HOW MUCH OF THE OUTGOING RADIATION CAN BE INTERCEPTED BY SCHWARZSCHILDEAN BLACK HOLES?

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The Schwarzschild spacetime is for electromagnetic waves like a nonuniform medium with a varying refraction index. A fraction of an outgoing radiation scatters off the curvature of the geometry and can be intercepted by a gravitational center. The amount of the intercepted energy is bounded above by the backscattered energy of an initially outgoing pulse of electromagnetic radiation, which in turn depends on the initial energy, the Schwarzschild radius and the pulse location. Its magnitude depends on the frequency spectrum: it becomes negligible in the short wave limit but can be significant in the long wave regime.

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Backscattering prevents waves from being transmitted exclusively along null cones. That aspect of waves propagation has been investigated since the beginning of XXth century (see [1] and, in the context of general relativity, [2–4]). Electromagnetic waves and their backscattered tails have been studied from the early seventies [4]. Much attention has been put into the explanation of various interference phenomena of backscattering tails [4, 5].

The energy loss in a single burst of radiation due to this effect has been assessed only recently in [6] and (for a scalar field) in [7]. It should be pointed out that there exist estimates that refer to a stationary radiation (*e.g.* Price *et al.* in [3]). It is clear, that in such a case the backscattered tails add to the radiation source and the backscattering effect is underestimated, in some cases quite significantly. This paper refines substantially the result of [6] concerning the backscattered energy but the main focus is on the issue of the dependence of the effect on the waves frequency.

Spherically symmetric geometry outside matter is given by a line element,

$$ds^2 = -\left(1 - \frac{2m}{R}\right) dt^2 + \frac{1}{1 - \frac{2m}{R}} dR^2 + R^2 d\Omega^2 \, , \tag{1}$$

where t is a time coordinate, R is a radial coordinate that coincides with the areal radius and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element on the unit sphere, $0 \le \phi < 2\pi$ and $0 \le \theta \le \pi$.

The Maxwell equations read, using a multipole expansion of the electromagnetic vector potential [2]

$$(-\partial_0^2 + \partial_{r^*}^2)\Psi_l = \left(1 - \frac{2m}{R}\right)\frac{l(l+1)}{R^2}\Psi_l\,. \tag{2}$$

Ψ's should be two-index functions, Ψ_{lM}, (where M is the projection of the angular momentum), but since the evolution equation is ϕ independent, the index M is suppressed. The variable $r^* \equiv R + 2m\ln(\frac{R}{2m} - 1)$ is the Regge–Wheeler tortoise coordinate. The backreaction exerted by the electromagnetic field onto the metric has been neglected in the present analysis. That assumption is justified for any gravitational sources other than black holes while for black holes this approximation holds true some distance away from its horizon [7]. In the rest of this paper only dipole radiation Ψ_1 will be considered. Consequently, all angular momentum related subscripts will be omitted.

It is convenient to seek a solution $\Psi(r^*, t)$ in the form

$$\Psi = \tilde{\Psi} + \delta\,, \tag{3}$$

where δ is an unknown function and

$$\tilde{\Psi}(r^*, t) = \partial_{r^*}\Big(-g(r^* + t) - f(r^* - t)\Big) + \frac{g(r^* + t) + f(r^* - t)}{R}\,. \tag{4}$$

Functions f and g can be uniquely determined from initial data. One can check that $\tilde{\Psi}$ solves Eq. (2) in Minkowski spacetime. The f-related and the g-related parts represent an outgoing or an ingoing radiation, respectively. In what follows it will be assumed that $g = 0$, *i.e.*, that the wave is initially outgoing.

Initially $\delta = \partial_0\delta = 0$. Dipole-type initial data are given by

$$\tilde{\Psi}(x(R)) = -\partial_{r^*} f(x(R)) + \frac{f(x(R))}{R(r^*)}\,, \tag{5}$$

where f is C^2-differentiable, has a support in the annulus $(a, b \le \infty)$ and $x(R) \equiv r^*(R) - r^*(a)$. The initial energy density is continuous and vanishes on the boundary a.

The assumption that initial data are (initially) purely outgoing is made only for the sake of convenience. The propagation of electromagnetic waves is a linear process as far as the backreaction can be neglected. Therefore the propagation of the initially outgoing radiation (or even of selected modes of the outgoing radiation) is independent of whether or not the ingoing radiation (or any other mode) is present.

The evolution of δ is ruled by

$$(-\partial_0^2 + \partial_{r^*}^2)\delta = \left(1 - \frac{2m}{R}\right)\left[\frac{2}{R^2}\delta + \frac{6mf}{R^4}\right]. \tag{6}$$

The energy $E_R(t)$ of the electromagnetic field Ψ contained in the exterior of a sphere of a radius R reads

$$E_R(t) = 2\pi \int_R^\infty dr \left(\frac{(\partial_0\Psi)^2}{1 - \frac{2m}{r}} + \left(1 - \frac{2m}{r}\right)(\partial_r\Psi)^2 + \frac{2(\Psi)^2}{r^2}\right). \tag{7}$$

$E_a(0)$ is the energy of the initial pulse. Let an outgoing null cone C_a originate from a point $(a, 0)$ of the initial hypersurface. In the Minkowski spacetime the outgoing radiation contained outside C_a does not leak inward and its energy remains constant. In a curved spacetime some energy will be lost from the main stream due to the diffusion of the radiation h_- through C_a. Most of the backscattered radiation will be intercepted by the gravitational center.

Theorem. Under the above assumptions, the fraction of the diffused energy $\delta E_a/E_a(0)$ satisfies the inequality

$$\frac{\delta E_a}{E_a(0)} \leq C\left(\frac{2m}{a}\right)^2 \left(\frac{1 + \sqrt{\frac{2m}{a-2m}}}{\left(1 - \frac{2m}{a}\right)^3}\right)^2 \left(1 - \frac{a}{b}\right), \tag{8}$$

where C is a constant depending on a, m and b. C decreases with the increase of a or the decrease of $(b - a)/a$, and it is bounded — $C < 10^2$ (a stricter estimate reads $C < 12$).

Sketch of the proof. Define the intensity of the backscattered radiation that is directed inward

$$h_-(R, t) = \frac{1}{1 - \frac{2m}{R}}(\partial_0 + \partial_{r^*})\delta\,. \tag{9}$$

The rate of the energy change along C_a is given by

$$(\partial_0 + \partial_{r^*})E_a = -2\pi\left(1 - \frac{2m}{R}\right)\left[\left(1 - \frac{2m}{R}\right)\left(h_- - \frac{f}{R^2}\right)^2 + \frac{2}{R^2}\left(\tilde{\Psi} + \delta\right)^2\right]. \tag{10}$$

The energy loss is equal to a line integral along C_a (where $f = \tilde{\Psi} = 0$),

$$\delta E_a \equiv E_a - E_\infty = 2\pi \int\limits_a^\infty dr \left[\left(1 - \frac{2m}{r}\right) h_-^2 + \frac{2\delta^2}{r^2} \right] . \tag{11}$$

The proof of (8) requires the derivation of estimates on h_- and δ. One obtains

$$\left| \frac{f(R, t=0)}{R} \right| = \left| \int\limits_a^R \partial_r \frac{f}{r} \right| = \left| - \int\limits_a^R dr \frac{\tilde{\Psi}}{(r-2m} + 2m \int\limits_a^R dr \frac{f}{r^2(r-2m)} \right| ; \tag{12}$$

the Schwarz inequality and the use of (7) imply

$$\left| \frac{f(R,0)}{R} \right| \le \sqrt{\frac{aE_a(0)}{4\pi(a-2m)}} + 2m \int\limits_a^R dr \frac{|f|}{r^2(r-2m} .$$

Define $X \equiv \int_a^R dr \frac{|f|}{r} \frac{1}{r(r-2m)}$; notice that $X(a) = 0$. The preceding bound of $|f|$ can be written in terms of X as $\frac{dX}{dR} \le \frac{\sqrt{E_a(0)/(4\pi)}}{R(R-2m)(1-2m/a)} \sqrt{R-a} + \frac{2mX}{R(R-2m)}$. The use of the method of differential inequalities yields $X \le 2\frac{1-2m/a}{1-2m/R} \sqrt{E_a(0)/(4\pi} \left[\frac{1}{\sqrt{a-2m}} - \frac{1}{\sqrt{R-2m}} \right]$.

Insertion of that into the former bound of f gives the bound,

$$\left| \frac{f(R,0)}{R} \right| \le \sqrt{E_a(0)/(4\pi)} \sqrt{R-a} \; \frac{1 + \sqrt{\frac{2m}{a-2m}}}{1 - \frac{2m}{a}} . \tag{13}$$

This bound of f is new but the next steps of the proof follow quite closely [6].

(i) One obtains an energy estimate on δ, using the energy method and equation (6).

(ii) The integration of (9) and the use of *(i)* yield a bound of h_-.

(iii) *(ii)* and again (6) improve a bound on δ.

(iv) In bounding the energy loss due to δ-related terms, one should use both types *(i)* and *(iii)* of estimates on δ; a variational type argument yields then the best evaluation of the constant C.

The above estimate is sharper than that of [6], especially in the regime $\kappa \equiv 1 - a/b \approx 1$, when the bound improves circa 300 times. Details will appear elsewhere.

When the support of the initial radiation is very narrow, *i.e.*, $\kappa \ll 1$, then $\frac{\delta E_a}{E_a(0)} \le C_1 \left(\frac{2m}{a}\right)^2 \kappa$, where C_1 is a constant. In the limit $\kappa \to 0$ the ratio $\frac{\delta E_a}{E_a(0)}$ becomes 0; the backscattering is negligible when a support of initial pulses of electromagnetic energy becomes very narrow. And conversely, the bound becomes bigger with the increase of the width of the radiation pulse.

The physical meaning of that can be deduced as follows. Let $a(t) = a + t + 2m \ln\left(\frac{a(t)}{2m} - 1\right)$ and $b(t) = b + t + 2m \ln\left(\frac{b(t)}{2m} - 1\right)$ be radial components of points lying on null cones Ω_a, Ω_b outgoing from $(a, t = 0)$ or $(b, t = 0)$, respectively. Let $t \gg b$; then $b(t)/a(t) \approx 1$. Then one can show that the energy content between $(a(t), b(t))$ of the transmitted pulse reads

$$E_{a(t)}(t) \approx 4\pi \int_{a(t)}^{b(t)} dr \left(\partial_r^2 f(r)\right)^2 . \tag{14}$$

An auxiliary lemma is needed.

Lemma. Let f be a twice differentiable function, $f(b(t) = 0$, $R \in (a(t), b(t))$ and $a(t) \approx b(t)$. Then

$$\int_{a(t)}^{b(t)} dr \frac{f^2}{r^2} \ll \int_{a(t)}^{b(t)} dr (\partial_r f)^2 . \tag{15}$$

Proof. Notice that $|f/R| = |\int_{b(t)}^R dr \partial_r (f/r)|$; that is bounded above (applying the Schwarz inequality and integrating) by $\sqrt{1/R}\left(|\int_{b(t)}^R dr (\partial_r f)^2|\right)^{1/2} + \sqrt{1/R}\left(|\int_{b(t)}^R dr f^2/r^2|\right)^{1/2}$. Using this one obtains

$$\int_{a(t)}^{b(t)} dr \frac{f^2}{r^2} \le 2 \ln \frac{b(t)}{a(t)} \left(\left| \int_{a(t)}^{b(t)} dr (\partial_r f)^2 \right| + \left| \int_{a(t)}^{b(t)} dr \frac{f^2}{r^2} \right| \right) , \tag{16}$$

which, taking into account $\ln b(t)/a(t) \approx 0$, immediately proves the Lemma.

If $t \gg b$ then terms $2m/r$ can be ignored and (7) becomes $E_{a(t)}(t)$ of (14) plus terms of the form $\int_{a(t)}^{b(t)} dr (a_0 f^2/r^4 + a_1 f \partial_r f/r^3 + a_2 f (\partial_r f)^2/r^2 +$

$a_3 f\partial_r^2 f/r^2 + a_4\partial_r^2 f\partial_r f/r)$, where a_i are some constants. But, applying several times the Lemma, one immediately shows that all these terms are much smaller than $E_{a(t)}$ if $t \gg b$ and therefore (14) is a valid approximation of the energy.

From the Parseval identity follows

$$E_{a(t)}(t) = 4\pi \int_{-\infty)}^{\infty} dkk^4|\hat{f}(k)|^2\,; \tag{17}$$

here $\hat{f}(k)$ is the Fourier transform of $f(r)$. The *similarity theorem* of the Fourier transform theory [8] states that compression of the support of a function corresponds to expansion of the frequency scale. In explicit terms, if $a(t)_N \equiv b(t) - (b(t) - a(t))/N$ then the Fourier transform of $f(r)_N \equiv f(b(t) - N(b(t) - r))$, $\hat{f}_N(k)$ satisfies $|\hat{f}_N(k)| = |\hat{f}(k/N)|/N$. The energy carried by the rescaled field in modes $\omega \le \Omega_0$ is $E(\Omega_0)^{(N)} \equiv 4\pi N^3 \int_{-\Omega_0/N)}^{\Omega_0/N} dkk^4|\hat{f}(k)|^2$ while the total energy $E_{a(t)}(t)^{(N)}$ is given by $4\pi N^3 \int_{-\infty)}^{\infty} dkk^4|\hat{f}(k)|^2 = N^3 E_{a(t)}(t)$.

The ratio of the two energies

$$\delta_N(\Omega_0) \equiv \frac{E(\Omega_0)^{(N)}}{E_{a(t)}(t)^{(N)}} = \frac{4\pi \int\limits_{-\Omega_0/N}^{\Omega_0/N} dkk^4|\hat{f}(k)|^2}{E_{a(t)}(t)} \tag{18}$$

vanishes in the limit $N \to \infty$. Thus if a support of initial data is made narrow, then the wavelengths scale of the pulse extends in the direction of short lengths, in the sense that most of the radiation comes in the high frequency band. That implies, in conjunction with the Theorem, that the high frequency radiation is essentially unhindered by the effect of backscattering while long waves can be backscattered.

It is of interest to determine ω_c — a frequency that is critical in the sense that waves with $\omega \le \omega_\mathrm{c}$ may be strongly backscattered while those with $\omega > \omega_\mathrm{c}$ can be only weakly backscattered. As this vague definition suggests, ω_c will be determined only up to an order of magnitude. The bound given in the Theorem shows that there exists a critical width; if $b - a$ of an initial pulse is of the order of the distance a from the gravitational center, then strong backscattering is not excluded, provided in addition that a is not much greater that the gravitational radius $R_\mathrm{S} = 2m$. Thus, in the imprecise sense of the former definition, the critical width is $b - a \approx 2m$. The sought **critical frequency** can be defined as the fundamental frequency $\omega_\mathrm{c} = \pi/R_\mathrm{S}$. One can show, for any pulse that is smoothly distributed within an annulus (a, b), that most ($\geq 80\%$) of its energy comes with frequencies $\omega \geq \omega_1/2 = \pi/(b-a)$; thus $\omega_{\rm c}$ is in fact critical in the sense defined above.

In order to exemplify the above statements, recall estimates of [6]. Assume the same location $a = 4R_{\rm S}$, of two radiative dipoles and *(i)* $\kappa = 1/8$ (*i.e.*, the fundamental wavelength $R_{\rm S}$ is simultaneously critical) for a pulse I; *(ii)* $\kappa = 1/128$ (*i.e.*, the fundamental wavelength $R_{\rm S}/8$ is much smaller than the critical one) for the pulse II. Then in the case I one obtains $\frac{\delta E_a}{E_a(0)} < 0.37$, while in the case II (of shorter waves, subcritical case) one gets $\frac{\delta E_a}{E_a(0)} < 0.001$. If the dipole radiation II is located at $a = 4m$ then $\frac{\delta E_a}{E_a(0)} \approx 0.77$, which demonstrates how sensitive the bound (and presumably the effect itself) is on the distance. This dependence of the backscattering on the wave length has been observed in the numerical investigation of the propagation of pulses of scalar massless fields [9].

The backscattering effect becomes negligible at distances much bigger than the Schwarzschild radius of a central mass. That rules out most stars as objects that can induce observable backscattering effects. For a star of a solar type and $\lambda \sim R_{\rm S}$, for instance, the ratio $\frac{\delta E_a}{E_a(0)}$ can be at most 10^{-20}. In the case of white dwarves and $\lambda \sim R_{\rm S}$ the above bound gives $\frac{\delta E_a}{E_a(0)} < 10^{-8}$. For long-wave radiation the bound is bigger — for white dwarves it becomes $\frac{\delta E_a}{E_a(0)} \sim 10^{-5}$ — but a sharper estimate would still lower that significantly.

Two astrophysical compact objects, neutron stars and black holes, can be of interest. They can intercept the backscattered radiation, which would possibly lead to the suppression of the total luminosity produced in accretion disks that exist in their vicinities. This effect would be probably weak since the most luminous regions of the disks are located at a distance of (at least) several Schwarzschild radii. More interesting can be "echoes" — aftermaths of flashy eruptions, produced by a radiation reflected from the close vicinity of a horizon of a black hole. Numerical calculations done in the massless scalar fields propagation suggest that the amplitude of the reflected long-wave radiation can constitute up to 20 % of the incident one.

The above results can be generalized into the case of higher order electromagnetic multipoles. An analysis similar to that of the present paper can be repeated also in the case of a weak gravitational radiation produced around Schwarzschildean black holes.

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