INDUCED SYMPLECTIC CONNECTION ON THE PHASE SPACE

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It is shown that the general theory of lifting the tensor fields from a Riemannian manifold M to its tangent bundle TM enables one to define in a natural manner the unique sympletic connection on the phase space T^*M which is induced by the Levi–Civita connection on M. This is exactly the symplectic connection given also by Bordemann, Neumaier and Waldmann *Commun. Math. Phys.* **198**, 363 (1998); *J. Geom. Phys.* **29**, 199 (1999). Relationship between the symplectic and Riemannian geometries on T^*M and M is considered.

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1. Introduction

The concept of deformation quantization was introduced in 1978 by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [3]. Briefly speaking the aim of deformation quantization is to understand quantization as a deformation of the usual product algebra of functions on the phase space and a deformation of the Poisson bracket algebra. The deformed product is called the $*$-*product* and the deformed Poisson bracket is usually called the *Moyal bracket.*

De Wilde and Lecomte [4] have shown that the $*$-product exists for any symplectic manifold. Recently one observes a great interest in deformation quantization [5]. This interest is evidently stimulated by outstanding works by Omori, Maeda and Yoshioka [6], Fedosov [7,8] and Kontsevich [9] where new approaches to the construction of the $*$-product are proposed. Especially beautiful is Fedosov's approach where the $*$-product for a symplectic manifold is defined in terms of the *Weyl algebras bundle geometry* and, on the other hand, Kontsevich's approach where some ideas of string theory enables one to construct the $*$-product for *any Poisson structure on* $\mathbb{R}^n$.

(See also the work by Kathotia [10] devoted to the relationship between the Kontsevich construction and the Campbell–Baker–Hausdorff formula).

In our work we deal with Fedosov's construction. We consider this construction in some details in the second part of the work. Here will be enough to observe that the essential point of the Fedosov $*$-product is the symplectic connection. It is known that any paracompact symplectic manifold can be endowed with a symplectic connection [8,11,12]. Then this connection defines the $*$-product. Although two different symplectic connections lead to equivalent (in the mathematical sense) quantizations [12] the physical content of these quantizations is different. In contrast to the case of Levi–Civita connection the symplectic connection is not uniquely defined and, in fact, the set of symplectic connections on a symplectic manifold is an infinite-dimensional affine space [11].

Therefore, from the physical point of view, the crucial point is to find some method which enables one to define the unique symplectic connection in a natural manner. In Ref. [13] such a method has been proposed. However it requires the definition of the preferred atlas on a symplectic manifold. Another approach has been proposed by Bourgeois, Cahen, Gutt and Rawnsley [14,15]. Roughly speaking, in this approach the symplectic connection comes from the variational principles of the Yang–Mills type.

Some time ago, Oziewicz [16] told us about the idea of lifting the geometrical objects from a manifold to its tangent or contangent bundle and that this idea could help one to geometrize some equations of mathematical physics. We have recognized very quickly that this idea could also enable us to define in a natural way the symplectic connection on a phase space. Indeed, if the Lagrangian of a given system of particles is a quadratic form of their velocities then it defines a Riemannian metric g on the configuration space M. Then one can lift this metric to the tangent bundle TM over M. Now as the metric g on M defines a natural isomorphism between TM and the cotangent bundle (*i.e.*, the phase space) T^*M over M we are able to transport the metric on TM to T^*M. Thus T^*M is not only endowed with the naturally defined symplectic form ω but also with a Riemannian metric $\widetilde{g}$ of signature $\Bigg(\underbrace{+,...,+}_{n},\underbrace{-,...,-}_{n}\Bigg)$, $n = \dim M$. Finally, the metric $\widetilde{g}$ defines the Levi–Civita connection on T^*M which in turn defines the symplectic connection $\nabla^{(\mathrm{S})}$ on T^*M. The crucial point of the proposed procedure lies in the fact that $\widetilde{g}, \nabla^{(\mathrm{S})}$ and the curvature of $\nabla^{(\mathrm{S})}$ are defined by the Levi–Civita connection of M. Thus *geometry of the phase space is induced by the geometry of the configuration space.*

In two distinguished papers Bordemann, Neumaier and Waldmann [1,2] used the "Fedosov type procedure" to find some homogeneous $*$-products on cotangent bundles over configuration spaces. Their construction required a homogenous in momenta symplectic connection which they have obtained by using some lifting procedures.

The aim of our paper is to give a geometrical interpretation of this lifted connection by showing that it fits into the general framework of theory of lifts developed by Yano and Ishihara [17].

Our paper is organized as follows. In Section 2 the Riemannian metric on the cotangent bundle T^*M induced by the metric on TM is given. Then the Levi–Civita connection and the curvature tensor for this metric on T^*M is found. Finally, the symplectic connection on T^*M induced by the Levi–Civita connection and the relationship between the symplectic geometry of T^*M and the Riemannian geometry of T^*M or M are considered in Section 3. (Note that we deal only with lifts to TM or T^*M. However, the general theory given by Yano and Ishihara [17] which enables one to define lifts from M to higher order tangent bundles T^rM $r \geq 1$ has found its application in self-dual gravity [18]).

2. Cotangent bundle as a Riemannian manifold

Let M be an n-dimensional smooth differentiable manifold and T^*_QM the cotangent space of M at a point $Q \in M$. Let $T^*M = \bigcup\limits_{Q\in M} T^*_QM$ be the cotangent bundle over M and $\Pi : T^*M \to M$ the bundle projection. Let, $\{U, (q^\alpha)\}$ be a coordinate neighborhood in M. (The Greek indices $\alpha, \beta, ...$ run through $1, ..., n$ and the Latin ones $i, j, ...$ run through $1, ..., 2n$). Then one can define in a natural way the coordinate neighborhood $\left\{\Pi^{-1}(U), \left(\widetilde{q}^i\right)\right\}$, i in T^*M as follows: if Q is a point of U of the coordinates $\left(q^1, ..., q^n\right)$ and $p = p_\alpha \left(dq^\alpha\right)_Q$ is a cotangent vector at Q then we assign the coordinates $\left(\widetilde{q}^i\right)$ to the point $(Q, p) \in \Pi^{-1}(Q)$ according to the rule

$$\widetilde{q}^\alpha := q^\alpha \quad \text{and} \quad \widetilde{q}^{\alpha+n} = p_\alpha, \quad \alpha = 1, ..., n\,. \tag{2.1}$$

Doing so for all points of U and all cotangent vectors on U one gets the coordinate neighborhood $\left\{\Pi^{-1}(U), (q^\alpha, p_\alpha)\right\}$ in T^*M which is called the *coordinate neighborhood in* T^*M *induced by* $\{U, (q^\alpha)\}$.

Having defined the induced coordinates in T^*M one can introduce the *basic 1-form* θ *on* T^*M *by*

$$\Gamma\left(T^*\left(T^*M\right)\right) \ni \theta := p_\alpha d(q^\alpha \circ \Pi) \tag{2.2}$$

for every induced coordinates in T^*M ($\Gamma(...)$ denotes the section of the respective bundle).

Then the exterior differential of θ gives the *symplectic form* ω *on* T^*M

$$\Gamma\left(\Lambda^2 T^*M\right) \ni \omega := d\theta = \tfrac{1}{2}\omega_{ij} d\widetilde{x}^i \wedge d\widetilde{x}^j = dp_\alpha \wedge d(q^\alpha \circ \Pi), \quad \omega_{ij} = -\omega_{ji}\,, \tag{2.3}$$

where $\left(\widetilde{x}^i\right)$ is a system of local coordinates in T^*M and $\Lambda^2 T^*M$ stands for the bundle of 2-forms on T^*M. From (2.3) one quickly finds that in terms of induced coordinates in T^*M

$$(\omega_{ij}) = \begin{pmatrix} 0 & -(\delta^\alpha_\beta) \\ (\delta^\alpha_\beta) & 0 \end{pmatrix}. \tag{2.4}$$

Consequently, (T^*M, ω) is a $2n$-dimensional symplectic manifold and the induced coordinates in T^*M appear to be exactly the *proper Darboux coordinates* considered in [13].

Assume now that the differential manifold M is endowed with a Riemannian metric g. This enables us to define the natural bundle isomorphism $\rho : TM \to T^*M$ as follows

$$\rho\left(\widetilde{\widetilde{Q}}\right) := \left(\pi\left(\widetilde{\widetilde{Q}}\right), \pi'\left(\widetilde{\widetilde{Q}}\right) \rfloor g_{\pi\left(\widetilde{\widetilde{Q}}\right)}\right) \in T^*M \tag{2.5}$$

for every $\widetilde{\widetilde{Q}} \in TM$. For local induced coordinates we get

$$\rho\left(q^\alpha, p^\alpha\right) = \left(q^\alpha, p_\alpha\right), \qquad p_\alpha = p^\beta g_{\beta\alpha}(q^\gamma)\,. \tag{2.6}$$

With the use of the isomorphism ρ we can send the objects defined on TM into the objects on T^*M. Especially we are interested in the pull-back of the *complete lift* g^C of the metric g

$$\left(g^C_{ij}\right) = \begin{pmatrix} (p^\gamma \partial_\gamma g_{\alpha\beta}) & (g_{\alpha\beta}) \\ (g_{\alpha\beta}) & 0 \end{pmatrix} \tag{2.7}$$

(see [17]: Eq. (5.14), the propositions 5.9 and 5.10, and the metric II on the page 138), and in the pull-back of the metric G

$$(G_{ij}) = \begin{pmatrix} (g_{\alpha\beta} + p^\gamma \partial_\gamma g_{\alpha\beta}) & (g_{\alpha\beta}) \\ (g_{\alpha\beta}) & 0 \end{pmatrix} \tag{2.8}$$

(see [17]: the metric I+II on the page 138). From (2.7) and (2.6) one easily finds that

$$\widetilde{g} := \rho^{-1*} g^C \Longrightarrow (\widetilde{g}_{ij}) = \begin{pmatrix} \left(-2p_\gamma \Gamma^\gamma_{\alpha\beta}\right) & \left(\delta^\alpha_\beta\right) \\ \left(\delta^\alpha_\beta\right) & 0 \end{pmatrix}, \tag{2.9}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the components of the *Levi–Civita connection (the Christoffel symbols) of the metric g on M i.e.*,

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\delta\beta} - \partial_\delta g_{\beta\gamma}\right). \tag{2.10}$$

It is evident that $\widetilde{g} \in \Gamma\left(T^0_2\left(T^*M\right)\right)$ is a symmetric, nondegenerate tensor field of the type $(0,2)$ on T^*M and the signature of $\widetilde{g}$ is $\Bigg(\underbrace{+,...,+}_{n},\underbrace{-,...,-}_{n}\Bigg)$.

Thus one arrives at the following theorem

Theorem 2.1 *Let (M,g) be an n-dimensional Riemannian manifold with g being a metric on M of an arbitrary signature. Then $(T^*M, \widetilde{g})$ is a 2n-dimensional Riemannian manifold and the metric $\widetilde{g}$ on T^*M has the signature* $\Bigg(\underbrace{+,...,+}_{n},\underbrace{-,...,-}_{n}\Bigg)$. ■

(This theorem corresponds to the proposition 5.9 in [17]).

The inverse metric to $\widetilde{g}$ reads

$$\left(\widetilde{g}^{ij}\right) = \begin{pmatrix} 0 & \left(\delta^\alpha_\beta\right) \\ \left(\delta^\alpha_\beta\right) & \left(2p_\gamma \Gamma^\gamma_{\alpha\beta}\right) \end{pmatrix}. \tag{2.11}$$

It is interesting to note that the tensor field (2.11) appears in a natural manner when the problem of the operator ordering within the Weyl–Wigner–Moyal formalism for the phase space $\mathbb{R}^{2n}$ is extended to curved phase space according to the Fedosov approach [1,2,13]. In other words the metric $\widetilde{g}$ on T^*M enters also into the definition of the "order of operators" in deformation quantization.

Moreover, the metric $\widetilde{g}$ has been introduced in the general theory of the Riemannian extensions of symmetric affine connections [17, 19]. In fact, according to this theory, the metric $\widetilde{g}$ *is the Riemann extension of the Levi–Civita connection on M to T^*M.* Observe that the metric (2.9) is also defined for any M endowed with a symmetric affine connection.

Consider now the pull-back of G to T^*M. From (2.8) and (2.6) we get

$$\widetilde{G} := \rho^{-1*}G \Longrightarrow \left(\widetilde{G}_{ij}\right) = \begin{pmatrix} \left(g_{\alpha\beta} - 2p_\gamma \Gamma^\gamma_{\alpha\beta}\right) & \left(\delta^\alpha_\beta\right) \\ \left(\delta^\alpha_\beta\right) & 0 \end{pmatrix}. \tag{2.12}$$

It is obvious that $\widetilde{G} \in \Gamma\left(T^0_2\left(T^*M\right)\right)$ is a symmetric, nondegenerate tensor field of the type (0,2) on T^*M. Consequently $\left(T^*M, \widetilde{G}\right)$ *is a 2n-dimensional Riemannian manifold.*

In the theory of Riemannian extensions $\widetilde{G}$ is called the *general Riemann extension of the Levi–Civita connection on M to T^*M.*

The inverse metric to $\widetilde{G}$ has the form of

$$\left(\widetilde{G}^{ij}\right) = \begin{pmatrix} 0 & \left(\delta^\alpha_\beta\right) \\ \left(\delta^\alpha_\beta\right) & \left(2p_\gamma \Gamma^\gamma_{\alpha\beta} - g_{\alpha\beta}\right) \end{pmatrix}. \tag{2.13}$$

Now we are in a position to find the Levi–Civita connection $\widetilde{\nabla}$ of the metric $\widetilde{g}$ on T^*M.

Substituting (2.9) and (2.11) into the formula

$$\widetilde{\Gamma}^i_{jk} = \frac{1}{2}\widetilde{g}^{il}\left(\partial_j \widetilde{g}_{lk} + \partial_k \widetilde{g}_{lj} - \partial_l \widetilde{g}_{jk}\right) \tag{2.14}$$

employing also (2.10) we get (see the formula (10.3) in [17])

$$\begin{aligned} \widetilde{\Gamma}^\alpha_{\beta\gamma} &= \Gamma^\alpha_{\beta\gamma}, \qquad \widetilde{\Gamma}^\alpha_{\beta\overline{\gamma}} = 0, \qquad \widetilde{\Gamma}^\alpha_{\overline{\beta}\overline{\gamma}} = 0\,, \\ \widetilde{\Gamma}^{\overline{\alpha}}_{\beta\gamma} &= p_\delta \left(\partial_\alpha \Gamma^\delta_{\beta\gamma} - \partial_\gamma \Gamma^\delta_{\alpha\beta} - \partial_\beta \Gamma^\delta_{\gamma\alpha} + 2\Gamma^\delta_{\tau\alpha}\Gamma^\tau_{\beta\gamma}\right), \\ \widetilde{\Gamma}^{\overline{\alpha}}_{\beta\overline{\gamma}} &= -\Gamma^\gamma_{\alpha\beta}\,, \qquad \text{and} \qquad \widetilde{\Gamma}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} = 0\,, \end{aligned} \tag{2.15}$$

where according to our convention $\overline{\alpha} = \alpha + n, ...$ *etc.*

(Remember also that $\widetilde{\Gamma}^i_{jk} = \widetilde{\Gamma}^i_{kj}$).

Then the straightforward calculations show that, the *Levi–Civita connection of the metric $\widetilde{G}$ on T^*M is the same as the Levi–Civita connection of $\widetilde{g}$.*

As we will see in the next section this result is of a great importance when the problem of the "natural symplectic connection" on T^*M is considered.

Finally, the curvature tensor of $\widetilde{\nabla}$ has the components

$$\widetilde{R}^i_{jkl} = \partial_k \widetilde{\Gamma}^i_{jl} - \partial_l \widetilde{\Gamma}^i_{jk} + \widetilde{\Gamma}^i_{mk}\widetilde{\Gamma}^m_{jl} - \widetilde{\Gamma}^i_{ml}\widetilde{\Gamma}^m_{jk}\,. \tag{2.16}$$

Substituting (2.15) into (2.16) one gets (see equation (10.6) in [17])

$$\begin{aligned}\widetilde{R}^{\overline{\alpha}}_{\beta\gamma\delta} &= p_\nu\left\{R^\nu_{\beta\gamma\delta;\alpha} - R^\nu_{\alpha\gamma\delta;\beta} + 2\Gamma^\nu_{\tau(\alpha}R^\tau_{\beta)\gamma\delta} + 2\Gamma^\nu_{\tau[\gamma}R^\tau_{\delta]\beta\alpha}\right\}\,,\\ \widetilde{R}^{\overline{\alpha}}_{\beta\gamma\overline{\delta}} &= R^\delta_{\gamma\beta\alpha}, \qquad \widetilde{R}^{\overline{\alpha}}_{\overline{\beta}\gamma\delta} = -R^\beta_{\alpha\gamma\delta}, \qquad \widetilde{R}^\alpha_{\beta\gamma\delta} = R^\alpha_{\beta\gamma\delta}\,,\end{aligned} \tag{2.17}$$

and all remaining independent components are zero. Here the symbol "; " denotes the covariant derivative with respect to the Levi–Civita connection $\Gamma^\alpha_{\beta\gamma}$ on M, $R^\alpha_{\beta\gamma\delta}$ are the components of the Riemannian curvature tensor field on $M, (\cdot,\cdot)$ and $[\cdot,\cdot]$ stand for the symmetrization and anti-symmetrization, respectively *i.e.*

$$A_{(\alpha}B_{\beta)} := \tfrac{1}{2}\left(A_\alpha B_\beta + A_\beta B_\alpha\right) \quad \text{and} \quad A_{[\alpha}B_{\beta]} := \tfrac{1}{2}\left(A_\alpha B_\beta - A_\beta B_\alpha\right).$$

From (2.17) one quickly finds the Ricci tensor field on T^*M $\widetilde{R}_{ij} := \widetilde{R}^k_{ikj}$ to read

$$\widetilde{R}_{\alpha\beta} = 2R_{\alpha\beta}, \qquad \widetilde{R}_{\alpha\overline{\beta}} = \widetilde{R}_{\overline{\beta}\alpha} = \widetilde{R}_{\overline{\alpha}\overline{\beta}} = 0\,, \tag{2.18}$$

where $R_{\alpha\beta} = R^\gamma_{\alpha\gamma\beta}$ is the Ricci tensor field on M.

3. Symplectic connection on T^*M induced by the Levi–Civita connection on M

Let M' be a $2n$-dimensional smooth differentiable manifold and ω a closed nondegenerate 2-form on M'. Then the pair (M',ω) is called the *symplectic manifold.*

From the famous Darboux theorem it is well known that for any point $P \in M'$ there exist local coordinates $\left(x^i\right)$ on a neighborhood of P such that

$$\omega = \tfrac{1}{2}\omega_{ij}dx^i \wedge dx^j; \qquad (\omega_{ij}) = \begin{pmatrix} 0 & -\left(\delta^\alpha_\beta\right) \\ \left(\delta^\alpha_\beta\right) & 0 \end{pmatrix}. \tag{3.1}$$

Any such coordinates are called the *Darboux coordinates.*

In Fedosov's approach to the deformation quantization [7,8] the fundamental role plays a symplectic connection. It is defined as follows:

A symplectic connection on M' is a symmetric affine connection $\nabla^{(\mathrm{S})}$ on M' such that $\nabla^{(\mathrm{S})}\omega = 0$.

Thus the symplectic connection is defined locally by

$$\nabla^{(\mathrm{S})}_k\omega_{ij} = \partial_k\omega_{ij} - \omega_{lj}\gamma^l_{ik} - \omega_{il}\gamma^l_{jk} = 0, \quad \gamma^i_{jk} = \gamma^i_{kj}, \quad \forall i,j,k \in \{1,...,2n\}\,, \tag{3.2}$$

where γ^i_{jk} are the local components of $\nabla^{(\mathrm{S})}$.

In Darboux coordinates we have

$$\begin{aligned}\gamma_{jik} - \gamma_{ijk} &= 0\,,\\ \gamma_{ijk} - \gamma_{ikj} &= 0\,,\end{aligned} \tag{3.3}$$

where

$$\gamma_{ijk} := \omega_{il}\gamma^l_{jk}\,. \tag{3.4}$$

From (3.3) one easily infers that the following proposition holds true.

Proposition 3.1 *Let (M', ω) be a symplectic manifold and ∇ an affine connection on M'. Then ∇ is a symplectic connection if and only if for every Darboux coordinate system in M' the components γ_{ijk} of ∇ are totally symmetric with respect to the indices (i,j,k) .* ■

It is known that every paracompact symplectic manifold admits a symplectic connection but, in contrary to the case of the Levi–Civita connection on a Riemannian manifold, the symplectic connection is not uniquely defined [8,11,12].

Indeed, from the proposition (3.1) it follows that if $\nabla^{(\mathrm{S})}$ is a symplectic connection on M' then $\nabla'^{(\mathrm{S})}$ is also a symplectic connection on M' if and only if

$$\nabla'^{(\mathrm{S})} = \nabla^{(\mathrm{S})} + T, \tag{3.5}$$

where $T \in \Gamma\left(T^1_2 M'\right)$ and the tensor field of the type (0,3) on M' defined locally by

$$T_{ijk} := \omega_{il}T^l_{jk} \tag{3.6}$$

is totally symmetric.

Let ∇ be any symmetric affine connection on M' and $\Gamma^i_{jk} = \Gamma^i_{kj}$ its local components. Then one can quickly verify that

$$\gamma^i_{jk} := \Gamma^i_{jk} - \tfrac{1}{3}\omega^{il}\left(\nabla_j\omega_{kl} + \nabla_k\omega_{jl}\right) \tag{3.7}$$

are the components of a symplectic connection [8].

(As usually, ω^{ij} stand for the components of the tensor inverse to the 2-form ω, *i.e.*, $\omega^{il}\omega_{lj} = \delta^i_j$.)

Then

$$\begin{aligned}\gamma_{ijk} &= \Gamma_{ijk} - \tfrac{1}{3}\left(\nabla_j\omega_{ki} + \nabla_k\omega_{ji}\right)\\ &= \tfrac{1}{3}\left(\Gamma_{ijk} + \Gamma_{kij} + \Gamma_{jki}\right) - \tfrac{1}{3}\left(\partial_j\omega_{ki} + \partial_k\omega_{ji}\right)\,,\end{aligned} \tag{3.8}$$

where, of course, $\gamma_{ijk} := \omega_{il}\gamma^l_{jk}$ and $\Gamma_{ijk} := \omega_{il}\Gamma^l_{jk}$.

Hence, in any Darboux coordinate system

$$\gamma_{ijk} = \tfrac{1}{3}\left(\Gamma_{ijk} + \Gamma_{kij} + \Gamma_{jki}\right) \tag{3.9}$$

and we call this symplectic connection the *symplectic connection on M' induced by the symmetric affine connection ∇ on M'.*

The formula (3.9) can be interpreted as follows: Let (M', ω) be a symplectic manifold and let M' be endowed with a symmetric affine connection. Then this connection defines in a natural manner the symplectic connection on M' given by (3.9).

Assume that M' is the cotangent bundle T^*M over M, where M is an n-dimensional differentiable manifold endowed with a Riemannian metric g. From the previous section we know that (T^*M, ω) with ω defined by (2.3) is a symplectic manifold, but also $(T^*M, \widetilde{g})$ with $\widetilde{g}$ defined by (2.9) is a Riemannian manifold and the Levi–Civita connection of the metric $\widetilde{g}$ on T^*M is given by (2.15).

This connection induces the symplectic connection on T^*M according to (3.9). However, both the metric $\widetilde{g}$ (2.9) and the Levi–Civita connection of $\widetilde{g}$ (2.15) are defined by the Levi–Civita connection of g on M. Consequently, we conclude that in the present case the symplectic connection on T^*M is induced by the Levi–Civita connection on M.

This is our fundamental result which shows how the *symplectic geometry of the phase space T^*M is determined by the Riemannian geometry of the configuration space M.* Analogous point of view is presented in [1,2]. This approach seems to have a spirit of the Einstein program of the geometrization of physics. (Another approach to the problem of the "natural" symplectic connection is presented in [13–15].)

From (2.15) and (3.9), employing also (2.3) and (2.4), one finds the components γ^i_{jk} of the symplectic connection $\nabla^{(\mathrm{S})}$ on T^*M with respect to the induced coordinates (*i.e.*, the proper Darboux coordinates) in terms of the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ of the metric g on M

$$
\begin{aligned}
\gamma^{\alpha}_{\beta\gamma} &= \gamma_{\overline{\alpha}\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}, \quad \gamma^{\alpha}_{\beta\overline{\gamma}} = \gamma_{\overline{\alpha}\beta\overline{\gamma}} = 0, \quad \gamma^{\alpha}_{\overline{\beta}\overline{\gamma}} = \gamma_{\overline{\alpha}\overline{\beta}\overline{\gamma}} = 0\,, \\
\gamma^{\overline{\alpha}}_{\beta\gamma} &= -\gamma_{\alpha\beta\gamma} \\
&= -\tfrac{1}{3}p_\delta(\partial_\alpha\Gamma^\delta_{\beta\gamma} + \partial_\gamma\Gamma^\delta_{\alpha\beta} + \partial_\beta\Gamma^\delta_{\gamma\alpha} - 2\Gamma^\delta_{\tau\alpha}\Gamma^\tau_{\beta\gamma} \\
&\quad -2\Gamma^\delta_{\tau\gamma}\Gamma^\tau_{\alpha\beta} - 2\Gamma^\delta_{\tau\beta}\Gamma^\tau_{\gamma\alpha})\,.
\end{aligned}
\tag{3.10}
$$

By the proposition 3.1 all remaining components can be obtained from the components given by (3.10). Note that the symplectic connection (3.10) can be also defined for any M endowed with symmetric affine connection $\Gamma^\alpha_{\beta\gamma}$ as it has been done within slightly another formalism by Bordemann, Neumaier and Waldmann [1].

Observe also that the components $\gamma^\alpha_{\beta\gamma}$, $\gamma^\alpha_{\beta\overline{\gamma}}$ and $\gamma^\alpha_{\overline{\beta}\overline{\gamma}}$ are exactly equal to those proposed in [13], where they have been found from some different considerations.

Now we are going to give some transparent geometric interpretation of the symplectic connection defined by (3.10). Let (M,g) be, as before, a Riemannian manifold and let $F:\mathbb{R}\rightarrow M$ be a smooth curve in M locally given by $q^{\alpha}=q^{\alpha}(t)$, $t\in\mathbb{R}$. We define the *lift of F to $T^{*}(M)$* to be the smooth curve $\widetilde{F}:\mathbb{R}\rightarrow T^{*}M$ in $T^{*}M$ locally given as follows: $\widetilde{q}^{i}=\widetilde{q}^{i}(t)$, where $q^{\alpha}=q^{\alpha}(t)$ and $p_{\alpha}=g_{\alpha\beta}\left(q^{\gamma}(t)\right)\frac{dq^{\beta}(t)}{dt}$.

The following theorem holds:

Theorem 3.1 *Let (M,g) be a Riemannian manifold and $(T^{*}M,\omega)$ the symplectic manifold defined as before. Then the symplectic connection $\nabla^{(\mathrm{S})}$ on $T^{*}M$ defined by (3.10) is the unique symplectic connection on $T^{*}M$ satisfying the following conditions*

(i) the projection on M of any geodesic in $T^{*}M$ with respect to the symplectic connection is a geodesic in M with respect to the Levi–Civita connection defined by the metric g and these two geodesics have the same affine parameters;

(ii) the lift of any geodesic in M with respect to the Levi–Civita connection of g to $T^{*}M$ is a geodesic in $T^{*}M$ with respect to the symplectic connection.

Proof. Assume that $\widetilde{\nabla}^{(\mathrm{S})}$ is a symplectic connection on $T^{*}M$ satisfying *(i)* and *(ii)*, and $\widetilde{\gamma}_{jk}^{i}$ are the local components of $\widetilde{\nabla}^{(\mathrm{S})}$. From *(i)* we have $\frac{d^{2}\widetilde{q}^{i}}{dt^{2}}+\widetilde{\gamma}_{jk}^{i}\frac{d\widetilde{q}^{j}}{dt}\frac{d\widetilde{q}^{k}}{dt}=0\Longrightarrow\frac{d^{2}q^{\alpha}}{dt^{2}}+\mathit{\Gamma}_{\beta\gamma}^{\alpha}\frac{dq^{\beta}}{dt}\frac{dq^{\gamma}}{dt}=0$, where $\mathit{\Gamma}_{\beta\gamma}^{\alpha}$ are the components of the Levi–Civita connection of g on M. Hence one quickly infers that: $\widetilde{\gamma}_{\beta\gamma}^{\alpha}=\mathit{\Gamma}_{\beta\gamma}^{\alpha}$, $\widetilde{\gamma}_{\beta\overline{\gamma}}^{\alpha}=0=\widetilde{\gamma}_{\overline{\beta}\overline{\gamma}}^{\alpha}$. Consequently, *(ii)* leads to the implication $\frac{d^{2}q^{\alpha}}{dt^{2}}+\mathit{\Gamma}_{\beta\gamma}^{\alpha}\frac{dq^{\beta}}{dt}\frac{dq^{\gamma}}{dt}=0\Longrightarrow\frac{d}{dt}\left(g_{\alpha\beta}\frac{dq^{\beta}}{dt}\right)+\widetilde{\gamma}_{\beta\gamma}^{\overline{\alpha}}\frac{dq^{\beta}}{dt}\frac{dq^{\gamma}}{dt}=0$, which gives $\widetilde{\gamma}_{\beta\gamma}^{\overline{\alpha}}=\gamma_{\beta\gamma}^{\overline{\alpha}}$.

Therefore one arrives at the conclusion that if $\widetilde{\nabla}^{(\mathrm{S})}$ satisfies *(i)* and *(ii)* then $\widetilde{\nabla}^{(\mathrm{S})}=\nabla^{(\mathrm{S})}$. But $\widetilde{\nabla}^{(\mathrm{S})}$ satisfies *(i)* and *(ii)*. Thus the proof is complete. ∎

Then, straightforward but tedious calculations lead to the following components K_{jkl}^{i} for the curvature tensor of the symplectic connection $\widetilde{\nabla}^{(\mathrm{S})}$ given by (3.10)

$$\begin{aligned}
K_{\beta\gamma\delta}^{\overline{\alpha}} &= -\tfrac{1}{3}p_{\nu}\left\{R_{\beta\gamma\delta;\alpha}^{\nu}+R_{\alpha\gamma\delta;\beta}^{\nu}-6\mathit{\Gamma}_{\tau(\alpha}^{\nu}R_{\beta)\gamma\delta}^{\tau}+4R_{(\alpha\beta)[\gamma}^{\tau}\mathit{\Gamma}_{\delta]\tau}^{\nu}\right\}=-K_{\alpha\beta\gamma\delta}\,, \\
K_{\beta\gamma\overline{\delta}}^{\overline{\alpha}} &= \tfrac{2}{3}R_{(\alpha\beta)\gamma}^{\delta}=-K_{\alpha\beta\gamma\overline{\delta}}\Longrightarrow K_{\beta\gamma\delta}^{\alpha}=R_{\beta\gamma\delta}^{\alpha}=K_{\overline{\alpha}\beta\gamma\delta}\,. \qquad (3.11)
\end{aligned}$$

All remaining independent components vanish (compare with [1]).

A symplectic manifold endowed with symplectic connections is called the *Fedosov manifold* [11]. Therefore, the triple $\left(T^*M, \omega, \nabla^{(\mathrm{S})}\right)$, where ω and $\nabla^{(\mathrm{S})}$ are defined by (2.3) and (3.10), respectively, is the Fedosov manifold. It is well known that the components $K_{ijkl} := \omega_{im} K^m_{jkl}$ of the curvature tensor of the symplectic connection for any Fedosov manifold satisfy the following conditions [11]

$$K_{ijkl} = -K_{ijlk}, \quad K_{ijkl} + K_{iljk} + K_{iklj} = 0, \quad K_{ijkl} = K_{jikl}\,. \tag{3.12}$$

These conditions imply

$$K_{ijkl} + K_{lijk} + K_{klij} + K_{jkli} = 0\,. \tag{3.13}$$

Employing (3.12) one quickly gets

$$\begin{aligned} K_{\overline{\delta}\alpha\beta\gamma} &= K_{\alpha\gamma\beta\overline{\delta}} - K_{\alpha\beta\gamma\overline{\delta}}\,, \\ K_{\alpha\beta\overline{\gamma}\overline{\delta}} &= K_{\alpha\overline{\gamma}\beta\overline{\delta}} - K_{\alpha\overline{\delta}\beta\overline{\gamma}}\,, \\ K_{\overline{\alpha}\overline{\beta}\gamma\delta} &= K_{\delta\overline{\alpha}\gamma\overline{\beta}} - K_{\gamma\overline{\alpha}\delta\overline{\beta}}\,. \end{aligned} \tag{3.14}$$

From the first relation of (3.14), written in terms of the induced coordinates we obtain

$$K^{\delta}_{\alpha\beta\gamma} = -K^{\overline{\alpha}}_{\gamma\beta\overline{\delta}} + K^{\overline{\alpha}}_{\beta\gamma\overline{\delta}}\,. \tag{3.15}$$

This formula leads to the last implication in (3.11).

The *symplectic Ricci tensor* is defined by

$$K_{ij} := K^k_{ikj}\,. \tag{3.16}$$

From (3.12) one easily infers that

$$K^k_{kij} = 0 \tag{3.17}$$

and

$$K_{ij} = K_{ji}\,. \tag{3.18}$$

Substituting (3.11) into (3.16) we get

$$K_{\alpha\beta} = \tfrac{2}{3} R_{\alpha\beta}, \quad K_{\alpha\overline{\beta}} = K_{\overline{\beta}\alpha} = K_{\overline{\alpha}\overline{\beta}} = 0 \tag{3.19}$$

or employing (2.18)

$$K_{ij} = \tfrac{1}{3} \widetilde{R}_{ij}\,. \tag{3.20}$$

Thus one arrives at the following propositions:

Proposition 3.2 *Fedosov manifold* $\left(T^*M, \omega, \nabla^{(\mathrm{S})}\right)$ *is symplectic Ricci flat* $(i.e., K_{ij} = 0)$ *if and only if the Riemannian manifold* (M, g) *is Ricci flat* $(i.e., R_{\alpha\beta} = 0)$.

Proposition 3.3 $\left(T^*M,\omega,\nabla^{(\mathrm{S})},\widetilde{g}\right)$ *is a symplectic Einstein manifold* $(i.e., K_{ij}=\lambda\widetilde{g}_{ij})$ *if and only if the Riemannian manifold* $(T^*M,\widetilde{g})$ *is an Einstein manifold* $\left(i.e., \widetilde{R}_{ij}=\lambda'\widetilde{g}_{ij}\right)$.

By straightforward but rather long calculations we find the covariant derivative $\nabla^{(\mathrm{S})}_m K^i_{jkl}$:

$$
\begin{aligned}
\nabla^{(\mathrm{S})}_\mu K^\alpha_{\beta\gamma\delta} &= R^\alpha_{\beta\gamma\delta;\mu}, \nabla^{(\mathrm{S})}_\mu K^{\overline{\alpha}}_{\beta\gamma\overline{\delta}} = \tfrac{2}{3}R^\delta_{(\alpha\beta)\gamma;\mu} \nabla^{(\mathrm{S})}_{\overline{\mu}} K^{\overline{\alpha}}_{\beta\gamma\delta} = -\tfrac{2}{3}R^\mu_{(\alpha|\gamma\delta|;\beta)}\,,\\
\nabla^{(\mathrm{S})}_\mu K^{\overline{\alpha}}_{\beta\gamma\delta} &= -\tfrac{2}{3}p_\nu(R^\nu_{(\alpha|\gamma\delta|;\beta)\mu} - 3\Gamma^\nu_{\tau(\alpha}R^\tau_{\beta)\gamma\delta;\mu} - \Gamma^\nu_{\tau\mu}R^\tau_{(\alpha|\gamma\delta|;\beta)}\\
&\quad -2\Gamma^\nu_{\tau[\gamma}R^\tau_{|(\alpha\beta)|\delta];\mu} + R^\tau_{\alpha\gamma\delta}R^\nu_{(\tau\beta)\mu} + R^\tau_{\beta\gamma\delta}R^\nu_{(\tau\alpha)\mu}\\
&\quad -\tfrac{2}{3}R^\tau_{(\alpha\beta)\gamma}R^\nu_{(\delta\tau)\mu} + \tfrac{2}{3}R^\tau_{(\alpha\beta)\delta}R^\nu_{(\gamma\tau)\mu})
\end{aligned}
\tag{3.21}
$$

with all remaining independent components being zero. From (3.21) one quickly infers that

$$\nabla^{(\mathrm{S})}_\mu K_{\alpha\beta} = \tfrac{2}{3}R_{\alpha\beta;\mu} \tag{3.22}$$

and remaining components of $\nabla^{(\mathrm{S})}_k K_{ij}$ are zero.

Recently Bourgeois, Cahen, Gutt and Rawnsley [14,15], using the variational principle for a symplectic connection on symplectic manifold (M',ω), dim $M'=2n$,

$$\delta\int\mathcal{A}\frac{\omega^n}{n!}=0 \tag{3.23}$$

have found that both $\mathcal{A}=\mu K_{ij}K^{ij}$ and $\mathcal{A}=\nu K_{ijkl}K^{ijkl}$ lead to the same system of differential equations

$$\nabla^{(\mathrm{S})}_{(i}K_{jk)}=0\,. \tag{3.24}$$

Any symplectic connection satisfying (3.24) is called a *preferred symplectic connection.* Comparing (3.22) and (3.24) we conclude that in our case, the symplectic connection on T^*M defined by (3.10) is preferred if and only if the Levi–Civita connection on M fulfills the following equations

$$R_{(\alpha\beta;\gamma)}=0\,. \tag{3.25}$$

(This result has been also found in [1].) For example this is so when (M,g) is an Einstein manifold *i.e.*, $R_{\alpha\beta}=\lambda g_{\alpha\beta}$.

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