A(x) + iC^A(x)\,, \tag{4.1a}$$

$$(W_2, W_3) \;\leftrightarrow\; \mu^{A'}(x) = X^{A'}(x) + iY^{A'}(x)\,. \tag{4.1b}$$

The suitability of the prescription (4.1) is partly due to the fact that the only twistor-derivative operators which actually enter into the ∇-expression (3.12a) are the ones defined with respect to the conformallyinvariant parts of the twistors which occur in Eqs. (3.1). The adequacy of $\tau^{\pm}$ is effectively made up by imposing both smoothness on the boundary $\partial\tau^{\pm}$ of its closure $\bar{\tau}^{\pm}$ and compactness on $\bar{\tau}^{\pm}$. For the volume of $\tau^{\pm}$, we thus have the (real) integral [5]

$$\vartheta[\tau^{\pm}] = \int\limits_{\tau^{\pm}} \mathfrak{D}\mathcal{B}\mathcal{C}\mathcal{X}\mathcal{Y}\,, \tag{4.2}$$

where $\mathfrak{D}\mathcal{B}\mathcal{C}\mathcal{X}\mathcal{Y}$ is an eight-form on $\tau^{\pm}$ given as the wedge product between the forms

$$\underset{\sim}{\mathcal{B}}(x) \;=\; \tfrac{1}{2}[dB_A(x) \wedge dB^A(x)]\,, \tag{4.3a}$$

$$\underset{\sim}{\mathcal{C}}(x) \;=\; \tfrac{1}{2}[dC_B(x) \wedge dC^B(x)]\,, \tag{4.3b}$$

and

$$\underset{\sim}{\mathcal{X}}(x) \;=\; \tfrac{1}{2}[dX_{A'}(x)\wedge dX^{A'}(x)]\,, \tag{4.3c}$$

$$\underset{\sim}{\mathcal{Y}}(x) \;=\; \tfrac{1}{2}[dY_{B'}(x)\wedge dY^{B'}(x)]\,. \tag{4.3d}$$

In setting up the dynamical principle, it appears to be necessary to replace the individual pieces of the CM-Lagrangian density with the kernels borne by the contour integrals of Section 2. This necessity stems from the uselessness of the result that arises when the contour integrals themselves are immediately coupled together. We thus arrive at the following set of field contributions

$$\begin{aligned}
&\phi_{AB}(x)\phi^{AB}(x):\\
&f_1(\hat{\boldsymbol{Z}}^2 f_3)\,,\quad (-2)(\hat{\boldsymbol{Z}} f_2)^2\,,\quad 2(\hat{\pi}_A f_2)(\hat{\pi}^A\hat{\boldsymbol{Z}} f_3)\,,\\
&\psi_{A'B'}(x)\psi^{A'B'}(x):\\
&g_1(\hat{\boldsymbol{W}}^2 g_3)\,,\quad (-2)(\hat{\boldsymbol{W}} g_2)^2\,,\quad 2(\hat{\lambda}_{A'} g_2)(\hat{\lambda}^{A'}\hat{\boldsymbol{W}} g_3)\,,
\end{aligned}$$

with the pertinent conjugation, $\hat{c}$, amounting to the combination of (3.13) with the interchange rules

$$Z^\alpha \leftrightarrow W_\beta\,,\quad I_{\mu\nu}\leftrightarrow I^{\rho\sigma}\,,\quad if_k\leftrightarrow(-ig_k)\,, \tag{4.4}$$

where k stands for the same label as before. We point out that the above-defined conjugation is still applicable to Eqs. (3.16) and (3.17). Making use of (2.2) and accounting for the Lorentz gauge, we obtain the alternative structures

$$\begin{aligned}
&\nabla_{A'A}\Phi^{A'}_{B}(x)\nabla^{A}_{B'}\Phi^{BB'}(x):\\
&\tfrac{1}{2}(\hat{\boldsymbol{W}} P_1)(\hat{\boldsymbol{Z}}\blacksquare P_3)\,,\quad \tfrac{1}{2}(\blacksquare P_2)^2\,,\quad (\hat{\pi}_A\hat{\boldsymbol{W}} P_2)(\hat{\pi}^A\blacksquare P_3)\,,
\end{aligned}$$

along with the $\hat{c}$-conjugate ones for

$$\nabla_{AA'}\Phi^{A}_{B'}(x)\nabla^{A'}_{B}\Phi^{B'B}(x)=\psi_{A'B'}(x)\psi^{A'B'}(x)\,. \tag{4.5}$$

We can construct the blocks for the interacting piece of (2.8) by implementing the identity $\mathcal{P}_{AA'}\equiv\mathcal{S}_{AA'}$ as well as the natural correspondence $P_k\to S_k$ between the twistor functions for the potential and the ones for the current density. In effect, we have the contributions

$$\begin{aligned}
&\Phi_{AA'}(x)J^{AA'}(x):\\
&(-\tfrac{1}{2})(\blacksquare P_3)S_1\,,\quad (-1)[(\hat{\boldsymbol{W}} P_2)(\hat{\boldsymbol{Z}} S_2)+(\hat{\boldsymbol{Z}} P_2)(\hat{\boldsymbol{W}} S_2)]\,,\\
&(\hat{\pi}_A P_2)(\hat{\pi}^A\hat{\boldsymbol{W}} S_3)+(\hat{\lambda}_{A'} P_2)(\hat{\lambda}^{A'}\hat{\boldsymbol{Z}} S_3)\,,\quad (\hat{\pi}_A\hat{\lambda}_{A'} P_3)(\hat{\pi}^A\hat{\lambda}^{A'} S_3)\,,
\end{aligned}$$

together with the structures which are obtained out of the ones just given by interchanging the kernel letters P and S. It should be noticed that such interchanges will leave the second and fourth ΦJ-blocks invariant if a suitable commutation assumption is effectively taken up.

We have thus been led to two dynamical sets of basic Lagrangian densities that carry the functional pieces for either

$$\{\phi_{AB}(x), \psi_{A'B'}(x), \Phi_{AA'}(x), J_{AA'}(x)\}$$

or

$$\{\Phi_{AA'}(x), J_{AA'}(x)\}.$$

It has become clear that the pieces of one density are related to the pieces of any other density through a trivial set of interchanges. Hence, supposing that the twistor functions and derivatives are all continuous in $\bar{\tau}^{\pm}$, and defining

$$\mathcal{L}_{\mathrm{TM}} = \mathcal{L}_{\mathrm{TM}}^{(\mathrm{FREE})} + \mathcal{L}_{\mathrm{TM}}^{(\mathrm{INT})}, \tag{4.6}$$

with $\mathcal{L}_{\mathrm{TM}}^{(\mathrm{FREE})}$ and $\mathcal{L}_{\mathrm{TM}}^{(\mathrm{INT})}$ denoting symbolically the free and interacting parts of any density $\mathcal{L}_{\mathrm{TM}}$, we write the action

$$\mathcal{S}[\mathcal{L}_{\mathrm{TM}}] = \int\limits_{\tau^{\pm}} (\mathcal{L}_{\mathrm{TM}}^{(\mathrm{FREE})} + \mathcal{L}_{\mathrm{TM}}^{(\mathrm{INT})}) \mathfrak{D}\mathcal{B}\mathcal{C}\mathcal{X}\mathcal{Y}, \tag{4.7}$$

along with the variational statement

$$\Delta\mathcal{S}[\mathcal{L}_{\mathrm{TM}}] = 0 = \Delta \int\limits_{\tau^{\pm}} (\mathcal{L}_{\mathrm{TM}}^{(\mathrm{FREE})} + \mathcal{L}_{\mathrm{TM}}^{(\mathrm{INT})}) \mathfrak{D}\mathcal{B}\mathcal{C}\mathcal{X}\mathcal{Y}. \tag{4.8}$$

The Δ-variation of Eq. (4.8) is prescribed so as to bear linearity and the Leibniz-rule property. Likewise, it preserves "reality" and does not involve any deformation of $\partial\tau^{\pm}$. In addition, it is required to commute with the conformally invariant twistor-derivative operators (3.12b) and, consequently, also with the Poincaré-invariant ones of (3.14) and (3.15) since Z^{α} and W_{β} are both held fixed when the variational process is put into effect. Presumably, the union of the singularity sets of the functional pieces carried by all the densities possesses a void intersection with $\tau^{\pm} \cup \partial\tau^{\pm}$, and $\Delta S_k \equiv 0$ throughout $\bar{\tau}^{\pm}$. We should emphasize that, in any case, the field and potential variations must be taken as arbitrary non-singular quantities on $\tau^{\pm} \cup \partial\tau^{\pm}$ which vanish on $\partial\tau^{\pm}$.

5. Equations of motion

The Δ-variation specified in the foregoing Section tells us that we have effectively to perform actual integrations by parts upon working out the statement (4.8). This fact implies that the Lagrangian blocks which involve the operators (3.12b) explicitly will become useless in case only scalar variations are utilized. Roughly speaking, the point is that such contributions will produce identically vanishing structures if they are brought into $\mathcal{S}[\mathcal{L}_{\rm TM}]$ together with scalar variations. At this stage, we will restrict ourselves to using scalar variations. The situation concerning specifically the utilization of spinor variations will be touched upon later in Section 6.

We thus consider the following operator splitting relations associated with Eqs. (3.15) and (4.1)

$$\hat{\boldsymbol{Z}} \;=\; \tfrac{1}{2}[(I^{\rho\sigma}W_\rho\hat{\boldsymbol{B}}_\sigma) - i(I^{\rho\sigma}W_\rho\hat{\boldsymbol{C}}_\sigma)] \doteqdot \tfrac{1}{2}(\hat{\boldsymbol{B}} - i\hat{\boldsymbol{C}})\,, \tag{5.1a}$$

$$\hat{\boldsymbol{W}} \;=\; \tfrac{1}{2}[(I_{\mu\nu}Z^\mu\hat{\boldsymbol{X}}^\nu) - i(I_{\mu\nu}Z^\mu\hat{\boldsymbol{Y}}^\nu)] \doteqdot \tfrac{1}{2}(\hat{\boldsymbol{X}} - i\hat{\boldsymbol{Y}})\,, \tag{5.1b}$$

which readily provide us with the auxiliary integral formulae

$$\int\limits_{\tau^\pm}(\hat{\boldsymbol{Z}}\Delta\mathfrak{G})\mathfrak{F}\mathfrak{D}\mathcal{B}\mathcal{C}\mathcal{X}\mathcal{Y} = \int\limits_{\partial\tau^\pm}(\Delta\mathfrak{G})\mathfrak{F}(\lambda_A d^7\Theta^A) - \int\limits_{\tau^\pm}(\Delta\mathfrak{G})(\hat{\boldsymbol{Z}}\mathfrak{F})\mathfrak{D}\mathcal{B}\mathcal{C}\mathcal{X}\mathcal{Y}\,, \tag{5.2}$$

$$\int\limits_{\tau^\pm}(\hat{\boldsymbol{W}}\Delta\mathfrak{G})\mathfrak{F}\mathfrak{D}\mathcal{B}\mathcal{C}\mathcal{X}\mathcal{Y} = \int\limits_{\partial\tau^\pm}(\Delta\mathfrak{G})\mathfrak{F}(\pi_{A'}d^7\,\Upsilon^{A'}) - \int\limits_{\tau^\pm}(\Delta\mathfrak{G})(\hat{\boldsymbol{W}}\mathfrak{F})\mathfrak{D}\mathcal{B}\mathcal{C}\mathcal{X}\mathcal{Y} \tag{5.3}$$

and

$$\int\limits_{\tau^\pm}(\blacksquare\Delta\mathfrak{G})\mathfrak{F}\mathfrak{D}\mathcal{B}\mathcal{C}\mathcal{X}\mathcal{Y} - 2\int\limits_{\partial\tau^\pm}(\Delta\mathfrak{G})(\hat{\boldsymbol{Z}}\mathfrak{F})(\pi_{A'}d^7\,\Upsilon^{A'}) + 2\int\limits_{\partial\tau^\pm}(\hat{\boldsymbol{W}}\Delta\mathfrak{G})\mathfrak{F}(\lambda_A d^7\Theta^A)$$

$$= \int\limits_{\tau^\pm}(\Delta\mathfrak{G})(\blacksquare\mathfrak{F})\mathfrak{D}\mathcal{B}\mathcal{C}\mathcal{X}\mathcal{Y}\,, \tag{5.4}$$

where $\mathfrak{F}$ and $\mathfrak{G}$ stand for any relevant functions, and

$$d^7\Theta^A(x) \;=\; \tfrac{1}{2}\left\{\left[dB^A(x)\wedge\underset{\sim}{\mathcal{C}}(x) - i\,\underset{\sim}{\mathcal{B}}(x)\wedge dC^A(x)\right]\wedge\underset{\sim}{\mathcal{X}}(x)\wedge\underset{\sim}{\mathcal{Y}}(x)\right\}, \tag{5.5a}$$

$$d^7\Upsilon^{A'}(x) \;=\; \tfrac{1}{2}\left\{\underset{\sim}{\mathcal{B}}(x)\wedge\underset{\sim}{\mathcal{C}}(x)\wedge\left[dX^{A'}(x)\wedge\underset{\sim}{\mathcal{Y}}(x) - i\,\underset{\sim}{\mathcal{X}}(x)\wedge dY^{A'}(x)\right]\right\}, \tag{5.5b}$$

define two independent elements of seven-hypersurface area of $\partial\tau^{\pm}$, with the integrations having been performed from right to left. In passing, we observe that the calculations leading to Eqs. (5.2)–(5.4) incidently bring about the conjugation correspondence

$$[\lambda_A d^7\,\Theta^A(x)] \leftrightarrow [\pi_{A'} d^7\,\Upsilon^{A'}(x)]\,. \tag{5.6}$$

Additionally, the explicit factor 1/2 of (5.5) should be implemented in place of the mistaken, but ultimately irrelevant, factor 1/4 which is carried by Eqs. (3.5) and (3.6) of Ref. [1].

The patterns of the free and interacting Lagrangian contributions naturally suggest writing the variational set for the system

$$\{\{\Delta P_1, \Delta P_3\}, \{\Delta P_2\}\}\,, \tag{5.7}$$

with all the entries involved being independent of each other. We thus have the particular densities

$$\mathcal{L}^{(1)}_{\rm TM} = \frac{1}{16\pi}[(\hat{\boldsymbol{W}} P_1)\hat{\boldsymbol{Z}}\blacksquare P_3 + (\hat{\boldsymbol{Z}} P_1)\hat{\boldsymbol{W}}\blacksquare P_3] - \frac{1}{4}[(\blacksquare P_3)S_1 + P_1\blacksquare S_3] \tag{5.8}$$

and

$$\mathcal{L}^{(2)}_{\rm TM} = \frac{1}{8\pi}(\blacksquare P_2)^2 - [(\hat{\boldsymbol{W}} P_2)\hat{\boldsymbol{Z}} S_2 + (\hat{\boldsymbol{Z}} P_2)\hat{\boldsymbol{W}} S_2]\,, \tag{5.9}$$

whose variations are given by

$$\Delta\mathcal{L}^{(1)}_{\rm TM} = \frac{1}{16\pi}\{[(\hat{\boldsymbol{W}}\,\Delta P_1)\hat{\boldsymbol{Z}}\blacksquare P_3 + (\hat{\boldsymbol{W}} P_1)\hat{\boldsymbol{Z}}\blacksquare\Delta P_3] + \hat{c}\ -\text{conjugate}\}$$

$$-\frac{1}{4}[(\blacksquare\Delta P_3)S_1 + (\Delta P_1)\blacksquare S_3] \tag{5.10}$$

and

$$\Delta\mathcal{L}^{(2)}_{\rm TM} = \frac{1}{4\pi}(\blacksquare\Delta P_2)\blacksquare P_2 - [(\hat{\boldsymbol{W}}\,\Delta P_2)\hat{\boldsymbol{Z}} S_2 + \hat{c}\ -\text{conjugate}]\,. \tag{5.11}$$

It follows that, invoking (4.8) and taking $\Delta P_k = 0$ on $\partial\tau^{\pm}$, after some manipulations we obtain the equations of motion

$$(\blacksquare\Pi_k - 4\pi\Sigma_k) = 0, \Pi_k = \blacksquare P_k, \Sigma_k = \blacksquare S_k\,, \tag{5.12}$$

where k runs over the same values as in Eqs. (3.16).

A glance at Eqs. (3.16), (5.8) and (5.9) shows that, in contradistinction to the CM-formulation [5], each of the equations of motion for $\{if_k, (-ig_k)\}$

can be derived from a manifestly field structure by suitably changing the choice of variations. In effect, for the case of (5.9), we have

$$\begin{aligned}\Delta\mathcal{L}^{(3)}_{\rm TM} &= \frac{1}{2\pi}\{[\hat{\boldsymbol{Z}}\Delta(if_2)]\hat{\boldsymbol{Z}}(if_2)+[\hat{\boldsymbol{W}}\Delta(-ig_2)]\hat{\boldsymbol{W}}(-ig_2)\}\\ &\quad+\{[\Delta(if_2)]\hat{\boldsymbol{Z}}S_2+[\Delta(-ig_2)]\hat{\boldsymbol{W}}S_2\}\,,\end{aligned}\tag{5.13}$$

with $\{\Delta(if_2),\Delta(-ig_2)\}$ thus taking over the dynamical role of ΔP_2. Hence, making use of the integration techniques that give rise to (5.12), we derive the equation

$$\hat{\boldsymbol{Z}}\left[\frac{1}{2\pi}\hat{\boldsymbol{Z}}(if_2)-S_2\right]+\hat{\boldsymbol{W}}\left[\frac{1}{2\pi}\hat{\boldsymbol{W}}(-ig_2)-S_2\right]=0\,,\tag{5.14}$$

which obviously represents the second half of the theory. Now, combining Eqs. (5.14) and (3.17) for $k=2$, we conclude that

$$(\hat{\boldsymbol{Z}}+\hat{\boldsymbol{W}})[\hat{\boldsymbol{Z}}(if_2)-2\pi S_2]=0=(\hat{\boldsymbol{Z}}+\hat{\boldsymbol{W}})[\hat{\boldsymbol{W}}(-ig_2)-2\pi S_2]\,.\tag{5.15}$$

The density (5.8) can be similarly worked out by adopting the variational set $\{\Delta P_1,\Delta F_3,\Delta G_3\}$, whose entries particularly involve the quantities

$$F_3\doteqdot\hat{\boldsymbol{Z}}(if_3)=\hat{\boldsymbol{W}}(-ig_3)\doteqdot G_3\,.\tag{5.16}$$

We thus obtain the field equations

$$\hat{\boldsymbol{Z}}(if_1)=2\pi S_1\;=\;\hat{\boldsymbol{W}}(-ig_1)\,,\tag{5.17a}$$

along with

$$\hat{\boldsymbol{W}}[\hat{\boldsymbol{Z}}(if_3)-2\pi S_3]\;=\;0\tag{5.17b}$$

and

$$\hat{\boldsymbol{Z}}[\hat{\boldsymbol{W}}(-ig_3)-2\pi S_3]\;=\;0\,.\tag{5.17c}$$

In order to represent the complete theory, it suffices to take

$$\blacksquare P_k\;=\;4\pi S_k\tag{5.18a}$$

and

$$\hat{\boldsymbol{Z}}(if_k)\;=\;2\pi S_k=\hat{\boldsymbol{W}}(-ig_k)\,.\tag{5.18b}$$

It is clear that Eqs. (5.18) are identical to the relations which emerge from contracting (2.11) and (2.9) with adequate kernels. As the formulation of (4.8) automatically balances the weights of any allowable scaling factors, the spinning character of the fields turns out to be reflected by the homogeneity degrees of the respective twistor functions.

6. Concluding remarks and outlook

One might claim that, in the first instance, inserting the contour integrals into the former field equations would not immediately yield a system of differential relations insofar as the contours involved are generally taken to be distinct from one another. In fact, this situation has been rectified in Section 3 by implementing a parametric form of the integrands which allows the definition of a single contour for the sets of integrals (3.5)–(3.10). The same parametrization procedure was used in Ref. [2], and has indeed been considered as a somewhat important part of our method. When utilized in conjunction with the product-space prescription given in Ref. [8], it may give rise to the construction of formal Green-function solutions to twistor wave equations. Moreover, it actually accounts for the eventual occurrence of the inner product $Z^{\mu}W_{\mu}$ in the denominators of the integrands, whence its combination with the techniques of Ref. [7] would enable one to write down an integral form of the theory in terms of twistor diagrams [9–16]. That it is consistent with the degrees of the standard differential forms can be rapidly established by considering the equivalent scaling-invariant relations

$$\int\limits_{\gamma_Z} F(a^{-1}Z^{\beta})\delta(a^{-1}Z) = \int\limits_{\gamma_Z} F(Z^{\beta})\delta Z$$

$$\Updownarrow$$

$$\int\limits_{\gamma_X} F(aX^{\beta})\delta(aX) = \int\limits_{\gamma_X} F(X^{\beta})\delta X,$$

with a being any (fixed) non-vanishing complex number and $Z^{\beta} = aX^{\beta}$.

One of the most notable results we have obtained above is related to the possibility of deriving directly from (4.8) the wave equations

$$\begin{aligned} \blacksquare(if_m) &= (-4\pi)\hat{\boldsymbol{W}}S_m\,, \\ \blacksquare(-ig_m) &= (-4\pi)\hat{\boldsymbol{Z}}S_m\,, \end{aligned}$$

which represent the (gauge-invariant) CM-equation

$$\Box F_{ab}(x) = 8\pi\nabla_{[a}J_{b]}(x)\,,$$

with m taking the values 2 and 3. The pertinent procedure for $m = 3$, for instance, consists first in re-expressing (5.8) as (see Eqs. (3.16))

$$\begin{aligned} \mathcal{L}^{(4)}_{\mathrm{TM}} &= \frac{(-1)}{16\pi}[(\hat{\boldsymbol{Z}}P_1)\blacksquare(if_3) + (\hat{\boldsymbol{W}}P_1)\blacksquare(-ig_3)] \\ &\quad -\frac{1}{4}\{[\hat{\boldsymbol{Z}}(if_3) + \hat{\boldsymbol{W}}(-ig_3)]S_1 + P_1\blacksquare S_3\}\,, \end{aligned}$$

and then picking up the piece of the statement (4.8) that carries ΔP_1. Hence, it seems to be of some interest to find out whether the wave equations for $\{if_1, (-ig_1)\}$ may be achieved in a similar fashion. In the event that explicit spinor relations involving the kernels for the current density have to be effectively required, the introduction of the pertinent contour integrals shall be based upon the charge-conservation law $\nabla_{AA'} J^{AA'}(x) = 0$, which accordingly will enter into the picture as a formal "field equation" for $J^{AA'}(x)$. Of course, the Lorentz-gauge feature of the twistor formulation would demand the implementation of the famous Gupta–Bleuler procedure if a quantum version of the theory were taken into consideration at the outset.

With regard to the choices of variations for the system, we must stress that our representation makes it viable to utilize spinor variations along with the Lagrangian densities which carry the pieces

$$(\hat{\pi}_A \hat{\boldsymbol{W}} P_2)(\hat{\pi}^A \blacksquare P_3), (\hat{\lambda}_{A'} \hat{\boldsymbol{Z}} P_2)(\hat{\lambda}^{A'} \blacksquare P_3)\,,$$

$$(\hat{\pi}_A P_2)(\hat{\pi}^A \hat{\boldsymbol{W}} S_3) + (\hat{\lambda}_{A'} P_2)(\hat{\lambda}^{A'} \hat{\boldsymbol{Z}} S_3)\,,$$

$$(\hat{\pi}^A \hat{\boldsymbol{W}} P_3)(\hat{\pi}_A S_2) + (\hat{\lambda}^{A'} \hat{\boldsymbol{Z}} P_3)(\hat{\lambda}_{A'} S_2)\,,$$

and

$$(\hat{\pi}_A \hat{\lambda}_{A'} P_3)(\hat{\pi}^A \hat{\lambda}^{A'} S_3)\,.$$

In some particular cases, these configurations yield a set of spinor equations of motion which involve twistor functions bearing different values of the label k, but their utilization does not provide any additional insights into the dynamical framework. A genuinely useful example is afforded by the density

$$\begin{aligned}\mathcal{L}^{(5)}_{\mathrm{TM}} \;=\; & \frac{1}{8\pi}\Big[(\hat{\pi}_A \hat{\boldsymbol{W}} P_2)(\hat{\pi}^A \blacksquare P_3) + (\hat{\lambda}_{A'} \hat{\boldsymbol{Z}} P_2)(\hat{\lambda}^{A'} \blacksquare P_3)\Big] \\ & + \frac{1}{2}\Big[(\hat{\pi}_A P_2)(\hat{\pi}^A \hat{\boldsymbol{W}} S_3) + (\hat{\lambda}_{A'} P_2)(\hat{\lambda}^{A'} \hat{\boldsymbol{Z}} S_3) + (\hat{\pi}_A \hat{\lambda}_{A'} P_3)(\hat{\pi}^A \hat{\lambda}^{A'} S_3)\Big]\end{aligned}$$

with the "mixed" prescription

$$\{\Delta(\hat{\pi}_A P_2), \Delta(\hat{\lambda}_{A'} P_2), \Delta P_3\}\,,$$

in which case we obtain the wave equation for P_3. It now becomes evident that the number of entries of each set of basic densities will get considerably reduced if only scalar variations are put into practice. We can therefore state that the exclusion of the densities which bear the potential functions

$$\{P_2, P_3\}, \{P_2, S_3\}, \{P_3, S_2\}, \{P_3, S_3\}\,,$$

would bring forward a non-trivial character of the whole representation without spoiling the completeness of the description.

Worthy of special consideration is the occurrence of the $\hat{c}$-conjugation property

$$\int\limits_{\partial\tau^{\pm}} (\Delta\mathfrak{G})(\hat{\boldsymbol{Z}}\mathfrak{F})\left(\pi_{A'}d^{7}\,\Upsilon^{A'}\right) - \int\limits_{\partial\tau^{\pm}} (\hat{\boldsymbol{W}}\Delta\mathfrak{G})\mathfrak{F}\left(\lambda_{A}d^{7}\,\Theta^{A}\right)$$
$$= \int\limits_{\partial\tau^{\pm}} (\Delta\mathfrak{G})(\hat{\boldsymbol{W}}\mathfrak{F})\left(\lambda_{A}d^{7}\,\Theta^{A}\right) - \int\limits_{\partial\tau^{\pm}} (\hat{\boldsymbol{Z}}\Delta\mathfrak{G})\mathfrak{F}\left(\pi_{A'}d^{7}\,\Upsilon^{A'}\right)\,,$$

which takes place when the reality of the $\blacksquare$-operator is incorporated into the construction of the device (5.4). Had the integrations by parts of Section 5 been performed from left to right the same volume integrals would have been obtained, but the overall signs of the boundary integrals of Eqs. (5.2)–(5.4) would have to be readjusted. It is obvious that the twistors $\{B^{\sigma}, C^{\sigma}, X_{\nu}, Y_{\nu}\}$ involved in Eqs. (5.1) have been introduced to define the real and imaginary parts of the operators (3.15). Nevertheless, it has not been strictly necessary to ascribe to them any group-theoretic meaning at all.

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