AN ALTERNATIVE: TWO-MIXING TEXTURE FOR THREE NEUTRINOS OR THREE-MIXING TEXTURE FOR FOUR NEUTRINOS*

Wojciech Królikowski

Institute of Theoretical Physics, Warsaw University
Hoża 69, 00–681 Warszawa, Poland

(*Received September 9, 2000*)

The alternative formulated in the title has a chance to be settled, when the existence of the LSND effect is experimentally excluded or confirmed. The first option, much discussed in literature, works in the case of three active neutrinos ν_e , ν_μ , ν_τ, when among their massive states ν_1 , ν_2 , ν_3 there is no direct mixing between ν_1 and ν_3, and the mass hierarchy $m_1^2 \lesssim m_2^2 \ll m_3^2$ holds. This option is consistent with the observed deficits of solar ν_e's and atmospheric ν_μ's, if $\Delta m_{21}^2 \leftrightarrow \Delta m_{\rm sol}^2$ and $\Delta m_{32}^2 \leftrightarrow \Delta m_{\rm atm}^2$. On the other hand, the second option is an extension of the idea of the former to the case of four neutrinos ν_s , ν_e , ν_μ , ν_τ (including one sterile neutrino ν_s), when among their massive states ν_0 , ν_1 , ν_2 , ν_3 there are no direct mixings between ν_0 and ν_2, ν_0 and ν_3, ν_1 and ν_3, and the mass hierarchy $m_0^2 \lesssim m_1^2 \ll m_2^2 \lesssim m_3^2$ is now valid. Such an option, belonging to a class of textures widely discussed in literature, may be consistent with the observed deficits of solar ν_e's and atmospheric ν_μ's as well as with the LSND appearance of ν_e's in the beam of accelerator ν_μ's, if now $\Delta m_{10}^2 \leftrightarrow \Delta m_{\rm sol}^2$, $\Delta m_{32}^2 \leftrightarrow \Delta m_{\rm atm}^2$ and $\Delta m_{21}^2 \leftrightarrow \Delta m_{\rm LSND}^2$ (however, in the case of solar ν_e's the role of ν_s's in the disappearance of ν_e's is recently questioned). In both options, *only the close neighbours in the hierarchies of massive neutrinos* ν_1 , ν_2 , ν_3 *and* ν_0 , ν_1 , ν_2 , ν_3, *respectively, mix directly.* This characteristic feature of the two-mixing texture for three neutrinos or the three-mixing texture for four neutrinos may be somehow physically significant.

PACS numbers: 12.15.Ff, 14.60.Pq, 12.15.Hh

* Supported in part by the Polish State Committee for Scientific Research (KBN) Grant 2 P03B 052 16.

1. Introduction

First of all, we would like to emphasize that the alternative formulated in the title of the paper has a chance to be settled, when the existence of the LSND effect [1] is experimentally excluded or confirmed. The first option of the alternative, much discussed in literature [2], works in the case of three active neutrinos ν_e , ν_μ , ν_τ, when among their massive states ν_1 , ν_2 , ν_3 there is no direct mixing between ν_1 and ν_3 [3], and the mass hierarchy $m_1^2 \lesssim m_2^2 \ll m_3^2$ holds. This option is consistent with the observed deficits of solar ν_e's [4] and atmospheric ν_μ's [5], if $\Delta m_{21}^2 \leftrightarrow \Delta m_{\rm sol}^2$ and $\Delta m_{32}^2 \leftrightarrow \Delta m_{\rm atm}^2$. On the other hand, the second option of the alternative is an extension of the idea [3] of the former to the case of four neutrinos ν_s , ν_e , ν_μ , ν_τ (including one sterile neutrino ν_s), when among their massive states ν_0 , ν_1 , ν_2 , ν_3 there are no direct mixings between ν_0 and ν_2, ν_0 and ν_3, ν_1 and ν_3, and the mass hierarchy $m_0^2 \lesssim m_1^2 \ll m_2^2 \lesssim m_3^2$ is now valid. Such an option, belonging to a class of neutrino textures widely discussed in literature [6], may be consistent with the observed deficits of solar ν_e's [4] and atmospheric ν_μ's [5] as well as with the LSND appearance of ν_e's in the beam of accelerator ν_μ's [1], if now $\Delta m_{10}^2 \leftrightarrow \Delta m_{\rm sol}^2$, $\Delta m_{32}^2 \leftrightarrow \Delta m_{\rm atm}^2$ and $\Delta m_{21}^2 \leftrightarrow \Delta m_{\rm LSND}^2$ (however, in the case of solar ν_e's the role of ν_s's in the disappearance of ν_e's is recently disputed [4,7]).

In both options, *only the close neighbours in the hierarchies of massive neutrinos* ν_1 , ν_2 , ν_3 [3] *and* ν_0 , ν_1 , ν_2 , ν_3*, respectively, mix directly.* This characteristic feature of the two-mixing texture for three neutrinos or the three-mixing texture for four neutrinos may be somehow physically significant, leading hopefully to a pertinent dynamical model for the neutrino texture.

2. The first option

If one conjectures that in the generic Cabibbo–Kobayashi–Maskawa-type matrix for leptons [8],

$$U = \left(\begin{array}{ccc} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta} & c_{13}s_{23} \\ s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta} & -s_{23}c_{12} - s_{13}c_{23}s_{12}e^{i\delta} & c_{13}c_{23} \end{array} \right) \quad (1)$$

with $s_{ij} = \sin\theta_{ij} > 0$ and $c_{ij} = \cos\theta_{ij} \geq 0$, $(i\,,\,j = 1,2,3)$, there is practically no direct mixing of massive neutrinos ν_1 and ν_3 (*i.e.*, $\theta_{13} = 0$), then U is reduced to the following two-mixing form much discussed previously [2]:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -c_{23}s_{12} & c_{23}c_{12} & s_{23} \\ s_{23}s_{12} & -s_{23}c_{12} & c_{23} \end{pmatrix} . \tag{2}$$

For the two-mixing option (2) the neutrino mixing formula $\nu_\alpha = \sum_i U_{\alpha i}\nu_i$ takes the form

$$\begin{aligned} \nu_e &= c_{12}\nu_1 + s_{12}\nu_2 \ , \\ \nu_\mu &= c_{23}(-s_{12}\nu_1 + c_{12}\nu_2) + s_{23}\nu_3 \ , \\ \nu_\tau &= -s_{23}(-s_{12}\nu_1 + c_{12}\nu_2) + c_{23}\nu_3 \ , \end{aligned} \tag{3}$$

while the inverse neutrino mixing formula $\nu_i = \sum_\alpha U^*_{\alpha i}\nu_\alpha$ gives

$$\begin{aligned} \nu_1 &= c_{12}\nu_e - s_{12}(c_{23}\nu_\mu - s_{23}\nu_\tau) \ , \\ \nu_2 &= s_{12}\nu_e + c_{12}(c_{23}\nu_\mu - s_{23}\nu_\tau) \ , \\ \nu_3 &= s_{23}\nu_\mu + c_{23}\nu_\tau \ . \end{aligned} \tag{4}$$

Note that Eq. (2) can be presented also in the form

$$U = \exp(i\lambda_7\theta_{23})\exp(i\lambda_2\theta_{12}),$$

where λ_2 and λ_7 are two of eight Gell-Mann 3×3 matrices.

In the representation, where the charged-lepton mass matrix is diagonal (and thus the corresponding diagonalizing matrix — unit), the lepton mixing matrix $U = (U_{\alpha i})$ $(\alpha = e\,,\mu\,,\tau\ ,\ i = 1,2,3)$ is, at the same time, the diagonalizing matrix for neutrino mass matrix $M = (M_{\alpha\beta})$ $(\alpha,\beta = e,\mu,\tau)$, $U^\dagger M U = \mathrm{diag}(m_1\,, m_2\,, m_3)$ with $m_1^2 \le m_2^2 \le m_3^2$, so that $M = \left(\sum_i U_{\alpha i}U^*_{\beta i}m_i\right)$. In this case, the orthogonal two-mixing form (2) of U leads to the real and symmetric

$$M= \begin{pmatrix} c_{12}^2m_1+s_{12}^2m_2 & (m_2-m_1)c_{12}s_{12}c_{23} & -(m_2-m_1)c_{12}s_{12}s_{23} \\ (m_2-m_1)c_{12}s_{12}c_{23} & s_{23}^2m_3+c_{23}^2(s_{12}^2m_1+c_{12}^2m_2) & (m_3-s_{12}^2m_1-c_{12}^2m_2)c_{23}s_{23} \\ -(m_2-m_1)c_{12}s_{12}s_{23} & (m_3-s_{12}^2m_1-c_{12}^2m_2)c_{23}s_{23} & c_{23}^2m_3+s_{23}^2(s_{12}^2m_1+c_{12}^2m_2) \end{pmatrix}. \tag{5}$$

Here, as is seen from Eq. (4), the values $c_{23} = 1/\sqrt{2} = s_{23}$ give maximal mixing of ν_μ and ν_τ: $(\nu_\mu \pm \nu_\tau)/\sqrt{2}$, and then $c_{12} \simeq 1/\sqrt{2} \simeq s_{12}$ — a nearly maximal mixing of ν_e and $(\nu_\mu - \nu_\tau)/\sqrt{2}$: approximately $[\nu_e \pm (\nu_\mu - \nu_\tau)/\sqrt{2}\,]/\sqrt{2}$.

From the familiar neutrino oscillation formulae

$$P(\nu_\alpha \to \nu_\beta) = |\langle \nu_\beta | \mathrm{e}^{iPL} | \nu_\alpha \rangle|^2 = \delta_{\alpha\beta} - 4 \sum_{j>i} U^*_{\beta j} U_{\alpha j} U_{\beta i} U^*_{\alpha i} \sin^2 x_{ji} \,, \quad (6)$$

with

$$x_{ji} = 1.27 \frac{\Delta m^2_{ji} L}{E} \,, \quad \Delta m^2_{ji} = m^2_j - m^2_i \quad (7)$$

(Δm^2_{ji}, L and E measured in eV^2, km and GeV, respectively) which is valid for $U^*_{\beta j} U_{\alpha j} U_{\beta i} U^*_{\alpha i}$ real (CP violation neglected), one infers in the case of two-mixing option (2) that

$$\begin{aligned} P(\nu_e \to \nu_e) &= 1 - (2c_{12}s_{12})^2 \sin^2 x_{21} \,, \\ P(\nu_\mu \to \nu_\mu) &= 1 - (2c_{12}s_{12}c_{23})^2 \sin^2 x_{21} - (2c_{23}s_{23})^2 (s^2_{12} \sin^2 x_{31} + c^2_{12} \sin^2 x_{32}) \\ &\simeq 1 - (2c_{23}s_{23})^2 \sin^2 x_{32} \,, \\ P(\nu_\mu \to \nu_e) &= (2c_{12}s_{12}c_{23})^2 \sin^2 x_{21} \,, \end{aligned} \quad (8)$$

where the final step in the second formula is valid for $x_{32} = x_{\mathrm{atm}} = \mathcal{O}(1)$, when $m^2_1 \lesssim m^2_2 \ll m^2_3$ or equivalently $\Delta m^2_{21} \ll \Delta m^2_{32} \lesssim \Delta m^2_{31}$.

The first formula (8) is consistent with the observed deficit of solar ν_e's if one applies the smaller-mass or larger-mass vacuum global solution or large-angle MSW global solution or finally LOW global solution [4] with $(2c_{12}s_{12})^2 \leftrightarrow \sin^2 2\theta_{\mathrm{sol}} \sim (0.72 \text{ or } 0.90 \text{ or } 0.79 \text{ or } 0.91)$ and $\Delta m^2_{21} \leftrightarrow \Delta m^2_{\mathrm{sol}} \sim (6.5 \times 10^{-11} \text{ or } 4.4 \times 10^{-10} \text{ or } 2.7 \times 10^{-5} \text{ or } 1.0 \times 10^{-7})\,\mathrm{eV}^2$, respectively. This gives $c^2_{12} \sim 0.5 + (0.26 \text{ or } 0.16 \text{ or } 0.23 \text{ or } 0.15)$ and $s^2_{12} \sim 0.5 - (0.26 \text{ or } 0.16 \text{ or } 0.23 \text{ or } 0.15)$, when taking $c^2_{12} \geq s^2_{12}$.

The second formula (8) describes correctly the observed deficit of atmospheric ν_μ's [5] if $(2c_{23}s_{23})^2 \leftrightarrow \sin^2 2\theta_{\mathrm{atm}} \sim 1$ and $\Delta m^2_{32} \leftrightarrow \Delta m^2_{\mathrm{atm}} \sim 3.5 \times 10^{-3}\,\mathrm{eV}^2$, since then $\Delta m^2_{21} \ll \Delta m^2_{32} \lesssim \Delta m^2_{31}$ for Δm^2_{21} determined as in the case of solar ν_e's. This implies that $c^2_{23} \sim 0.5 \sim s^2_{23}$ and $m^2_3 \sim 3.5 \times 10^{-3}\,\mathrm{eV}^2$, because $m^2_1 \lesssim m^2_2 \ll m^2_3$.

Then, the third formula (8) shows that no LSND effect for accelerator ν_μ's [1] should be observed, $P(\nu_\mu \to \nu_e) \sim 0$, since with $\Delta m^2_{21} \leftrightarrow \Delta m^2_{\mathrm{sol}} \sim (10^{-10} \text{ or } 10^{-10} \text{ or } 10^{-5} \text{ or } 10^{-7})\,\mathrm{eV}^2 \ll \Delta m^2_{\mathrm{LSND}} \sim 1\,\mathrm{eV}^2$, one gets $\sin^2 (x_{12})_{\mathrm{LSND}} \sim (10^{-21} \text{ or } 10^{-19} \text{ or } 10^{-9} \text{ or } 10^{-14}) \ll \sin^2 x_{\mathrm{LSND}} \sim 1$, while $(2c_{12}s_{12}c_{23})^2 \sim (0.72 \text{ or } 0.90 \text{ or } 0.79 \text{ or } 0.91) \times 0.5 > \sin^2 2\theta_{\mathrm{LSND}} \sim 10^{-2}$.

In the case of Chooz experiment looking for oscillations of reactor $\bar{\nu}_e$'s [9], where it happens that $(x_{32})_{\mathrm{Chooz}} = 1.27 \Delta m^2_{32} L_{\mathrm{Chooz}} / E_{\mathrm{Chooz}} \sim 1$ for $\Delta m^2_{32} \leftrightarrow \Delta m^2_{\mathrm{atm}}$, the first formula (8) leads to $P(\bar{\nu}_e \to \bar{\nu}_e) \sim 1$, since $(x_{21})_{\mathrm{Chooz}} \ll (x_{32})_{\mathrm{Chooz}} \sim 1$ for $\Delta m^2_{21} \leftrightarrow \Delta m^2_{\mathrm{sol}}$ ($U_{e3} = 0$ in our case). This is consistent with the negative result of Chooz experiment. We can see, however, that for the actual lepton counterpart of Cabibbo–Kobayashi–Maskawa matrix the nonzero entry U_{e3} may be a potential correction to the two-mixing option (2) ($|U_{e3}| < 0.1$ according to the estimation in Chooz experiment).

Further on, we will put $c_{23} \simeq 1/\sqrt{2} \simeq s_{23}$. Then, from Eq. (5) we infer that approximately

$$M = \begin{pmatrix} c_{12}^2 m_1 + s_{12}^2 m_2 & \frac{(m_2 - m_1)c_{12}s_{12}}{\sqrt{2}} & \frac{-(m_2 - m_1)c_{12}s_{12}}{\sqrt{2}} \\ \frac{(m_2 - m_1)c_{12}s_{12}}{\sqrt{2}} & \frac{m_3 + s_{12}^2 m_1 + c_{12}^2 m_2}{2} & \frac{m_3 - s_{12}^2 m_1 - c_{12}^2 m_2}{2} \\ \frac{-(m_2 - m_1)c_{12}s_{12}}{\sqrt{2}} & \frac{m_3 - s_{12}^2 m_1 - c_{12}^2 m_2}{2} & \frac{m_3 + s_{12}^2 m_1 + c_{12}^2 m_2}{2} \end{pmatrix} . \tag{9}$$

Here, $M_{e\mu} = -M_{e\tau}$, $M_{\mu\mu} = M_{\tau\tau}$ and

$$\begin{aligned} M_{ee} &= c_{12}^2 m_1 + s_{12}^2 m_2 \,, & M_{ee} + M_{\mu\mu} - M_{\mu\tau} &= m_1 + m_2 \,, \\ M_{\mu\mu} + M_{\mu\tau} &= m_3 \,, & M_{e\mu} &= \frac{(m_2 - m_1)c_{12}s_{12}}{\sqrt{2}} \,. \end{aligned} \tag{10}$$

Assuming that $M_{ee} = 0$, we get from Eq. (10) the relations $M_{\mu\mu} = (m_3 + m_2 + m_1)/2$, $M_{\mu\tau} = (m_3 - m_2 - m_1)/2$, $M_{e\mu} = (s_{12}/c_{12})m_2/\sqrt{2}$, and

$$\frac{m_1}{m_2} = -\frac{s_{12}^2}{c_{12}^2} \,, \qquad \Delta m_{21}^2 \equiv m_2^2 - m_1^2 = m_2^2 \frac{c_{12}^2 - s_{12}^2}{c_{12}^4} \tag{11}$$

or

$$m_1 = -\sqrt{\Delta m_{21}^2}\frac{s_{12}^2}{\sqrt{c_{12}^2 - s_{12}^2}} \,, \quad m_2 = \sqrt{\Delta m_{21}^2}\frac{c_{12}^2}{\sqrt{c_{12}^2 - s_{12}^2}} \,, \tag{12}$$

when taking $m_1 \leq m_2$. For instance, applying to Eq. (12) the LOW solar solution [4] *i.e.*, $s_{12}^2 \sim 0.5 - 0.15$, $c_{12}^2 \sim 0.5 + 0.15$ and $\Delta m_{21}^2 \sim 1.0 \times 10^{-7}\,\mathrm{eV}^2$, we estimate

$$m_1 \sim -2.0 \times 10^{-4}\,\mathrm{eV} \,, \quad m_2 \sim 3.8 \times 10^{-4}\,\mathrm{eV} \,, \tag{13}$$

while the Super–Kamiokande result $\Delta m_{32}^2 \sim 3.5 \times 10^{-3}\,\mathrm{eV}^2$ [5] leads to the estimation

$$m_3 \sim 5.9 \times 10^{-2}\,\mathrm{eV} \,, \tag{14}$$

what shows explicitly that $|m_1| \lesssim m_2 \ll m_3$. Thus, in this case

$$\begin{aligned} M_{ee} &= 0 \,, & M_{\mu\mu} &= M_{\tau\tau} \sim 3.0 \times 10^{-2}\mathrm{eV} \,, \\ M_{e\mu} &= -M_{e\tau} \sim 1.9 \times 10^{-4}\mathrm{eV} \,, & M_{\mu\tau} &\sim 3.0 \times 10^{-2}\mathrm{eV} \,, \end{aligned} \tag{15}$$

where $M_{\mu\mu} \gtrsim M_{\mu\tau} \gg M_{e\mu}$.

In conclusion, the two-mixing texture of three (Dirac or Majorana) active neutrinos $\nu_\alpha (\alpha = e, \mu, \tau)$, described by the formulae (2) and (5), is neatly consistent with the observed solar and atmospheric neutrino deficits, but it predicts no LSND effect whose confirmation should imply, therefore, the existence of at least one sterile neutrino ν_s, mixing with ν_e. This might be either one extra, light (Dirac or Majorana) sterile neutrino ν_s [6,10] or one of three conventional, light Majorana sterile neutrinos $\nu^{(\mathrm{s})}_\alpha = \nu_{\alpha \mathrm{R}} + (\nu_{\alpha \mathrm{R}})^c$ $(\alpha = e\,, \mu\,, \tau)$ [11,12] existing in this case beside three light Majorana active neutrinos $\nu^{(\mathrm{a})}_\alpha = \nu_{\alpha \mathrm{L}} + (\nu_{\alpha \mathrm{L}})^c$ $(\alpha = e\,, \mu\,, \tau)$ [of course, $\nu^{(\mathrm{a})}_{\alpha \mathrm{L}} = \nu_{\alpha \mathrm{L}}$ and $\nu^{(\mathrm{s})}_{\alpha \mathrm{L}} = (\nu_{\alpha \mathrm{R}})^c$].

The essential agreement of the observed neutrino oscillations with the two-mixing option (2) for U (provided there is really no LSND effect) suggests that the conjecture of absence of direct mixing of massive neutrinos ν_1 and ν_3, leading to U of the form (2), is somehow physically important. This absence tells us that *only the close neighbours,* ν_1 *and* ν_2*,* ν_2 *and* ν_3*, in the hierarchy of massive neutrinos* $\nu_1\,, \nu_2\,, \nu_3$ *mix directly.*

3. The second option

When we want to introduce one sterile neutrino mixing with three active neutrinos ν_e, ν_μ, ν_τ (thus leading to four massive neutrino states $\nu_0, \nu_1, \nu_2, \nu_3$), we ought to extend properly the two-mixing formula (2) of the previous 3×3 mixing matrix U. A natural form of such a new 4×4 mixing matrix seems to be

$$U = \begin{pmatrix} c_{01} & s_{01} & 0 & 0 \\ -s_{01} & c_{01} & 0 & 0 \\ 0 & 0 & c_{23} & s_{23} \\ 0 & 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{12} & s_{12} & 0 \\ 0 & -s_{12} & c_{12} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} c_{01} & s_{01}c_{12} & s_{01}s_{12} & 0 \\ -s_{01} & c_{01}c_{12} & c_{01}s_{12} & 0 \\ 0 & -c_{23}s_{12} & c_{23}c_{12} & s_{23} \\ 0 & s_{23}s_{12} & -s_{23}c_{12} & c_{23} \end{pmatrix}, \tag{16}$$

if in the hierarchy of massive neutrinos $\nu_0, \nu_1, \nu_2, \nu_3$ the new massive neutrino ν_0 mixes directly only with its close neighbour ν_1 ($c_{01} = \cos\theta_{01}$ and $s_{01} = \sin\theta_{01}$). Then, *only the close neighbours,* ν_0 *and* ν_1*,* ν_1 *and* ν_2*,* ν_2 *and* ν_3*, in the hierarchy of massive neutrinos* $\nu_0\,, \nu_1\,, \nu_2\,, \nu_3$ *mix directly.* In the limiting case of $\theta_{01} = 0$ the three-mixing form (16) of 4×4 mixing matrix is reduced to the two-mixing form (2) of 3×3 mixing matrix. It is interesting to observe that in this four-neutrino texture the sterile small-angle MSW global solution [4] leads to a small value $\theta_{01} \simeq s_{01} \sim 0.0017$ (*cf.* the first relation

(19) later on). If, however, a considerable or even nearly maximal mixing of ν_0 and ν_1 can work effectively for solar ν_e's, such a small value of θ_{01} may be replaced by a considerable θ_{01} or even $\theta_{01} \simeq \pi/4$: $c_{01} \gtrsim 1/\sqrt{2} \gtrsim s_{01}$. On the other hand, a small mixing of ν_1 and ν_2 may be sufficient to explain the possible LSND effect (or its modified version), while the nearly maximal mixing of ν_2 and ν_3 still works well for atmospheric ν_μ's. Thus, putting in Eq. (16) $c_{23} \simeq 1/\sqrt{2} \simeq s_{23}$ and $c_{12} \simeq 1 \gg s_{12} \simeq \varepsilon\sqrt{2} > 0$, we get approximately from Eq. (16)

$$U = \begin{pmatrix} c_{01} & s_{01} & s_{01}\varepsilon\sqrt{2} & 0 \\ -s_{01} & c_{01} & c_{01}\varepsilon\sqrt{2} & 0 \\ 0 & -\varepsilon & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \varepsilon & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \tag{17}$$

The mixing matrix (17) gives through Eqs. (6) with (7), where now $\alpha, \beta = s, e, \mu, \tau$ and $i, j = 0, 1, 2, 3$, the following neutrino oscillation probabilities:

$$\begin{aligned} P(\nu_e \to \nu_e) &= 1 - (2c_{01}s_{01})^2 \sin^2 x_{10} - 8c_{01}^2\varepsilon^2(s_{01}^2 \sin^2 x_{21} + c_{01}^2 \sin^2 x_{31}) \\ &\simeq 1 - (2c_{01}s_{01})^2 \sin^2 x_{10} - 4c_{01}^2\varepsilon^2 , \\ P(\nu_\mu \to \nu_\mu) &= 1 - \sin^2 x_{23} - 2\varepsilon^2(\sin^2 x_{21} + \sin^2 x_{31}) \simeq 1 - \sin^2 x_{23} - 2\varepsilon^2 \\ P(\nu_\mu \to \nu_e) &= 4\, c_{01}^2\varepsilon^2 \sin^2 x_{21} . \end{aligned} \tag{18}$$

The second step in the first and second formula (18) is valid for $x_{10} = x_{\rm sol} = \mathcal{O}(1)$ and $x_{32} = x_{\rm atm} = \mathcal{O}(1)$, respectively, where now in this four-neutrino texture $m_0^2 \lesssim m_1^2 \ll m_2^2 \lesssim m_3^2$ or equivalently $\Delta m_{10}^2 \ll \Delta m_{21}^2 \lesssim \Delta m_{20}^2$ and $\Delta m_{32}^2 \ll \Delta m_{21}^2 \lesssim \Delta m_{31}^2$.

Then, the formulae (18) are consistent with experimental data for solar ν_e's [4], atmospheric ν_μ's [5] and LSND accelerator ν_μ's [1], if

$$\begin{aligned} (2c_{01}s_{01})^2 \leftrightarrow \sin^2 2\theta_{\rm sol} &\sim \begin{cases} 6.6\times 10^{-3} \text{ or} \\ 0.72 \text{ or} \\ 0.90 \end{cases} , \\ \Delta m_{10}^2 \leftrightarrow \Delta m_{\rm sol}^2 &\sim \begin{cases} 4.0\times 10^{-6}\ \text{eV}^2 \text{ or} \\ 6.5\times 10^{-11}\ \text{eV}^2 \text{ or} \\ 4.4\times 10^{-10}\ \text{eV}^2 \end{cases} , \\ 1 \leftrightarrow \sin^2 2\theta_{\rm atm} &\sim 1 , \\ \Delta m_{32}^2 \leftrightarrow \Delta m_{\rm atm}^2 &\sim 3.5\times 10^{-3}\ \text{eV}^2 , \\ 4c_{01}^2\varepsilon^2 \leftrightarrow \sin^2 2\theta_{\rm LSND} &\sim 10^{-2} , \\ \Delta m_{21}^2 \leftrightarrow \Delta m_{\rm LSND}^2 &\sim 1\ \text{eV}^2 , \end{aligned} \tag{19}$$

respectively. Here, in the case of solar ν_e's we apply the sterile small-angle MSW global solution or, just for an illustration, smaller-mass or larger-mass vacuum global solution [4] (however, in the case of solar ν_e's the role of ν_s's in the disappearance of ν_e's is recently disputed [4,7]). Then, $c^2_{01} \sim$ (1 or 0.76 or 0.66) and $s^2_{01} \sim$ (0.0017 or 0.24 or 0.34). From Eqs. (19) we obtain readily the estimations $m_2 \sim 1$ eV, $m_3 \sim 1$ eV and $m_1 \sim (2.0 \times 10^{-3}$ or 8.1×10^{-6} or $2.1 \times 10^{-5})$ eV, the last if we conjecture that $m_0 = 0$, and $\varepsilon \sim$ (5.0 or 5.7 or 6.2) $\times 10^{-2}$. This shows explicitly that $m_0 \lesssim m_1 \ll m_2 \lesssim m_3$.

In the case of mixing matrix (17), the 4×4 mass matrix $M = (M_{\alpha\beta})$ $(\alpha, \beta = s, e, \mu, \tau)$, takes, up to $\mathcal{O}(\varepsilon^2)$, the form

$$M = \left(\begin{array}{cccc} c^2_{01}m_0 + s^2_{01}m_1 & c_{01}s_{01}(m_1 - m_0) & \varepsilon s_{01}(m_2 - m_1) & \varepsilon s_{01}(m_1 - m_2) \\ c_{01}s_{01}(m_1 - m_0) & s^2_{01}m_0 + c^2_{01}m_1 & \varepsilon c_{01}(m_2 - m_1) & \varepsilon c_{01}(m_1 - m_2) \\ \varepsilon s_{01}(m_2 - m_1) & \varepsilon c_{01}(m_2 - m_1) & \frac{m_2 + m_3}{2} & \frac{m_3 - m_2}{2} \\ \varepsilon s_{01}(m_1 - m_2) & \varepsilon c_{01}(m_1 - m_2) & \frac{m_3 - m_2}{2} & \frac{m_2 + m_3}{2} \end{array} \right), \tag{20}$$

since $M = \left(\sum_i U_{\alpha i} U^*_{\beta i} m_i \right)$. Hence, up to $\mathcal{O}(\varepsilon^2)$,

$$m_{0,1} = \frac{M_{ss} + M_{ee}}{2} \mp \sqrt{\left(\frac{M_{ee} - M_{ss}}{2} \right)^2 + M^2_{se}}\,, \; m_{2,3} = M_{\mu\mu} \mp M_{\mu\tau}\,, \tag{21}$$

where $M_{ee} = M_{ss} + (c_{01}/s_{01} - s_{01}/c_{01})M_{se}$, $M_{ss} = (s_{01}/c_{01})M_{se}$ (when $m_0 = 0$), $M_{se} \sim c_{01}s_{01}(2.0 \times 10^{-3}$ or 8.1×10^{-6} or $2.1 \times 10^{-5})$ eV (when $m_0 = 0$) and $M_{\tau\tau} = M_{\mu\mu} \sim 1$ eV, $M_{\mu\tau} \sim (3.5/4) \times 10^{-3}$ eV.

If eventually the LSND effect turns out to be confirmed, then at least one sterile neutrino mixing with three active neutrinos ought to exist. The second option discussed here is a natural candidate for its texture. If there are more sterile neutrinos mixing with active neutrinos, the neutrino texture would be effectively more extended [13].

REFERENCES

[1] C. Athanassopoulos *et al.* (LSND Collaboration), *Phys. Rev. Lett.* **75**, 2650 (1995); *Phys. Rev.* **C54**, 2685 (1996); *Phys. Rev. Lett.* **77**, 3082 (1996); **81**, 1774 (1998); G. Mills, talk at Neutrino 2000, Sudbury, Canada, June 2000.

[2] *Cf. e.g.* F. Feruglio, *Acta Phys. Pol.* **B31**, 1221 (2000); and references therein.

[3] W. Królikowski, `hep-ph/0007255`.

[4] *Cf. e.g.* J.N. Bahcall, P.I. Krastev, A.Y. Smirnov, *Phys. Lett.* **B477**, 401 (2000); `hep-ph/0002293`.

[5] Y. Fukuda *et al.* (Super-Kamiokande Collaboration), *Phys. Rev. Lett.* **81**, 1562 (1998) [E. **81**, 4279 (1998)]; **82**, 1810 (1999); **82**, 2430 (1999).

[6] *Cf. e.g.* G. Giunti, M.C. Gonzales–Garcia, C. Peña–Garay, *Phys. Rev.* **D62**, 013005 (2000); and references therein.

[7] *Cf. e.g.* W. Barger, *et al.*, `hep-ph/0008019`.

[8] Z. Maki, M. Nakagawa, S. Sakata, *Progr. Theor. Phys.* **28**, 870 (1962).

[9] M. Appolonio *et al.* (Chooz Collaboration), *Phys. Lett.* **B420**, 397 (1998); **B466**, 415 (1999).

[10] *Cf. e.g.* W. Królikowski, *Nuovo Cim.* **A111**, 1257 (1999); and references therein.

[11] *Cf. e.g.* A. Geiser, `hep-ph/9901433`; and references therein.

[12] *Cf. e.g.* W. Królikowski, *Nuovo Cim.* **A112**, 893 (1999), also `hep-ph/9904489`; *Acta Phys. Pol.* **B31**, 663 (2000); and references therein.

[13] W. Królikowski, `hep-ph/0004222`; and references therein.