A CONCEPT OF STRONG NECESSARY CONDITION IN NONLINEAR FIELD THEORY

K. Sokalski[a,b], T. Wietecha[b] and Z. Lisowski[b]

[a]Institute of Computer Science, Technical University of Częstochowa, Poland
[b]Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Kraków, Poland

(Received March 21, 2001; revised version received July 9, 2001)

Concept of the strong necessary condition for existence of the extremum of functional is discussed as an alternative to the Euler equation. This concept leads to field equations of the order lower than the order of the Euler equation. They appear as duality equations: Bogomolny decomposition or Bäcklund transformations. The derived formalism is presented and tested on some examples: nonlinear σ-model, nonlinear Klein-Gordon equations (both hyperbolic and elliptic) and nonlinear Schrödinger equation.

PACS numbers: 02.30.Wd, 02.60.Lj

1. Introduction

Two methods applicable in nonlinear field theories: the Bogomolny decomposition [1–3] and the Bäcklund transformation [4–7] seem to have common background resulting from the concept of invariance of the field equation with respect to the gauge transformation of the action functional [8–10]. Therefore, it should be possible to derive one formalism which generates both the Bogomolny decomposition and the Bäcklund transformation as particular types of duality equations. How to unify derivation of both these methods? It is assumed that the field equation is the necessary condition for existence of an extremum of the action functional (Euler equation). The gauge invariance of the field equations is equivalent to the invariance of the necessary conditions. But if one formulates the variational problem in such a way that the gauge transformation of the action functional breaks the invariance of the necessary condition then this transformation can contribute to the field equation. If this new variational problem guarantees that the set of solutions of new equation is included in the solution set resulting from the Euler equation, then we derive a new tool for the nonlinear field theory [11, 12].

The aim of this paper is to derive the field equations by applying a new necessary condition to the gauge transformed action functional. We replace the Euler theorem by the strong necessary condition. This concept together with the gauge transformations lead to the Bogomolny decomposition or to the Bäcklund transformation. Analyzing different nonlinear field models with the aid of the strong necessary condition we derive some conclusions concerning similarities and differences between the Bogomolny decomposition and the Bäcklund transformation. On the basis of these conclusions we draw a hypothesis about existence of other types of "duality equations".

The paper is organized in the following way: in Sec. 2 we introduce a concept of strong necessary conditions in the problem of extremum of a functional. Sec. 3 contains the test of the derived formalism. We re-derive the Bogomolny decomposition for the two-dimensional σ-model, the Bäcklund transformation for the two classes of nonlinear Klein–Gordon equations (hyperbolic and elliptic) and for the nonlinear Schrödinger equation. In the last section we discuss the existence of the higher types of the duality equations.

2. Strong necessary condition for extremum of a functional

We will limit all considerations to the one type of extremum, the minimum. If necessary, all considerations of this section can be easily extended to the maximum. Let us consider the functional of the following form:

$$\Phi[y] = \int\limits_X F(x,y,y')dx\,, \tag{1}$$

where $y \in C^1$ and $F \in C^2$. The functional $\Phi[y]$ reaches a local minimum for $y = y^*$ if there exists a neighborhood of the point y^*

$$K(y^*,\varepsilon) = \{y \in C^1,\ \|y - y^*\| < \varepsilon\}, \tag{2}$$

in which $\Phi[y] \geq \Phi[y^*]$ for all $y \in K(y^*,\varepsilon)$. We consider functionals $\Phi[y]$ on a set of differentiable functions. These functions belong to the space C^1, where the norm is defined by:

$$\|y\|_{C^1} = \max_x |y(x)| + \max_x |y^{'}(x)|\,. \tag{3}$$

A field equation derived from the least action principle for the fixed boundaries has the form of Euler's equation:

$$F_{,y} - \frac{d}{dx}F_{,y'} = 0\,. \tag{4}$$

The idea of the Bogomolny decomposition and the Bäcklund transformation is to derive simpler equations than (4) for the extremals of (1). The orders of equations resulting from these concepts are usually lower. The order of (4) is usually higher then the highest order of the derivative of $y(x)$ appearing in (1). Therefore, in order to derive the Bogomolny decomposition or the Bäcklund transformation from the least action principle one should not apply the Euler equation as the necessary condition.

Let us analyze the variation of Φ:

$$\delta\Phi = \int\limits_X \left(F_{,y}h + F_{,y'}h'\right) dx\,, \tag{5}$$

where $h = \delta y(x)$ is the increment of the function $y(x)$. In order to satisfy the necessary condition $\delta\Phi = 0$ in a different way than (4) we set the following condition:

$$F_{,y'} = 0\,, \tag{6}$$

which implies

$$F_{,y} = 0\,. \tag{7}$$

We call (6) and (7) the strong necessary conditions. All solutions of (6) and (7) satisfy the Euler equation (4) but in most cases the set of solutions of (6) and (7) is trivial ($y = 0$) or empty. In order to extend this set to a nontrivial subset of the solutions of (4) we use the gauge transformation of (1)

$$\Phi \to \Phi + I \tag{8}$$

and instead of (4) we apply (6) and (7). For the variational problem (1) the gauge transformation is generated by the following functional:

$$I = \int\limits_X G(y)y'\,dx\,. \tag{9}$$

Eq. (9) is the topological invariant with respect to the local variation of $y(x)$:

$$\delta I \equiv 0\,, \tag{10}$$

i.e. $I[y]$ remains constant while the field $y(x)$ varies continuously preserving its boundary conditions. Vanishing of δI under assumption about the local variation of $y(x)$ leads to the operational definition of the topological invariant expressed by its density $G(y)y'$:

$$\frac{\partial[G(y)y']}{\partial y} - \frac{d}{dx}\frac{\partial[G(y)y']}{\partial y'} \equiv 0\,. \tag{11}$$

Therefore, the Euler equations resulting from the extremum of Φ and the extremum of $\Phi + I$ are equivalent. However, (6) and (7) are not invariant with respect to $\Phi \to \Phi + I$, *i.e.* the gauge transformation contributes to the strong necessary condition. This contribution can extend the subset of solutions to the nontrivial one. In this way we derive a simpler differential equation for extremals of Φ which solutions form a subset of the solutions of the Euler equation. Applying the necessary condition: $\delta\Phi = 0$ to the gauge transformed functional we obtain:

$$\lambda_0 \int\limits_X \left[F_{,y}\left(x,y,y^{'}\right)h(x) + F_{,y^{'}}\left(x,y,y^{'}\right)h^{'}(x)\right]dx$$
$$+\lambda_1 \int\limits_X \left[\left(G(y)y^{'}\right)_{,y} h(x) + \left(G(y)y^{'}\right)_{,y^{'}} h^{'}(x)\right]dx = 0\,. \tag{12}$$

Applying the strong necessary condition to (12) we derive the following field equations:

$$\begin{aligned} \lambda_0 F_{,y}(x,y,y^{'}) + \lambda_1 G_{,y}(y)y^{'} &= 0\,, \\ \lambda_0 F_{,y^{'}}(x,y,y^{'}) + \lambda_1 G(y) &= 0\,. \end{aligned} \tag{13}$$

Formulas (13) establish a simultaneous set of equations for $y(x)$ and $G(y)$. It must be stressed again that any solution of (13) satisfies (4).

2.1. Some generalizations

For the purpose of realistic field theories we generalize our considerations in two directions:

1. y depends on n independent variables: $y = y(x_1, x_2, ..., x_n)$.
2. Φ depends on p independent functions and their derivatives:

$$\Phi = \Phi[y_1, y_{1,x_1}, ...y_{1,x_n}, y_{1,x_1x_1}, y_{1,x_1x_2}, ..., y_2, y_{2,x_1}, ..., y_p, y_{p,x_1}, ...]\,. \tag{14}$$

As an illustrative example we consider a functional of two arguments:

$$\Phi[u,v] = \int\limits_{E^2} F(u, u_{,x}, u_{,t}, u_{,xx}, v, v_{,x}, v_{,t}, v_{,xx})dxdt\,, \tag{15}$$

where

$$\begin{aligned} &u(\cdot,t) \in C^2, \qquad u(x,\cdot) \in C^1\,, \\ &v(\cdot,t) \in C^2, \qquad v(x,\cdot) \in C^1. \end{aligned} \tag{16}$$

The gauge transformation of (15) is generated using the following topological invariants:

$$I_1 = \int_{E^2} G_1(u,v)(u_{,x}v_{,t} - u_{,t}v_{,x})\, dxdt\,, \tag{17}$$

$$I_2 = \int_{E^2} D_x G_2(u,v,u_{,x},v_{,x})\, dxdt\,, \tag{18}$$

$$I_3 = \int_{E^2} D_t G_3(u,v)\, dxdt\,. \tag{19}$$

Since the densities in (18) and (19) are total derivatives of G_2 and G_3, therefore, we call them divergence terms. They can be constructed for a smooth map $M_1^m \to M_2^n$ for arbitrary dimensions m and n, where M_1^m and M_2^n are compact orientable manifolds. Non-divergence term (17) in the case $m = n$ is the degree of a smooth map (winding number, topological charge). This quantity plays crucial role in the construction of topologically stable solutions. For the functional (15) the list of invariants (17)–(19) may not be sufficient. In such a case one must extend dependence of G_2 and G_3 on the higher derivatives of u and v. In the general case the list of invariants depends on:

(i) the number of independent variables,

(ii) the number of dependent variables,

(iii) the degrees of derivatives of the dependent variables present in (14).

The strong necessary conditions for extremum of (15) are:

$$\begin{array}{llll} F_{,u} = 0\,, & F_{,v} = 0\,, & F_{,u_{,x}} = 0\,, & F_{,v_{,x}} = 0\,, \\ F_{,u_{,t}} = 0\,, & F_{,v_{,t}} = 0\,, & F_{,u_{,xx}} = 0\,, & F_{,v_{,xx}} = 0\,. \end{array} \tag{20}$$

In order to extend (20) to the nontrivial set of equations we apply the following gauge transformation:

$$\Phi^* = \Phi + I_1 + I_2 + I_3\,. \tag{21}$$

Applying the strong necessary condition to Φ^* we obtain:

$$u: \quad \lambda_0 F_{,u} + G_{1,u}(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_x G_{2,u} + D_t G_{3,u} = 0\,, \tag{22}$$

$$v: \quad \lambda_0 F_{,v} + G_{1,v}(u_{,x}v_{,t} - u_{,t}v_{,x}) + D_x G_{2,v} + D_t G_{3,v} = 0\,, \tag{23}$$

$$u_{,x}: \quad \lambda_0 F_{,u_{,x}} + G_1 v_{,t} + (G_{2,u} + D_x G_{2,u_{,x}}) = 0\,, \tag{24}$$

$$v_{,x}: \quad \lambda_0 F_{,v_{,x}} - G_1 u_{,t} + (G_{2,v} + D_x G_{2,v_{,x}}) = 0\,, \tag{25}$$

$$u_{,t}: \qquad \lambda_0 F_{,u_{,t}} - G_1 v_{,x} + G_{3,u} = 0\,, \tag{26}$$
$$v_{,t}: \qquad \lambda_0 F_{,v_{,t}} + G_1 u_{,x} + G_{3,v} = 0\,, \tag{27}$$
$$u_{,xx}: \qquad \lambda_0 F_{,u_{xx}} + G_{2,u_{,x}} = 0\,, \tag{28}$$
$$v_{,xx}: \qquad \lambda_0 F_{,v_{xx}} + G_{2,v_{,x}} = 0\,, \tag{29}$$

where λ_0 is a Lagrange multiplier. In the simplification process of (22)–(29) the following commutators have been used:

$$\left[\frac{\partial}{\partial u}, D_x\right] = 0\,, \qquad \left[\frac{\partial}{\partial v}, D_x\right] = 0\,,$$
$$\left[\frac{\partial}{\partial u_{,x}}, D_x\right] = \frac{\partial}{\partial u}, \qquad \left[\frac{\partial}{\partial v_{,x}}, D_x\right] = \frac{\partial}{\partial v}\,,$$
$$\left[\frac{\partial}{\partial u_{,xx}}, D_x\right] = \frac{\partial}{\partial u_{,x}}\,, \qquad \left[\frac{\partial}{\partial v_{,xx}}, D_x\right] = \frac{\partial}{\partial v_{,x}}\,.$$

Equations (22)–(29) establish a set of simultaneous equations for $u(x,t)$, $v(x,t)$, G_1, G_2 and G_3.

3. Applications

This section is devoted to the two classes of applications of the derived formalism: the Bogomolny decompositions and the Bäcklund transformations. In the first section we present derivation of the "duality equations" for the $\pi_1(S^1)$ and $\pi_2(S^2)$ models. In the second section we give all the details of derivations of the Bäcklund transformations for the class of nonlinear Klein–Gordon equations, both hyperbolic and elliptic. Finally, we test our formalism on the nonlinear Schrödinger equation.

3.1. Bogomolny decomposition

In the study of some nonlinear models solutions can be obtained by considering the first order differential equations (Bogomolny equations), instead of more complicated Euler–Lagrange equations [1, 13, 14]. The traditional method of deriving Bogomolny equations is based on transforming an expression of the energy of a field configuration to the positive determined form which the lower bound has topological nature. In the study of topological solitons the Bogomolny equations play the special role. In recent years there have been numerous studies on the soliton solutions of Chern–Simons gauge theories, Landau–Ginzburg model and the Maxwell–Chern–Simons theory [1, 15, 16]. Since the Bogomolny method is based on the minimization of the energy the derived solutions are static. Physically one can think of this property as reflecting the absence of static forces between well-separated single solitons. However, some time dependent solutions can be derived. Defining the space of static soliton solutions of soliton number n (moduli space M_n) the interacting dynamics of several vortices has been constructed [17]. More powerful method then the traditional one is $N = 2$ supersymmetric extension of the investigated model [18–20]. In this formalism the energy of field configuration is bounded below by the central charge of the supersymmetric algebra. The Bogomolny equations arise as algebraic results from the following algorithm [21]. Let us consider a theory with a conserved topological charge. (The central charge for $N = 2$ supersymmetric version is equal to the topological charge [22].) We shall construct a supersymmetric extension of this theory. A topological conservation law, if true in the original theory, will remain in the extended theory. The energy functional of extended theory should reduce to that of the original theory when the extra physical fields are eliminated. Conditions for such a reduction appear to be the Bogomolny equations.

In this section we present derivation of the Bogomolny equations (Bogomolny decomposition) resulting from the strong necessary condition concept. In contrast to the above mentioned approaches this derivation does not require any bounds for the energy or action functional.

3.1.1. Models associated with $\pi_1(S^1)$ homotopy group

Following [1, 2] one can derive the above Bogomolny decomposition for the one-dimensional scalar field theory. We are looking for the lowest possible static energy in disconnected sectors characterized by the different possibilities for the asymptotic behavior of the finite energy configurations. Let the sectors be classified by the elements of the homotopy group $\pi_1(S^1)$ and let the static energy be of the following form:

$$H[y] = \int\limits_X \left[\frac{1}{2} \left(\frac{dy(x)}{dx} \right)^2 + U[y(x)] \right] dx\,. \tag{30}$$

Then the ground state minimizing (30) must satisfy the associated Euler's equation:

$$y(x)_{,xx} = U_{,y}[y(x)]\,. \tag{31}$$

Following the Bogomolny decomposition one splits $H[y]$:

$$H[y] = \frac{1}{2} \int\limits_X \left[\frac{dy(x)}{dx} \pm \sqrt{2(U[y] - C)} \right]^2 dx + I_0\,, \tag{32}$$

where

$$I_0 = \mp \int\limits_X \frac{dy(x)}{dx} \sqrt{2(U[y] - C)} dx + \int\limits_X C dx \tag{33}$$

is the topological invariant. C is a constant determinig the origin of the energy scale and satisfying the following condition:

$$\left| \int_X C dx \right| < \infty \,. \tag{34}$$

It results from (32) that, $y(x)$ is the minimum of (30) if and only if $y(x)$ satisfies the first order differential equation:

$$\frac{dy(x)}{dx} \pm \sqrt{2(U[y] - C)} = 0 \,. \tag{35}$$

When (35) is derived by integration of the Euler equation then C plays the role of integration constants. Now we present derivation of (35) from the strong necessary condition concept. Let us transform (30) to $H^*[y] = H[y] + I$, where I and H are given by (9) and (30), respectively. Applying (13) we derive:

$$\begin{aligned} U_{,y} + G_{,y} y_{,x} &= 0 \,, \\ y_{,x} + G &= 0 \,. \end{aligned} \tag{36}$$

Eliminating G from (36) we obtain (35).

3.1.2. Field equations associated with $\pi_2(S^2)$ homotopy group

Less trivial problems correspond to a mapping of the two-dimensional space of independent variables into a two-dimensional sphere. Let us assume that all possible values of a continuous field establish a manifold isomorphic to S^2. This assumption is equivalent to the assumption of a constant boundary conditions at infinity. Therefore, any continuous field function satisfying the boundary conditions can be classified by the homotopy class. The set of all these classes (and the rules of superposition) establish the homotopy group $\pi_2(S^2)$ [3, 23]. This information is very important from the point of view of the strong necessary condition concept. It determines the main topological invariants used for the construction of the gauge transformation. There are several important field models classified by $\pi_2(S^2)$: static two-dimensional classical Heisenberg model [24, 25], field models in (1+1)-dimensional space generating soliton equations [26]. In this section we present some results for the Heisenberg model (σ-model). Belavin and Polyakov [24, 25] using the Bogomolny decomposition, found all topological solutions of the static two-dimensional Heisenberg model. In this section we will re-derive their results by applying the strong necessary condition concept. In order to make direct use of the results from Sec. 2 we identify independent variables (x_1, x_2) with (x, t), respectively. We derive their results by the use of procedure (15)–(29). The model is governed by the following differential equation:

$$\Delta w - \frac{2w^*(\nabla w)^2}{1 + ww^*} = 0\,, \tag{37}$$

where $w = (S^x + iS^y)/(1 + S^z), S^x, S^y, S^z$ are components of the classical Heisenberg spin, normalized to a constant value: $(S^x)^2 + (S^y)^2 + (S^z)^2 = \text{const.}$ (37) results from the least "action" principle, where the "action" is represented by the integral of energy:

$$H = \int\limits_{E^2} \frac{\nabla w \nabla w^*}{(1 + ww^*)^2} dx_1 dx_2\,, \tag{38}$$

where w is a complex field on E^2. It is sufficient to consider only the invariant I_1 from (17)–(19):

$$I_1 = \int\limits_{E^2} G_1(w, w^*)(w_{,x_1} w^*_{,x_2} - w_{,x_2} w^*_{,x_1})\, dx_1 dx_2\,. \tag{39}$$

We derive the following necessary conditions:

$$\frac{w^*_{,x_1}}{(1 + ww^*)^2} + w^*_{,x_2} G_1(w, w^*) = 0\,, \tag{40}$$

$$\frac{w^*_{,x_2}}{(1 + ww^*)^2} - w^*_{,x_1} G_1(w, w^*) = 0\,, \tag{41}$$

$$-\frac{2w^* \nabla w \nabla w^*}{(1 + ww^*)^3} + G_{1,w}(w, w^*)(w_{,x_1} w^*_{,x_2} - w_{,x_2} w^*_{,x_1}) = 0\,, \tag{42}$$

and c.c.

Equations (40)–(42) and the complex conjugated ones must be self consistent. This requirement determines $G_1(w, w^*)$ uniquely:

$$G_1(w, w^*) = -i\, \frac{1}{(1 + ww^*)^2}\,. \tag{43}$$

Substituting (43) into (41) and (42) we obtain the Belavin–Polyakov result:

$$w^*_{,x_1} - i w^*_{,x_2} = 0\,,$$
$$w_{,x_1} + i w_{,x_2} = 0\,.$$

3.2. The Bäcklund transformations

Bäcklund transformation arose in the 19th century and still remains the only hope to construct sufficiently complicated exact solutions of nonlinear equations. The main idea of the Bäcklund transformation is the following. Let $S(u) = 0$ and $T(v) = 0$ be two uncoupled partial differential equations, in two independent variables x and t, for the two functions u and v. Let $R_i = 0$ be a pair of relations:

$$R_i(u, v, u_{,x}, v_{,x}, u_{,t}, v_{,t}, ...; x, t) = 0 \tag{44}$$

between the two functions u and v, where $i = 1, 2$. Then $R_i = 0$ is a Bäcklund transformation if it is integrable for v when $S(u) = 0$ and if resulting v is a solution of $T(v) = 0$, and vice versa. This approach to the solutions of the equations $S(u) = 0$ and $T(v) = 0$ is useful if the relations (44) are simpler then the original equations. The existence of the Bäcklund transformations is usually taken as a criterion for complete integrability. There are different ways in which the Bäcklund transformations may be achieved [4–7]. The newest one is connected with the zero curvature formulation [27, 28]. For the large class of equations admitting a zero curvature representation, the auto-Bäcklund transformations can be recovered from the Darboux matrix concept. Very recently new variational approach to the Bäcklund transformations has been derived on the basis of the strong necessary condition concept [12]. This method is of the same structure as the variational derivation of the Bogomolny decomposition described in Sec. 3.1. Below we present some applications of the strong necessary condition to derivation of the Bäcklund transformations.

3.2.1. A class of nonlinear Klein–Gordon equations

We illustrate applicability of our method on the well known class of equations [7]:

$$u_{,xt} = P(u)\,, \tag{45}$$

where u satisfies (16) and $\exists\, p : p_{,u} = P(u)$. Eq. (45) possesses poor topology, *i.e.* all configurations of the field $u(x,t)$ satisfying the boundary conditions are classified by the elements of the homotopy group $\pi_2(S^1)$, which is trivial. In order to construct topological invariant (17) we have to consider a model corresponding to the nontrivial homotopy group $\pi_2(S^2)$. Thus we combine two independent equations of (45) type:

$$u_{,xt} = P(u)\,, \quad v_{,xt} = Q(v)\,, \tag{46}$$

where u and v satisfy (16) and $\exists\, p : p_{,u} = P(u), \exists\, q : q_{,v} = Q(v)$. The nonlinear Klein–Gordon system enables us to formulate the following problem:

What are the forms of $P(u)$ and $Q(v)$ for which (46) possesses the Bäcklund transformation? We start from writing down the action functional:

$$\Phi[u,v] = \int_{E^2} \left[\tfrac{1}{2}u_{,x}u_{,t} + p(u) + \lambda_0\left(\tfrac{1}{2}v_{,x}v_{,t} + q(v)\right)\right] dxdt\,. \tag{47}$$

For the purpose of the strong necessary condition concept we generate the gauge transformation with (17)–(19), where for simplicity we reduce G_2 to the following form: $G_2 = G_2(u,v)$. Applying the concept of the strong necessary condition to the gauge transformed $\Phi^* = \Phi + I_1 + I_2 + I_3$ we derive the following field equations:

$$u:\ P(u)+G_{1,u}(u_{,x}v_{,t}-u_{,t}v_{,x})+G_{2,uu}u_{,x}+G_{2,uv}v_{,x}+G_{3,uu}u_{,t}+G_{3,uv}v_{,t}=0\,, \tag{48}$$

$$v:\ \lambda_0 Q(v)+G_{1,v}(u_{,x}v_{,t}-u_{,t}v_{,x})+G_{2,uv}u_{,x}+G_{2,vv}v_{,x}+G_{3,vu}u_{,t}+G_{3,vv}v_{,t}=0\,, \tag{49}$$

$$u_{,x}:\quad \frac{1}{2}u_{,t} + G_1 v_{,t} + G_{2,u} = 0\,, \tag{50}$$

$$v_{,x}:\quad \frac{\lambda_0}{2}v_{,t} - G_1 u_{,t} + G_{2,v} = 0\,, \tag{51}$$

$$u_{,t}:\quad \frac{1}{2}u_{,x} - G_1 v_{,x} + G_{3,u} = 0\,, \tag{52}$$

$$v_{,t}:\quad \frac{\lambda_0}{2}v_{,x} + G_1 u_{,x} + G_{3,v} = 0\,. \tag{53}$$

First of all (48)–(53) must be self consistent. Formally, we have six simultaneous equations for the five unknown functions: u, v, G_1, G_2, G_3. Therefore, in the first step we must decrease the number of equations to four by making (48)–(53) linearly dependent. We achieve this by the following conditions:

$$\frac{\lambda_0}{4} + G_1^2 = 0\,, \tag{54}$$

$$2G_1G_{2,u} + G_{2,v} = 0\,, \tag{55}$$

$$2G_1G_{3,u} - G_{3,v} = 0\,. \tag{56}$$

(54)–(56) imply that G_1 must be constant: $G_1 = \pm\frac{\sqrt{-\lambda_0}}{2}$ while $\lambda_0 < 0$ and

$$G_2 = G_2\left(\frac{u}{\sqrt{-\lambda_0}} - v\right), \tag{57}$$

$$G_3 = G_3\left(\frac{u}{\sqrt{-\lambda_0}} + v\right). \tag{58}$$

For further calculations we choose $\lambda_0 = -1$ and $G_1 = \frac{1}{2}$. Taking into account (55) and (56) we reduce (48)–(53) to the set of four equations:

$$\begin{aligned} P(u) &= -G_{2,uu}(u_{,x} - v_{,x}) - G_{3,uu}(u_{,t} + v_{,t})\,, \\ Q(u) &= -G_{2,uu}(u_{,x} - v_{,x}) + G_{3,uu}(u_{,t} + v_{,t})\,, \end{aligned} \tag{59}$$

$$\begin{aligned} u_{,x} - v_{,x} &= -2G_{3,u}\,, \\ u_{,t} + v_{,t} &= -2G_{2,u}\,. \end{aligned} \tag{60}$$

It will appear below that equations (60) establish the Bäcklund transformation for equations (46). Using (60) we eliminate from (59) the derivatives with respect to x and t:

$$\tfrac{1}{4}(P(u) + Q(v)) = G_{2,uu}G_{3,u}\,, \tag{61}$$

$$\tfrac{1}{4}(P(u) - Q(v)) = G_{3,uu}G_{2,u}\,. \tag{62}$$

Taking into account relations between the derivatives of the second order: $G_{2,uu} = -G_{2,uv}, G_{3,uu} = G_{3,uv}$ we derive from (61) and (62) the following conditions for G_2 and G_3:

$$\tfrac{1}{2}P(u) = \frac{\partial}{\partial u}(G_{2,u}G_{3,u})\,, \tag{63}$$

$$-\tfrac{1}{2}Q(v) = \frac{\partial}{\partial v}(G_{2,u}G_{3,u})\,. \tag{64}$$

It results from (63) and (64) that:

$$G_{2,u}G_{3,u} = \tfrac{1}{2}[p(u) - q(v)] + \text{const.} \tag{65}$$

According to (57) and (58) we introduce the following notation: $G_{2,u}=f(u-v)$ and $G_{3,u} = g(u+v)$. Then (65) becomes:

$$f(u-v)g(u+v) = \tfrac{1}{2}[p(u) - q(v)] + \text{const.} \tag{66}$$

In order to determine admissible set of solutions of (66) we remove $p(u)$ and $q(v)$ from (66) by differentiation with respect to u and v:

$$\frac{f''(\xi)}{f(\xi)} = \frac{g''(\eta)}{g(\eta)} = \omega\,, \tag{67}$$

where $\xi = u - v$, $\eta = u + v$. The separation constants ω labels the solutions of the one parameter family consisting of three disconnected sets ($\omega > 0$, $\omega = 0$, $\omega < 0$) corresponding to different forms of the right-hand sides of the

TABLE I

Dependence of type of the Bäcklund transformation on the forms of the right-hand sides of the initial equations (46).

ω	$f(\xi)$ and $g(\eta)$	$P(u)$ and $Q(v)$	A,B,C,D
$\omega>0$	$f(\xi)=A\exp\sqrt{\omega}\xi$ $+B\exp-\sqrt{\omega}\xi$	$P(u)=4\sqrt{\omega}\,(AC\exp 2\sqrt{\omega}u$ $-BD\exp -2\sqrt{\omega}u)$	auto-Bä: $C^2+D^2>0$
	$g(\eta)=C\exp\sqrt{\omega}\eta$ $+D\exp-\sqrt{\omega}\eta$	$Q(v)=4\sqrt{\omega}(-BC\exp 2\sqrt{\omega}v$ $+AD\exp -2\sqrt{\omega}v)$	$A=-B\neq 0$
			Bä: in other cases
$\omega=0$	$f(\xi)=A\xi+B$	$P(u)=4ACu$	auto-Bä:
	$g(\eta)=C\eta+D$	$Q(v)=4ACv$	$\forall A,C\in\mathbf{R}$
$\omega<0$	$f(\xi)=A\cos\sqrt{-\omega}\xi$ $+B\sin\sqrt{-\omega}\xi$	$P(u)=4\sqrt{-\omega}[(BD-AC)\sin 2\sqrt{-\omega}u$ $+(AD+BC)\cos 2\sqrt{-\omega}u]$	auto-Bä: $A=B, C=D$
	$g(\eta)=C\cos\sqrt{-\omega}\eta$ $+D\sin\sqrt{-\omega}\eta$	$Q(v)=4\sqrt{-\omega}[(BD+AC)\sin 2\sqrt{-\omega}v$ $+(-AD+BC)\cos 2\sqrt{-\omega}v]$	Bä: in other cases

original equations. Depending on A,B,C and D parameters we obtain auto-Bäcklund (auto-Bä) or Bäcklund (Bä) transformation inside each subset of ω space (Table I).

Moreover, if we admit more general forms for P and Q in (46):

$$u_{,xt}=P(u,x,t)\,,\qquad v_{,xt}=Q(v,x,t)\,,$$

then the auto-Bäcklund transformation exists if and only if $P=P_is(x)r(t)$ and $Q=Q_is(x)r(t)$, where P_i and Q_i are given in Table I and $s(x)$, $r(t)$ are arbitrary functions of x and t belonging to the class C^1. Similar result was obtained by Byrnes in [30].

3.2.2. The nonlinear inhomogeneous elliptic Klein–Gordon equation

In this section we apply the strong necessary condition to the elliptic nonlinear inhomogeneous Klein–Gordon equation:

$$u_{,xx}+u_{,tt}=\gamma(x,t,u)\,, \tag{68}$$

where x,t are the independent variables, u is the dependent variable, whereas γ is a function of x, t and u. The Lagrangian density which leads to equation (68) has the form:

$$\mathcal{L}=\tfrac{1}{2}(u_{,x}^2+u_{,t}^2)+\Gamma(x,t,u)\,, \tag{69}$$

where

$$\frac{\partial \Gamma(x,t,u)}{\partial u} = \gamma(x,t,u)\,. \tag{70}$$

In order to derive the Bäcklund transformation from the strong necessary condition concept we define the following gauge transformed functional:

$$\Phi^*[u,v] = \int\limits_{E^2} \left\{ \left[\tfrac{1}{2}(u_{,x}^2+u_{,t}^2) + \Gamma_1(x,t,u)\right] + \lambda_0 \left[\tfrac{1}{2}(v_{,x}^2+v_{,t}^2) + \Gamma_2(x,t,v)\right] \right\} dxdt$$
$$+I_1 + I_2 + I_3\,, \tag{71}$$

where I_1, I_2, I_3 are the topological invariants taken as:

$$I_1 = \int\limits_{E^2} G_1(u,v)(u_{,x}v_{,t} - u_{,t}v_{,x})dxdt\,, \tag{72}$$

$$I_2 = \int\limits_{E^2} D_x G_2(x,t,u,v)dxdt\,, \tag{73}$$

$$I_3 = \int\limits_{E^2} D_t G_3(x,t,u,v)dxdt\,. \tag{74}$$

In formulas (72)–(74) G_1, G_2 and G_3 are arbitrary functions of the given arguments. Following the strong necessary condition concept we obtain the set of equations:

$$u: \quad \gamma_1(x,t,u) + G_{1,u}(u_{,x}v_{,t} - u_{,t}v_{,x}) + (G_{2,xu} + G_{2,uu}u_{,x} + G_{2,uv}v_{,x})$$
$$+(G_{3,tu} + G_{3,uu}u_{,t} + G_{3,uv}v_{,t}) = 0\,, \tag{75}$$

$$v: \quad \lambda_0\gamma_2(x,t,v) + G_{1,v}(u_{,x}v_{,t} - u_{,t}v_{,x}) + (G_{2,xv} + G_{2,uv}u_{,x} + G_{2,vv}v_{,x})$$
$$+(G_{3,tv} + G_{3,uv}u_{,t} + G_{3,vv}v_{,t}) = 0\,, \tag{76}$$

$$u_{,x}: \quad u_{,x} + G_1 v_{,t} + G_{2,u} = 0\,, \tag{77}$$

$$v_{,x}: \quad G_1 u_{,x} + \lambda_0 v_{,t} + G_{3,v} = 0\,, \tag{78}$$

$$u_{,t}: \quad u_{,t} - G_1 v_{,x} + G_{3,u} = 0\,, \tag{79}$$

$$v_{,t}: \quad -G_1 u_{,t} + \lambda_0 v_{,x} + G_{2,v} = 0\,. \tag{80}$$

We perform the reduction procedure for equations (77)–(80). Assuming $G_1 = 1$, $\lambda_0 = 1$ and

$$G_{3,u} = -G_{2,v}\,, \tag{81}$$

$$G_{3,v} = G_{2,u}\,, \tag{82}$$

we obtain two equations:

$$u_{,x} + v_{,t} = -G_{2,u}\,, \tag{83}$$

$$u_{,t} - v_{,x} = G_{2,v}\,. \tag{84}$$

Introducing complex variables: $z = x + it$, $\bar{z} = x - it$ as the independent variables and the new dependent variables: $\alpha = \frac{1}{2}(u + iv)$, $\beta = \frac{1}{2}(u - iv)$ we express (83) and (84) in the form:

$$\alpha_{,z} = -\tfrac{1}{4} G_{2,\beta}\,, \tag{85}$$

$$\beta_{,\bar{z}} = -\tfrac{1}{4} G_{2,\alpha}\,. \tag{86}$$

Equations (75) and (76) in those independent and dependent variables are:

$$\gamma_1(z, \bar{z}, \alpha + \beta) + (G_{2,z\alpha} + G_{2,\bar{z}\beta}) - \tfrac{1}{4}(G_{2,\alpha\alpha} G_{2,\beta} + G_{2,\beta\beta} G_{2,\alpha}) = 0\,, \tag{87}$$

$$\gamma_2(z, \bar{z}, \frac{\alpha - \beta}{i}) + i(G_{2,z\alpha} - G_{2,\bar{z}\beta}) - \tfrac{1}{4}\, i\,(G_{2,\alpha\alpha} G_{2,\beta} - G_{2,\beta\beta} G_{2,\alpha}) = 0\,. \tag{88}$$

In derivation of equations (87) and (88) we have used the fact that $G_{2,\alpha\beta} = 0$, which results from (81) and (82). Following that property the function G_2 may be presented in the form:

$$G_2(z, \bar{z}, \alpha, \beta) = A(z, \bar{z}, \alpha) + B(z, \bar{z}, \beta)\,. \tag{89}$$

Substituting (89) into (87) and (88) we obtain:

$$\gamma_1(z, \bar{z}, \alpha + \beta) + (A_{,z\alpha} + B_{,\bar{z}\beta}) - \tfrac{1}{4}(A_{,\alpha\alpha} B_{,\beta} + B_{,\beta\beta} A_{,\alpha}) = 0\,, \tag{90}$$

$$\gamma_2(z, \bar{z}, \frac{\alpha - \beta}{i}) + i(A_{,z\alpha} - B_{,\bar{z}\beta}) - \tfrac{1}{4}\, i\,(A_{,\alpha\alpha} B_{,\beta} - B_{,\beta\beta} A_{,\alpha}) = 0\,. \tag{91}$$

Differentiating equation (90) with respect to α and β, and subtracting the obtained equations from each other we have:

$$A_{,z\alpha\alpha} - \tfrac{1}{4} A_{,\alpha\alpha\alpha} B_{,\beta} = B_{,\bar{z}\beta\beta} - \tfrac{1}{4} A_{,\alpha} B_{,\beta\beta\beta}\,. \tag{92}$$

Equation (92) may be separated if we assume that: $A_{,z} = 0, B_{,\bar{z}} = 0$. Therefore,

$$A_{,\alpha} = a_1(\bar{z})\, \mathrm{e}^{\sqrt{\lambda}\alpha} + a_2(\bar{z})\, \mathrm{e}^{-\sqrt{\lambda}\alpha}\,, \tag{93}$$

$$B_{,\beta} = b_1(z)\, \mathrm{e}^{\sqrt{\lambda}\beta} + b_2(z)\, \mathrm{e}^{-\sqrt{\lambda}\beta}\,, \tag{94}$$

where a_1, a_2, b_1, b_2 are arbitrary functions of the given arguments. Repeating the described above procedure for equation (91) we obtain the same solutions for $A_{,\alpha}$ and $B_{,\beta}$. Substituting (93) and (94) into (90) and (91) we obtain:

$$\gamma_1(z,\bar{z},\alpha+\beta) = \frac{\sqrt{\lambda}}{2}\left[a_1(\bar{z})b_1(z)\,\mathrm{e}^{\sqrt{\lambda}(\alpha+\beta)} - a_2(\bar{z})b_2(z)\,\mathrm{e}^{-\sqrt{\lambda}(\alpha+\beta)}\right], \quad (95)$$

$$\gamma_2(z,\bar{z},\frac{\alpha-\beta}{i}) = \frac{i\sqrt{\lambda}}{2}\left[a_1(\bar{z})b_2(z)\,\mathrm{e}^{\sqrt{\lambda}(\alpha-\beta)} - a_2(\bar{z})b_1(z)\,\mathrm{e}^{-\sqrt{\lambda}(\alpha-\beta)}\right]. \quad (96)$$

Thus, we have obtained that the Bäcklund transformation exists if the functions γ_1 and γ_2 are given by (95) and (96). Following (85) and (86) the Bäcklund transformation has the form:

$$\alpha_{,z} = -\tfrac{1}{4}\left[b_1(z)\,\mathrm{e}^{\sqrt{\lambda}\beta} + b_2(z)\,\mathrm{e}^{-\sqrt{\lambda}\beta)}\right], \quad (97)$$

$$\beta_{,\bar{z}} = -\tfrac{1}{4}\left[a_1(\bar{z})\,\mathrm{e}^{\sqrt{\lambda}\alpha} + a_2(\bar{z})\,\mathrm{e}^{-\sqrt{\lambda}\alpha)}\right]. \quad (98)$$

In the special case if $\lambda = -1$, $a_1(\bar{z}) = -a_2(\bar{z}) = -ia(\bar{z})$ and $b_1(z) = -b_2(z) = -ib(z)$ we obtain:

$$\alpha_{,z} = -\frac{b(z)}{2}\sin\beta\,, \quad (99)$$

$$\beta_{,\bar{z}} = -\frac{a(\bar{z})}{2}\sin\alpha\,. \quad (100)$$

Formulas (95) and (96) take the form:

$$\gamma_1(z,\bar{z},\alpha+\beta) = a(\bar{z})b(z)\sin(\alpha+\beta)\,, \quad (101)$$

$$\gamma_2(z,\bar{z},\frac{\alpha-\beta}{i}) = a(\bar{z})b(z)\sinh(\frac{\alpha-\beta}{i})\,. \quad (102)$$

It results from the general form given by (97) and (98) that equation (68) admits the Bäcklund transformation if the right-hand side depends on independent variables in a special manner and the function of u has the form of exponential, trigonometric sine and cosine, and hyperbolic sine and cosine functions. The obtained result corresponds to those considered in [31–35] for the case if $\gamma_1(x,t,u) = \sin u$ and $\gamma_2(x,t,u) = \sinh u$.

3.2.3. The nonlinear Schrödinger equation

We consider the Nonlinear Schrödinger Equation (NLS) in the canonical form:

$$iu_{,t} + u_{,xx} + 2u \mid u \mid^2 = 0\,, \tag{103}$$

where $u(x,t)$ is a complex valued function. Eq. (103) results from the least action principle for the following Lagrangian density:

$$\mathcal{L}[u] = -u^*_{,x}u_{,x} - \frac{i}{2}(u^*_{,t}u - u^*u_{,t}) + \mid u^*u \mid^2 . \tag{104}$$

The field functions take values in the one-dimensional Complex Projective space CP^1. The homotopy group $\pi_2(\mathrm{CP}^1)$ is equivalent to $\pi_2(S^2)$ and in principle it is possible to perform the Bogomolny decomposition. In order to derive the Bäcklund transformation we have to combine two independent NLS equations:

$$iu_{,t} + u_{,xx} + 2u \mid u \mid^2 = 0\,, \tag{105}$$

$$iv_{,t} + v_{,xx} + 2v \mid v \mid^2 = 0\,, \tag{106}$$

where all values of $u(x,t)$ and $v(x,t)$ compose CP^2 manifold. The identity of (105) and (106) limits our considerations to the auto-Bäcklund transformation. The action functional for (105) and (106) has the following form:

$$\Phi[u,v] = \int_{E^2} (\mathcal{L}[u] + \lambda_0 \mathcal{L}[v])dxdt\,, \tag{107}$$

where λ_0 is a Lagrange multiplier. In order to create the topological invariant we assume that CP^2 possesses the structure of the Kähler manifold [36]. Then the gauge transformation for (107) is generated by the following set of the topological invariants:

$$\begin{aligned} J_1 &= i\int_{E^2} \Gamma_1(u+v, u^*+v^*)\Big[u_{,x}u^*_{,t} - u^*_{,x}u_{,t} + v_{,x}v^*_{,t} - v^*_{,x}v_{,t} \\ &\quad + u_{,x}v^*_{,t} - u^*_{,x}v_{,t} + v_{,x}u^*_{,t} - v^*_{,x}u_{,t}\Big]dxdt\,, \end{aligned} \tag{108}$$

$$J_2 = \int_{E^2} D_x\Gamma_2(u,v,u^*,v^*,u_{,x},v_{,x},u^*_{,x},v^*_{,x})dxdt\,, \tag{109}$$

$$J_3 = \int_{E^2} D_t\Gamma_3(u,v,u^*,v^*)dxdt\,, \tag{110}$$

where $\Gamma_1(\cdot, z^*) \in C^1$, $\Gamma_1(z, \cdot) \in C^1$ and $\Gamma_1 : (z, z^*) \to \Gamma_1(z, z^*) \in \boldsymbol{R}$. Γ_2 and Γ_3 are the mappings of the C^2 class with respect to all their arguments. (For convenience we introduce the imaginary units on the front of the integral in (108).) The gauge transformation

$$\Phi^* = \Phi - I \tag{111}$$

is defined by $I = J_1 + J_2 + J_3$. Applying the strong necessary conditions to (111) we obtain the following field equations:

$$u_{,t}: \quad \frac{i}{2}u^* = -i\Gamma_1(u^*_{,x} + v^*_{,x}) + \Gamma_{3,u}\,. \tag{112}$$

$$u^*_{,t}: \quad -\frac{i}{2}u = i\Gamma_1(u_{,x} + v_{,x}) + \Gamma_{3,u^*}\,. \tag{113}$$

$$v_{,t}: \quad \lambda_0\frac{i}{2}v^* = -i\Gamma_1(v^*_{,x} + u^*_{,x}) + \Gamma_{3,v}\,. \tag{114}$$

$$v^*_{,t}: \quad -\lambda_0\frac{i}{2}v = i\Gamma_1(u_{,x} + v_{,x}) + \Gamma_{3,v^*}\,. \tag{115}$$

$$u_{,x}: \quad u^*_{,x} = -i\Gamma_1(u^*_{,t} + v^*_{,t}) - \Gamma_{2,u}\,. \tag{116}$$

$$u^*_{,x}: \quad u_{,x} = i\Gamma_1(u_{,t} + v_{,t}) - \Gamma_{2,u^*}\,. \tag{117}$$

$$v_{,x}: \quad -\lambda_0 v^*_{,x} = i\Gamma_1(v^*_{,t} + u^*_{,t}) + \Gamma_{2,v}\,. \tag{118}$$

$$v^*_{,x}: \quad \lambda_0 v_{,x} = i\Gamma_1(u_{,t} + v_{,t}) - \Gamma_{2,v^*}\,. \tag{119}$$

$$u: \quad -\frac{i}{2}u^*_{,t} + 2u(u^*)^2 = D_t\Gamma_{3,u} + D_x\Gamma_{2,u} + i\partial_1\Gamma_1\Omega\,. \tag{120}$$

$$u^*: \quad \frac{i}{2}u_{,t} + 2u^*u^2 = D_t\Gamma_{3,u^*} + D_x\Gamma_{2,u^*} + i\partial_2\Gamma_1\Omega\,. \tag{121}$$

$$v: \quad -\lambda_0\frac{i}{2}v^*_{,t} + 2\lambda_0 v(v^*)^2 = D_t\Gamma_{3,v} + D_x\Gamma_{2,v} + i\partial_1\Gamma_1\Omega\,. \tag{122}$$

$$v^*: \quad \lambda_0\frac{i}{2}v_{,t} + 2\lambda_0 v^*v^2 = D_t\Gamma_{3,v^*} + D_x\Gamma_{2,v^*} + i\partial_2\Gamma_1\Omega\,, \tag{123}$$

where $\Omega = (u+v)_{,x}(u+v)^*_{,t} - (u+v)^*_{,x}(u+v)_{,t}$. The symbols preceding colons indicate the arguments of formal differentiations leading to the respective equation. Eqs. (112)–(123) establish the simultaneous system of equations for the nine unknown functions: u, v, u^*, v^*, Γ_1, Γ_2, Γ_2^*, Γ_3, Γ_3^*. The fundamental problem is to constitute (112)–(123) to be self consistent and then to reduce them to the two relations of the two complex functions u and v. Let us assume that $\Gamma^*_{3,u} = \Gamma_{3,u^*}$, $\Gamma^*_{3,v} = \Gamma_{3,v^*}$ and $\lambda_0 = -1$. Then, (112) and (114) are the complex conjugated to (113) and (115), respectively. Additionally we assume that:

$$\Gamma_3(u, v, u^*, v^*) = \frac{i}{2}(uv^* - vu^*)\,, \tag{124}$$

which is consistent with the earlier assumptions for Γ_3. This form finally reduces (112)–(115) to one equation:

$$u - v = -2\Gamma_1 (u+v)_{,x} \; . \tag{125}$$

We are able to assume the following form for Γ_1:

$$\Gamma_1 = -\tfrac{1}{2} \left(\kappa^2 - (u+v)(u^* + v^*)\right)^{-1/2} , \tag{126}$$

where κ is real and $\mid u+v \mid$ is constrained by $\Gamma_1^* = \Gamma_1$. (125) takes the following final form:

$$(u+v)_{,x} = (u-v) \left(\kappa^2 - (u+v)(u^* + v^*)\right)^{1/2} . \tag{127}$$

In the next step we form the consistency between the remaining equations (116)–(119). In order to reduce this set we assume the following constraints for Γ_2: $\Gamma^*_{2,u} = \Gamma_{2,u^*}$ and $\Gamma^*_{2,v} = \Gamma_{2,v^*}$. Then (116)–(119) are reduced to:

$$u_{,x} \;=\; i\Gamma_1 (u+v)_{,t} - \Gamma_{2,u^*} \, , \tag{128}$$

$$v_{,x} \;=\; -i\Gamma_1 (u+v)_{,t} + \Gamma_{2,v^*} \, , \tag{129}$$

where Γ_1 is defined by (126).
In order to obtain the consistency between (128) and (129) we put:

$$\begin{aligned} \Gamma_{2,u^*} \;=\; & -u_{,x} + \tfrac{1}{2}(u_{,x} - v_{,x}) + \tfrac{1}{4}(u+v)^2 (u+v)^* \Psi^{-1/2} \\ & + \tfrac{1}{4}(u+v) \mid u_{,x} + v_{,x} \mid^2 \Psi^{-3/2} , \end{aligned} \tag{130}$$

where $\Psi = \kappa^2 - (u+v)(u^* + v^*)$. Integrating (130) with respect to u^* we obtain:

$$\begin{aligned} \Gamma_2 \;=\; & \tfrac{1}{2}(u_{,x} + v_{,x})(v^* - u^*) - \tfrac{1}{2}\Psi^{1/2} \left[\tfrac{2}{3}\kappa^2 + \tfrac{1}{3}(u+v)(u^* + v^*) \right] \\ & + \tfrac{1}{2}\Psi^{-1/2} \mid u_{,x} + v_{,x} \mid^2 + \phi(u, v, u_{,x}, v_{,x}) \, , \end{aligned} \tag{131}$$

where $\phi(u, v, u_{,x}, v_{,x})$ is an arbitrary function of class C^2. The forms of terms appearing in (131) enable us to determine ϕ in the following form:

$$\phi = \tfrac{1}{2}(v^*_{,x} + u^*_{,x})(v - u) \, . \tag{132}$$

Substituting (132) into (131) we satisfy the constraints for Γ_2. Inserting (131) into (128) and (129) we receive two identical equations of the following form:

$$(u+v)_{,t} = i\,(u_{,x} - v_{,x})\Psi^{1/2} + \frac{i}{2}(u+v)\Big[\mid u+v \mid^2 + (u+v)_{,x} (u^* + v^*)_{,x} \, \Psi^{-1} \Big] . \tag{133}$$

Using (124), (126), (131) and (132) we reduce (120), (121), (122) and (123) to the following single equation:

$$i(u+v)_{,t} + (u+v)_{,xx} + 2(u \mid u \mid^2 + v \mid v \mid^2) = 0\,. \tag{134}$$

Lemma. The following equations: (134), (127) and (133) are dependent.
Proof. Differentiating (127) with respect to x and eliminating $u_{,x} + v_{,x}$ and $u^*_{,x} + v^*_{,x}$ by the use of (127) and its complex conjugated one, we obtain:

$$(u+v)_{,xx} = (u-v)_{,x}\, \Psi^{1/2} - \tfrac{1}{2}\left[(u-v)^2(u^*+v^*) + (u-v)(u+v)(u^*-v^*)\right]. \tag{135}$$

Substituting (135) to (134) and performing some evaluations we derive:

$$(u+v)_{,t} = i(u-v)_{,x}\, \Psi^{1/2} + \frac{i}{2}(u+v)(2uu^* + 2vv^*)\,. \tag{136}$$

The direct calculations show that

$$2uu^* + 2vv^* = \mid u+v \mid^2 + (u+v)_{,x}(u^*+v^*)_{,x}\, \Psi^{-1}\,. \tag{137}$$

Therefore, the remaining independent relations: (127) and (133) establish the desired auto-Bäcklund transformation of the nonlinear Schrödinger equation. If one compares (127) and (133) with the well known form of the Bäcklund transformation [37] then (127) is to be substituted into (133):

$$(u+v)_{,t} = i\,(u_{,x} - v_{,x})\, \Psi^{1/2} + \frac{i}{2}(u+v)\left[\mid u+v \mid^2 + \mid u-v \mid^2\right]. \tag{138}$$

Finally, (127) and (138) represent the Bäcklund transformation in the canonical form.

4. Conclusions

The introduced concept of the strong necessary condition leads to the duality equation associated with the Euler equation. The highest order of the duality equations is lower then the order of the Euler equation. On principle each solution of the duality equation satisfies the Euler one, therefore, $O_{\rm D} \subseteq O_{\rm E}$, where $O_{\rm D}$ and $O_{\rm E}$ are sets of solutions of the duality and the Euler equations, respectively. This can be easily proved by transforming the duality equations into the Euler one by applying a differentiation. It would be a great advantage to extend the above relation to $O_{\rm D} \equiv O_{\rm E}$. In the general case this is not possible. In the worst case $O_{\rm D}$ contains only a trivial solution or $O_{\rm D} = \emptyset$. In order to extend $O_{\rm D}$ to the nontrivial subset of $O_{\rm E}$ we apply two methods: gauge transformation of the action functional and/or softening of the strong necessary condition [12]. If one succeeds to eliminate all G_i (or in the case of complex fields all Γ_i) functions from the duality equations without any differentiation process and without any constraint for the field functions then the final equations establish one of the following forms: the Bogomolny decomposition, the Bäcklund transformation or a duality equation of the higher type. Duality equations correspond to the Bogomolny decomposition if they govern only the field function appearing in the considered differential equation (45), whereas they correspond to the Bäcklund transformation if they govern two independent field functions being unknowns of the two independent differential equations (46). In the general case the higher type duality equations govern more then two independent field functions. When does each of these types appear? The necessary condition to perform the Bogomolny decomposition of the action functional is a nontrivial homotopy group associated with the considered model (see Sec. 3.1.1, 3.1.2 and [29]). In the case of the trivial homotopy group we combine two (or more) independent models in such a way that the resulting homotopy group is nontrivial. As it was already mentioned above, such a procedure leads to the Bäcklund transformation or to the higher types of the duality equations (Sec. 3.2.1, 3.2.2, 3.2.3 and [12]).

REFERENCES

[1] E.B. Bogomolny, *Sov. J. Nucl. Phys.* **24**, 861 (1976).

[2] B. Felsager, *Geometry, Particles and Fields*, Odense University Press 1981, p. 140.

[3] A.S. Schwarz, *Quantum Field Theory and Topology* (in Russian), Nauka, Moscow 1989, sec. II.

[4] A.V. Bäcklund, *Einiges über Curven und Flächentransformationen*, Lund Universitets Arsskrift **10**, (1875).

[5] C. Rogers, *Soliton Theory: A Survey of Results*, Ed. A.P. Fordy, Manchester University Press 1990, p. 97.

[6] B.K. Harrison, in Lecture Notes in Physics 226, *Non-Linear Equations in Classical and Quantum Field Theory*, Ed. N. Sanchez, Springer Verlag, Berlin–Heidelberg–New York–Tokyo 1985, p. 45.

[7] H. Rund, in Lecture Notes in Mathematics 515, *Bäcklund Transformations, the Inverse Scattering Method, Solitons, and their Applications*, Ed. R.M. Miura, Springer Verlag, Berlin–Heidelberg–New York 1976, p. 199.

[8] K. Sokalski, *Acta Phys. Pol.* **A56**, 571 (1979).

[9] K. Sokalski, *Phys. Lett.* **A81**, 102 (1981).

[10] P.T. Jochym, K. Sokalski, *J. Phys.* **A26**, 3837 (1993).

[11] K. Sokalski, *Acta Phys. Pol.* **A60**, 3 (1981).

[12] K. Sokalski, T. Wietecha, Z. Lisowski, *Acta Phys. Pol.* **B32**, 17 (2001).

[13] A.A. Belavin, A.M. Polyakov, A.S. Schwartz, Yu.S. Tyupkin, *Phys. Lett.* **B59**, 85 (1975).
[14] H.J. de Vega, F.A. Schaposnik, *Phys. Rev.* **D14**, 1100 (1976).
[15] J.Hong, Y. Kim, P.Y. Pac, *Phys. Rev. Lett.* **64**, 2230 (1990).
[16] R. Jackiw, E.J. Weinberg, *Phys.Rev. Lett.* **64**, 2234 (1990).
[17] T.M. Samols, *Commun. Math. Phys.* **145**, 149 (1992).
[18] D. Olive, E. Witten, *Phys. Lett.* **B78**, 97 (1978).
[19] P. West, *Introduction to Supersymmetry and Supergravity*, World Scientific, 1986.
[20] B. Damski, *Acta Phys. Pol.* **B31**, 637 (2000).
[21] Z. Hlousek, D. Spector, *Nucl. Phys.* **B397**, 173 (1993).
[22] J. Edelstein, C. Núñez, F. Schaposnik, *Phys. Lett.* **B329**, 39 (1994).
[23] R. Shankar, *J. Phys.* **38**, 1405 (1977).
[24] A.A. Belavin, A.M. Polyakov, *JETP Lett.* **22**, 245 (1975).
[25] B. Felsager, *Geometry, Particles and Fields*, Odense University Press, 1981, p. 570.
[26] B. Felsager, *Geometry, Particles and Fields*, Odense University Press, 1981, p. 127.
[27] J. Cieśliński, *J. Math. Phys.* **36**, 5670 (1995).
[28] F. Brandt, `hep-th/9405064`.
[29] K. Sokalski, *Acta Phys. Pol.* **A65**, 457 (1984).
[30] S.G. Byrnes, *J. Math. Phys.* **17**, 836 (1976).
[31] G. Leibbrandt, *Phys. Rev.* **B15**, 3353 (1977).
[32] G. Leibbrandt, *Phys. Rev.* **D16**, 970 (1977).
[33] G. Leibbrandt, *J. Math. Phys.* **19**, 960 (1977).
[34] A.B. Borisov, V.V. Kiseliev, *Physica* **D31**, 49 (1988).
[35] R. Ferre, *Phys. Status Solidi* **B199**, 535 (1997).
[36] A.M. Perelomov, *Commun. Math. Phys.* **63**, 237 (1978).
[37] P.G. Drazin, R.S. Johnson, *Cambridge Texts in Applied Mathematics, Solitons: an Introduction*, Cambridge University Press, Cambridge, New York, New Rochelle, Melbourne, Sydney 1989, p. 126.