ON THRESHOLD AMPLITUDES I*

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This is the first paper of the series devoted to threshold amplitudes in quantum field theory. We consider here some aspects of tree approximation. The careful discussion of relevant generating functionals including the problem of boundary conditions is given. The general rules for constructing the field matrix elements between threshold states are rederived. Some features of amplitudes for all particles at the threshold are discussed. They are related to the properties of reduced classical Newton systems. In particular, the nullification and divergence of amplitudes are interrelated and explained in terms of dynamics of classical point particles.

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1. Introduction

The problem of multiparticle production has attracted much attention in the past decade [1]. Originally, it concerned electroweak barion and lepton numbers violating processes in the instanton sector [2]. It has been found that, contrary to the naive expectations, the relevant cross sections are not so strongly suppressed if a large number of bosons is present in a final state. Later topologically trivial sector has been considered with similar conclusions: it has been shown that the tree amplitudes for n-particle production in scalar Φ^4-theory behave like $n!\lambda^{\frac{n}{2}}$ so they are not suppressed even in a weakly coupled theory [3]. It appeared also that a very detailed knowledge concerning amplitudes is possible for special kinematics: that of off-shell, vanishing fourmomenta [4] or when all final particles are at the threshold [4–9]; other kinematics were also considered [10].

The results concerning threshold amplitudes are very interesting because, being physically relevant, they provide at the same time a rare example of exact calculations in quantum field theory ("exact" means exact in some

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parameter — here the number of bosons produced). Moreover, they are a starting point for other interesting results; for example, knowing threshold amplitudes one can make estimates, based on unitary, beyond threshold [11]. Another interesting phenomenon is the nullification of certain amplitudes on the threshold. For example, for the process $2 \to n$ in Φ^4 unbroken theory, all amplitudes vanish at threshold, except $n = 2$ and $n = 4$; if the symmetry $\Phi \to -\Phi$ is broken the only nonvanishing amplitude is $2 \to 2$ [10, 12, 13]. Other theories were also analysed from this point of view and the nullification of tree $2 \to n$ amplitudes at the threshold was discovered in the bosonic sector of electroweak model [14] and in the linear σ-model [15]; these results in general do not extend to the one-loop level [16] (see, however, Ref. [17]). In more complicated theories the nullification takes place only provided some relations between parameters are satisfied [8, 15]. Other interesting examples of nullification are provided by the amplitudes with both initial and final particles at the threshold. It appears that in some theories almost all tree amplitudes of that kind vanish; the most prominent example is the O(2)-symmetric theory with two fields in defining representation of O(2) and the symmetry softly broken by the mass term [18]. The nullification of tree amplitudes is here ultimately related to integrability of some classical dynamical systems. This can be explained as follows [18]. The generating functional for tree amplitudes obeys the classical field equations. For threshold amplitudes the translational invariance is restored and dynamical equations take the form of Newton equations for some systems of finite degrees of freedom. One can show [18] that the amputated Green functions are nonvanishing only provided in the course of solving perturbatively the dynamical equations the resonances do appear. However, this is excluded if our reduced system posses a certain kind of symmetry (like O(2)-theory mentioned above [19]). The relation between nullification of amplitudes and integrable systems can be understood within general framework of modern theory of integrable systems [20]. However, a more traditional approach based on cancellations due to Ward identities is also possible [21]. These and other properties of threshold amplitudes make the whole subject very interesting and deserving a more detailed study. The present paper is the first of the series devoted to such a study, both at the tree and at the loop levels. It is organized as follows. In Sec. 2 we discuss the generating functionals for Green functions and amputated Green functions in the tree approximation. The boundary effects are treated with a special care and shown to modify the form of the relevant functionals. The general rules for building tree-graph matrix elements of T-ordered field products between states with arbitrary number of particles are derived in Sec. 3 using functional methods. Then, in Sec. 4 these considerations are specified to the case of threshold asymptotic states. Finally, in Sec. 5 we give a general discussion of nullification of threshold-to-threshold tree-graph amplitudes in the context of classical mechanics of Newtonian systems.

2. Generating functions for tree amplitudes

For definiteness we consider the Φ^4-theory defined by the action

$$\begin{aligned} S_J[\Phi] &\equiv S[\Phi] + \int d^4x J(x)\Phi(x) \equiv \int d^4x L(\Phi(x)) + \int d^4x J(x)\Phi(x)\,, \\ L(\Phi) &\equiv \frac{1}{2}(\partial_\mu\Phi\partial^\mu\Phi - m^2\Phi^2) - \frac{\lambda}{4!}\Phi^4\,; \end{aligned} \tag{1}$$

the results are valid, *mutatis mutandis*, also for more complicated theories.

The basic quantity we start with is the generating functional for connected Green functions

$$W[J] \equiv -i\,\ln(\langle 0\mathrm{out} \mid 0\mathrm{in}\rangle_J)\,; \tag{2}$$

here $\langle 0\mathrm{out} \mid 0\mathrm{in}\rangle_J$ is the vacuum-to-vacuum amplitude in the presence of external current $J(x)$. We are interested in the tree-graph approximation. It is well known [22] that, within this approximation, $\Phi(x \mid J) \equiv \delta W/\delta J(x)$ obeys classical field equations in the presence of external current $J(x)$,

$$(\Box + m^2)\Phi(x \mid J) + \frac{\lambda}{3!}\Phi^3(x \mid J) = J(x)\,. \tag{3}$$

The proof is very simple. One converts Eq. (3) into integral equation with appropriate, *i.e.* Feynman boundary conditions

$$\Phi(x \mid J) = \int d^4y \Delta_{\mathrm{F}}(x-y)J(y) - \frac{\lambda}{3!}\int d^4y \Delta_{\mathrm{F}}(x-y)\Phi^3(y \mid J)\,. \tag{4}$$

Iteractive solution of this equation reproduces the tree-graph expansion of one-point Green function. One can then calculate the generating functional

$$W[J] = \int_0^1 d\alpha \int d^4x J(x)\Phi(x \mid \alpha J)\,. \tag{5}$$

Integrating by parts and using

$$\frac{\delta S}{\delta\Phi(x \mid \alpha J)} = -\alpha J(x)$$

one gets

$$\begin{aligned} W[J] &= \int d^4x\alpha J(x)\Phi(x \mid \alpha J) \mid_{\alpha=0}^{\alpha=1} - \int\limits_0^1 d\alpha \int d^4x\alpha J(x)\frac{d\Phi(x \mid \alpha J)}{d\alpha} \\ &= \int d^4xJ(x)\Phi(x \mid J) + \int\limits_0^1 d\alpha \int d^4x \frac{\delta S}{\delta\Phi(x \mid \alpha J)}\frac{d\Phi(x \mid \alpha J)}{d\alpha} \\ &= \int d^4xJ(x)\Phi(x \mid J) + \int\limits_0^1 d\alpha \frac{dS(\alpha)}{d\alpha} = S[\Phi] + \int d^4xJ(x)\Phi(x \mid J)\,. \end{aligned} \tag{6}$$

This derivation is slightly formal due to the fact that $\delta S/\delta\Phi(x)$ involves integration by parts which is not justified because $\Phi(x \mid J)$ does not vanish at infinity. One can show that this results in slight modification of $S[\Phi]$; namely, $S[\Phi]$ is understood as

$$S[\Phi] = \int d^4x\Big(-\frac{1}{2}\Phi(x \mid J)(\Box + m^2)\Phi(x \mid J) - \frac{\lambda}{4!}\Phi^4(x \mid J)\Big)\,. \tag{7}$$

Indeed, consider $S(\alpha)$ entering Eq. (6),

$$\begin{aligned} S(\alpha) &= \int d^4x\Big(\frac{1}{2}(\partial_\mu\Phi(x \mid \alpha J)\partial^\mu\Phi(x \mid \alpha J) \\ &\quad -m^2\Phi^2(x \mid \alpha J)) - \frac{\lambda}{4!}\Phi^4(x \mid \alpha J)\Big) \end{aligned} \tag{8}$$

we have

$$\begin{aligned} \frac{dS(\alpha)}{d\alpha} &= \int d^4x\Big(\partial_\mu\Big(\frac{d\Phi(x \mid \alpha J)}{d\alpha}\Big)\partial^\mu\Phi(x \mid \alpha J) - m^2\frac{d\Phi(x \mid \alpha J)}{d\alpha}\Phi(x \mid \alpha J) \\ &\quad -\frac{\lambda}{3!}\frac{d\Phi(x \mid \alpha J)}{d\alpha}\Phi^3(x \mid \alpha J)\Big) = \int d^4x\Big(\partial_\mu\Big(\frac{d\Phi(x \mid \alpha J)}{d\alpha}\partial^\mu\Phi(x \mid \alpha J)\Big) \\ &\quad -\frac{d\Phi(x \mid \alpha J)}{d\alpha}\Big((\Box + m^2)\Phi(x \mid \alpha J) + \frac{\lambda}{3!}\Phi^3(x \mid \alpha J)\Big)\Big) \\ &= \int d^4x\Big(\partial_\mu\Big(\frac{d\Phi(x \mid \alpha J)}{d\alpha}\partial^\mu\Phi(x \mid \alpha J)\Big) - \alpha J(x)\frac{d\Phi(x \mid \alpha J)}{d\alpha}\Big)\,. \end{aligned} \tag{9}$$

Therefore, according to the derivation given above, the proper formula is

$$W[J] = S[\Phi] + \int d^4xJ\Phi - \int\limits_0^1 d\alpha \int d^4x\partial_\mu\Big(\frac{d\Phi}{d\alpha}\partial^\mu\Phi\Big)\,. \tag{10}$$

On the other hand one can write

$$\begin{aligned} S[\Phi] &= \int d^4x \left(\frac{1}{2}(\partial_\mu\Phi\partial^\mu\Phi - m^2\Phi^2) - \frac{\lambda}{4!}\Phi^4\right) \\ &= \int d^4x \left(-\frac{1}{2}\Phi(\Box + m^2)\Phi - \frac{\lambda}{4!}\Phi^4\right) + \frac{1}{2}\int d^4x\partial_\mu(\Phi\partial^\mu\Phi) \\ &= \int d^4x \left(-\frac{1}{2}\Phi(\Box + m^2)\Phi - \frac{\lambda}{4!}\Phi^4\right) \\ &\quad + \frac{1}{2}\int\limits_0^1 d\alpha \int d^4x\partial_\mu\left(\frac{d\Phi}{d\alpha}\partial^\mu\Phi + \Phi\partial^\mu\left(\frac{d\Phi}{d\alpha}\right)\right) \end{aligned} \tag{11}$$

which, together with Eq. (10) implies

$$\begin{aligned} W[J] &= \int d^4x \left(-\frac{1}{2}\Phi(\Box + m^2)\Phi - \frac{\lambda}{4!}\Phi^4\right) + \int d^4xJ\Phi \\ &\quad + \frac{1}{2}\int\limits_0^1 d\alpha \int d^4x\partial_\mu\left(\Phi\partial^\mu\frac{d\Phi}{d\alpha} - \frac{d\Phi}{d\alpha}\partial^\mu\Phi\right) . \end{aligned} \tag{12}$$

Last term equals

$$\frac{1}{2}\int\limits_0^1 d\alpha \int d^4x\left(\Phi(\Box + m^2)\frac{d\Phi}{d\alpha} - \frac{d\Phi}{d\alpha}(\Box + m^2)\Phi\right) . \tag{13}$$

Now, using

$$\Phi(x \mid \alpha J) = \int d^4y\Delta_{\mathrm{F}}(x-y)\left(\alpha J(y) - \frac{\lambda}{3!}\Phi^3(y \mid \alpha J)\right) \tag{14}$$

and

$$\frac{d\Phi\,(x \mid \alpha J)}{d\alpha} = \int d^4y\Delta_{\mathrm{F}}(x-y)\left(J(y) - \frac{\lambda}{2!}\Phi^2(y \mid \alpha J)\frac{d\Phi(y \mid \alpha J)}{d\alpha}\right) \tag{15}$$

as well as $\Delta_{\mathrm{F}}(-x) = \Delta_{\mathrm{F}}(x)$ we find that (13) vanishes.

Let us now pass to the S-matrix elements. It is easy to see that the standard LSZ formulae can be summarized as follows. To get the generating functional for S-matrix elements one takes $W[J]$ and makes the replacement $J(x) \to \Phi_0(x)\overrightarrow{(\Box_x + m^2)}$, where $\Phi_0(x)$ is a classical free field. Let $\{f_i(x)\}$

be a complete set of normalized positive energy solutions of K–G equations; put

$$\Phi_0(x) = \sum_i (\beta_i f_i(x) + \overline{\beta_i f_i(x)})\,. \tag{16}$$

Then

$$S(\beta) = W[J] \mid_{J \to \Phi_0 \overrightarrow{(\Box + m^2)}} \tag{17}$$

is the generating functional for S-matrix elements: $\partial/\partial\beta_i(\partial/\partial\overline{\beta_i})$ produces initial (final) state described by the wave function $f_i(x)$. More generally, if the above replacement is made after taking a number of derivatives with respect to $J(x)$ one obtains the generating functional for the arbitrary matrix elements of time-ordered field products; for example, the derivatives $\partial/_{\partial\beta_i}(\partial/_{\partial\overline{\beta_i}})$ of

$$\frac{\delta^2 W}{\delta J(x_1)\delta J(x_2)}\Big|_{J \to \Phi_0 \overrightarrow{(\Box + m^2)}}$$

generate matrix elements of $T(\Phi(x_1)\Phi(x_2))$. It is not difficult to find the relevant equation for $\Phi(x \mid \Phi_0) \equiv \Phi(x \mid J) \mid_{J \to \Phi_0 \overrightarrow{(\Box + m^2)}}$. Making the replacement $J(x) \to \Phi_0(x)\overrightarrow{(\Box_x + m^2)}$ in Eq. (4) one obtains

$$\Phi(x \mid \Phi_0) = \Phi_0(x) - \frac{\lambda}{3!}\int d^4y \varDelta_{\mathrm{F}}(x-y)\Phi^3(y \mid \Phi_0)\,. \tag{18}$$

This equation implies that $\Phi(x \mid \Phi_0)$ obeys

$$(\Box + m^2)\Phi(x \mid \Phi_0) + \frac{\lambda}{3!}\Phi^3(x \mid \Phi_0) = 0 \tag{19}$$

together with

$$\Phi(x \mid \Phi_0) \mid_{\lambda=0} = \Phi_0(x)\,. \tag{20}$$

The above derivation is slightly formal. However, the validity of Eq. (18) can be confirmed by solving it recursively. One obtains then tree-graph expansion of matrix elements of $\Phi(x)$.

Let us now calculate the generating functional $S(\beta) = S[\Phi_0]$. Naive approach would be to use Eqs. (6), (7) and (18) to get

$$\begin{aligned} S[\Phi_0] \;=\; & \int d^4x \left(-\frac{1}{2}\Phi(x \mid \Phi_0)(\Box + m^2)\Phi(x \mid \Phi_0) - \frac{\lambda}{4!}\Phi^4(x \mid \Phi_0)\right) \\ & + \int d^4x \Phi_0(x)(\Box + m^2)\Phi(x \mid \Phi_0)\,. \end{aligned} \tag{21}$$

However, this is wrong [23, 24]. To see this it is sufficient to calculate $\Phi(x \mid \Phi_0)$ to the first order in λ and then $S[\Phi_0]$ to the same order [24].

In order to get a right answer we have to consider carefully the kinetic term. One has

$$\begin{aligned}&\int d^4x\Phi(x \mid J)(\Box + m^2)\Phi(x \mid J)\\ &= \int d^4x d^4y \Big(J(x) - \frac{\lambda}{3!}\Phi^3(x \mid J)\Big) \Delta_{\mathrm{F}}(x-y) \Big(J(y) - \frac{\lambda}{3!}\Phi^3(y \mid J)\Big) .\end{aligned} \tag{22}$$

Let us insert now $J(x) \to \Phi_0(x)\overrightarrow{(\Box_x + m^2)}$, $J(y) \to \Phi_0(y)\overrightarrow{(\Box_y + m^2)}$; then (22) attains the form

$$\int d^4x \left(\frac{-2\lambda}{3!}\right) \Phi^3(x \mid \Phi_0)\Phi_0(x) + \left(-\frac{\lambda}{3!}\right)^2 \int d^4x d^4y \Phi^3(x \mid \Phi_0)\Delta_{\mathrm{F}}(x-y)\Phi^3(y \mid \Phi_0). \tag{23}$$

On the other hand

$$\begin{aligned}&\int d^4x\Phi(x \mid \Phi_0)(\Box + m^2)\Phi(x \mid \Phi_0)\\ &= \int d^4x \left(\Phi_0(x) - \frac{\lambda}{3!}\int d^4y \Delta_f(x-y)\Phi^3(y \mid \Phi_0)\right) \left(\frac{-\lambda}{3!}\right) \Phi^3(\mid \Phi_0) .\end{aligned} \tag{24}$$

Eqs. (23), (24) differ by the term

$$\int d^4x\Phi_0(x) \left(-\frac{\lambda}{3!}\right) \Phi^3(x \mid \Phi_0) = \int d^4x\Phi_0(x)(\Box + m^2)\Phi(x \mid \Phi_0). \tag{25}$$

Taking all that into account one obtains final answer [23, 24].

$$\begin{aligned} S[\Phi_0] \;=\; &\int d^4x \left(-\frac{1}{2}\Phi(x \mid \Phi_0)(\Box + m^2)\Phi(x \mid \Phi_0) - \frac{\lambda}{4!}\Phi^4(x \mid \Phi_0)\right)\\ &+\frac{1}{2}\int d^4x\Phi_0(x)(\Box + m^2)\Phi(x \mid \Phi_0) .\end{aligned} \tag{26}$$

To check this formula let us calculate

$$\begin{aligned}\frac{\partial S}{\partial \beta_i} \;=\; &\int d^4x \left(-\frac{1}{2}\frac{\partial \Phi(x)}{\partial \beta_i}(\Box + m^2)\Phi(x) - \frac{1}{2}\Phi(x)(\Box + m^2)\frac{\partial \Phi(x)}{\partial \beta_i}\right.\\ &\left.-\frac{\lambda}{3!}\Phi^3(x)\ \frac{\partial \Phi(x)}{\partial \beta_i}\right) + \frac{1}{2}\int d^4x f_i(x)(\Box + m^2)\Phi(x)\\ &+\frac{1}{2}\int d^4x\Phi_0(x)(\Box + m^2)\frac{\partial \Phi(x)}{\partial \beta_i} ;\end{aligned} \tag{27}$$

here $\Phi(x) \equiv \Phi(x \mid \Phi_0)$. Using field equation for $\Phi(x \mid \Phi_0)$ we can write

$$\begin{aligned}\frac{\partial S}{\partial \beta_i} &= \frac{1}{2}\int d^4x f_i(x)(\Box + m^2)\Phi(x) + \frac{1}{2}\int d^4x \Phi_0(x)(\Box + m^2)\frac{\partial \Phi(x)}{\partial \beta_i} \\ &+\frac{1}{2}\int d^4x \left(\frac{\partial \Phi(x)}{\partial \beta_i}(\Box + m^2)\Phi(x) - \Phi(x)(\Box + m^2)\frac{\partial \Phi(x)}{\partial \beta_i}\right). \end{aligned} \tag{28}$$

Using

$$\frac{\partial \Phi(x)}{\partial \beta_i} = f_i(x) - \frac{\lambda}{2!}\int d^4y \Delta_{\mathrm{F}}(x-y)\Phi^2(y)\frac{\partial \Phi(y)}{\partial \beta_i} \tag{29}$$

we arrive after some manipulations at

$$\frac{\partial S}{\partial \beta_i} = \int d^4x f_i(x)(\Box + m^2)\Phi(x \mid \Phi_0). \tag{30}$$

Analogously

$$\frac{\partial S}{\partial \overline{\beta_i}} = \int d^4x \overline{f_i(x)}(\Box + m^2)\Phi(x \mid \Phi_0) \tag{31}$$

which confirms validity of Eq. (26).

3. Matrix elements

As it has been explained above the derivatives of $W[J]$ produce connected matrix elements of T-ordered field products. Indeed, taking an appropriate number of derivatives with respect to $J(x)$ and making the replacement $J(x) \to \Phi_0(x)\overrightarrow{(\Box + m^2)}$ one gets the generating functional for relevant matrix elements. Starting from $\Phi(x \mid \Phi_0)$ one obtains the matrix elements $\langle \mathrm{out} \mid \Phi(x) \mid \mathrm{in}\rangle$ with in- (out-) states specified by $\partial/\partial\beta_i$ $(\partial/\partial\overline{\beta_i})$ derivatives. This element can be further reduced by applying K–G operator or used as an input for calculating the amplitude for a larger process.

Let us now pass to the matrix elements $\langle \mathrm{out} \mid T(\Phi(x)\Phi(y)) \mid \mathrm{in}\rangle$. Let us define

$$G(x, y \mid J) \equiv \frac{\delta \Phi(x \mid J)}{\delta J(y)};$$

by differentiating Eq. (3) with respect to $J(y)$ we arrive at the following equation

$$(\Box_x + m^2)G(x, y \mid J) + \frac{\lambda}{2!}\Phi^2(x \mid J)G(x, y \mid J) = \delta^{(4)}(x-y). \tag{32}$$

Making standard replacement one converts Eq. (32) into

$$(\Box_x + m^2)G(x,y \mid \Phi_0) + \frac{\lambda}{2!}\Phi^2(x \mid \Phi_0)G(x,y \mid \Phi_0) = \delta^{(4)}(x-y) \qquad (33)$$

with

$$G(x,y \mid \Phi_0) \equiv G(x,y \mid J) \mid_{J \to \Phi_0 \overrightarrow{(\Box + m^2)}} .$$

The corresponding integral equation which accommodates proper boundary conditions reads

$$G(x,y \mid \Phi_0) = \Delta_{\mathrm{F}}(x-y) - \frac{\lambda}{2!}\int d^4z \Delta_{\mathrm{F}}(x-z)\Phi^2(z \mid \Phi_0)G(z,y \mid \Phi_0) \,. \qquad (34)$$

Iteractive solution of Eq. (33) produces perturbative expansion for $G(x,y \mid \Phi_0)$. If one is able to solve Eq. (34) in closed form all matrix elements $\langle \mathrm{out} \mid T(\Phi(x)\Phi(y) \mid \mathrm{in}\rangle$ are attainable. They can serve, for example, to calculate the S-matrix elements for processes where two particles, initial or final, are distinguished (the main application, in the present context, is the scattering of hard particle in the presence of the arbitrary number of soft ones).

Let us note that Eq. (33) defines Green function for quantum field coupled to the classical external field $\frac{1}{2}\Phi^2(x \mid \Phi_0)$, the coupling constant being λ. Therefore, in order to calculate the S-matrix elements obtained by reducing the fields from T-product one can use the results from scattering theory in classical external field. For example, the relevant scattering amplitude can be calculated from the soution of homogeneous counterpart of Eq. (32) [25]. Let $g(x)$ be the positive energy solution of K–G equation. Define the Feynman wave function [25]

$$\Psi_g(x) \equiv \int d^4y G(x,y \mid \Phi_0)\overleftarrow{(\Box_y + m^2)}g(y) \,. \qquad (35)$$

Then, due to Eq. (33) and $(\Box + m^2)g(x) = 0$,

$$\left(\Box + m^2 + \frac{\lambda}{2}\Phi^2(x \mid \Phi_0)\right)\Psi_g(x) = 0 \qquad (36)$$

and

$$\Psi_g(x) \mid_{\lambda=0} = g(x) \,. \qquad (37)$$

If $f(x)$ is another positive energy solution, the amplitude of the scattering $g \to f$ in the presence of external field $\frac{1}{2}\Phi^2(x \mid \Phi_0)$ is calculated by finding the amplitude that $f(x)$ enters $\Psi_g(x)$ in distant future [25]. The only difference between our case and of actual external field scattering is that our amplitudes are already properly normalized contrary to the external field problem [25]. In Appendix we demonstrate that this technique coincides with that used in Ref. [8].

One can continue with higher T-products. Define

$$G(x,y,z \mid J) = \frac{\delta G(x,y \mid J)}{\delta J(z)}\,. \tag{38}$$

Taking the derivative of Eq. (32) one obtains

$$\begin{aligned}&(\Box_x + m^2)G(x,y,z \mid J) + \frac{\lambda}{2!}\Phi^2(x \mid J)G(x,y,z \mid J)\\ &+\lambda\Phi(x \mid J)G(x,z \mid J)G(x,y \mid J) = 0\end{aligned} \tag{39}$$

and

$$\begin{aligned}&(\Box_x + m^2)G(x,y,z \mid \Phi_0) + \frac{\lambda}{2!}\Phi^2(x \mid \Phi_0)G(x,y,z \mid \Phi_0)\\ &+\lambda\Phi(x \mid \Phi_0)G(x,z \mid \Phi_0)G(x,y \mid \Phi_0) = 0\,.\end{aligned} \tag{40}$$

There are no connected three-point functions in quantum theory of particles in external classical field. Using this as a boundary conditions we can solve Eq. (40)

$$G(x,y,z \mid \Phi_0) = -\lambda\int d^4u G(x,u \mid \Phi_0)\Phi(u \mid \Phi_0)G(u,y \mid \Phi_0)G(u,z \mid \Phi_0)\,. \tag{41}$$

This procedure can be continued. By differentiating Eq. (40) with respect to $J(\omega)$ one obtains the equation for four-point function which again is explicitly solvable. It is not difficult to verify that the tree approximation to the Green functions $G(x_1,\ldots,x_n \mid \Phi_0)$ which generate matrix elements of $T(\Phi(x_1),\ldots\Phi(x_n))$ is determined by the Lagrangian

$$\begin{aligned}\widetilde{L}(\Phi) \;=\; &\frac{1}{2}\left(\partial_\mu\Phi(x)\partial^\mu\Phi(x) - m^2\Phi^2(x) - \frac{\lambda}{2}\Phi^2(x \mid \Phi_0)\Phi^2(x)\right)\\ &-\frac{\lambda}{3!}\Phi(x \mid \Phi_0)\Phi^3(x) - \frac{\lambda}{4!}\Phi^4(x)\end{aligned} \tag{42}$$

obtained from $L(\Phi)$ by making a shift $\Phi(x) \to \Phi(x) + \Phi(x \mid \Phi_0)$ (*cf.* also Ref. [1]). This result can be extended to the loop amplitudes (using, for example, path integral representation) [1].

4. Threshold amplitudes

Let us consider the matrix elements (in particular — amplitudes) of the form $\langle \text{out} \mid T(\Phi(x)...) \mid \text{in}\rangle$ where all incoming and outgoing particles have vanishing threemomenta. Then the problem becomes translationally invariant. In particular, $\Phi_0(x) \equiv \Phi_0(t)$ can be chosen as follows

$$\Phi_0(t) = \beta \mathrm{e}^{-imt} + \overline{\beta}\mathrm{e}^{imt} \tag{43}$$

and $\Phi(x \mid \Phi_0) \equiv \Phi(t \mid \Phi_0)$ becomes a function of time only. Eqs. (19), (20) take the form

$$(\partial_t^2 + m^2)\Phi(t \mid \Phi_0) + \frac{\lambda}{3!}\Phi^3(t \mid \Phi_0) = 0\,, \tag{44}$$

$$\Phi(t \mid \Phi_0) \mid_{\lambda=0} = \Phi_0(t) \tag{45}$$

while Eq. (18) is converted into

$$\Phi(t \mid \Phi_0) = \Phi_0(t) - \frac{\lambda}{3!}\int\limits_{-\infty}^{\infty} dt' D_{\mathrm{F}}(t-t')\Phi^3(t' \mid \Phi_0) \tag{46}$$

with

$$D_{\mathrm{F}}(t) \equiv \frac{1}{2\Pi}\int\limits_{-\infty}^{\infty} d\rho \frac{\mathrm{e}^{-ip(t-t')}}{m^2 - p^2 - i\varepsilon}\,. \tag{47}$$

The problem becomes now tractable: Eqs. (44), (45) define one-dimensional anharmonic (quatric) oscillator. The same applies to all matrix elements $\langle \text{out} \mid T(\Phi(x)\ldots) \mid \text{in}\rangle$. In fact, for translational invariant in- and out- states these elements are translational invariant so one can use Fourier transform and reduce the relevant partial differential equations to the ordinary ones. In some cases an explicit solution is then available. Let us note that β and $\overline{\beta}$ in Eq. (43) need not to be complex conjugated; $\Phi_0(t)$ and, consequently, $\Phi(t \mid \Phi_0)$ become then complex. In particular, if we are interested in amplitudes with only initial (final) particles at the rest we can assume $\overline{\beta} = 0$ $(\beta = 0)$. This property is very important because it happens often that the amplitudes with both initial and final particles at threshold are divergent while those with such particles in one state only, initial or final, are finite (see below).

5. All particles at the threshold

We shall first consider the amplitudes for the processes with all, initial as well as final, particles at the threshold. For one component Φ^4-theory this means that we are dealing with $n\Phi \to n\Phi$ amplitudes. To calculate them we have to solve first Eqs. (44), (45), then to compute $\frac{\partial^{2n-1}\Phi(t|\Phi_0)}{\partial\beta^{n-1}\partial\overline{\beta}^n}|_{\beta=\overline{\beta}=0}$ $\left(\text{or } \frac{\partial^{2n-1}\Phi(t|\Phi_0)}{\partial\beta^n\partial\overline{\beta}^{n-1}}\right)$ and finally reduce the remaining field from the matrix element.

According to Eqs. (44), (45) we are, therefore, looking for the solution to the anharmonic oscillator problem with prescribed harmonic limit. Let us write the λ-expansion of $\Phi(t \mid \Phi_0)$

$$\Phi(t \mid \Phi_0) = \sum_{n=0}^{\infty} \Phi_n(t)\lambda^n; \tag{48}$$

$\Phi_n(t)\lambda^n$ is the sum of all graphs of order n in λ with all but one external propagators amputated and replaced by Φ_0. Expansion (48), when inserted into Eq. (44), gives

$$(\partial_t^2 + m^2)\Phi_{n+1}(t) + \frac{1}{3!}\sum_{k+l+m=n} \Phi_k(t)\Phi_l(t)\Phi_m(t) = 0 \tag{49}$$

which allows to solve recusively for $\Phi_n(t)$ once $\Phi_0(t)$ is known.

The frequency corresponding to the unamputated external line is integer (negative or positive) multiple of m. In order to obtain a nonvanishing amplitude after amputating the last remaining external propagator the relevant frequency should be $-m$ (initial line) or m (final line).

In the succesive step of perturbative solution one has to solve harmonic oscillator equation corresponding to a given external force. The first term on the left-hand side of Eq. (49) gives the sum of amputated trees of $n+1$-st order. We see that the amplitudes are nonvanishing in this order if and only if the frequency of external force is $\pm m$ *i.e.* we are faced with resonances.

Let us analyse Eq. (49) in more detail. We assume first that the frequency ω of "external force" differs from $\pm m$. Inserting the expansion (49) into the integral equation (46) we conclude that the particular solution of Eq. (49) we should use is the one obtained by dividing the external force by $m^2 - \omega^2$. Let us note that the choice of $D_{\mathrm{F}}(t)$ is irrelevant as long as $m^2 \neq \omega^2$: any choice of Green function would produce the same result. This property is reflected by the corresponding property of tree graphs — the relevant amplitudes do not depend on the choice of $i\varepsilon$-prescription as long as all internal momenta are off-shell.

The situation changes drastically once the resonances occur. Then, as it is well known, the solution is no longer periodic in time; rather, it has the form of the polynomial of first degree times a periodic function. More generally, (i) if the external force is a polynomial of degree k times periodic function of frequency $\pm m$, the solution is a polynomial of degree $k+1$ times periodic function while (ii) if the force is a polynomial of degree k times periodic function of the frequency different from $\pm m$, the solution is a periodic function times polynomial of the same degree. One can combine this statement with the tree-graph interpretation of Eq. (49); the main point here is that the occurrence of resonance correspond to the divergence of unamputated propagator. Taking all that into account we conclude that following: in each step of pertubative solution of Eq. (49) the degree of the polynomial in t multiplying the term $\mathrm{e}^{ikmt}, k \in Z$, equals the maximal number of on-shell propagators in the tree graphs with unamputated propagator carrying the energy km, contributing to that order. We see that threshold tree amplitudes are in general divergent and this divergence may be related to the structure of solution to the classical dynamical equations for anharmonic oscillator.

It is not difficult to recognize the origin of polynomial terms in perturbative expansion. Assume that β and $\overline{\beta}$ are complex conjugated so that $\Phi_0(t)$ is real and, consequently $\Phi(t \mid \Phi_0)$ also. Obviously $\Phi(t \mid \Phi_0)$ is periodic with the period $\omega(E,\lambda)$ depending in general on energy and coupling constant; the boundary condition is $\omega(E,\lambda=0)=m$. The coordinate $\Phi(t \mid \Phi_0)$ can be developed in Fourier series

$$\Phi(t \mid \Phi_0) = \sum_{k=-\infty}^{\infty} \Phi_k(E,\lambda)\mathrm{e}^{ik\omega(E,\lambda)t} \tag{50}$$

with $\overline{\Phi_k} = \Phi_{-k}$ and $\Phi_k(E,\lambda=0) = \beta\delta_{1k} + \overline{\beta}\delta_{-1k}$. By expanding in coupling constant we get

$$\begin{aligned}\Phi(t \mid \Phi_0) &= \sum_{k=-\infty}^{\infty} \left(\Phi_k^{(0)} + \Phi_k^{(1)}\lambda + \ldots\right) \mathrm{e}^{ikt\left(m + \frac{\partial\omega(E,\lambda)}{\partial\lambda}\Big|_{\lambda=0} \lambda + \ldots\right)} \\ &= \sum_{k=-\infty}^{\infty} \left(\Phi_k^{(0)} + \Phi_k^{(1)}\lambda + \ldots\right) \left(1 + ik\frac{\partial\omega(E,\lambda)}{\partial\lambda}\Big|_{\lambda=0} \lambda\, t + \ldots\right) \mathrm{e}^{ikmt}\end{aligned} \tag{51}$$

The terms linear, quadratic *etc.* in time appear due to the λ-dependence of the frequency of motion (it could *a priori* happen that in order to fulfil the boundary conditions $\Phi(t \mid \Phi_0) \mid_{\lambda=0} = \Phi_0(t)$ one has to take the energy E as λ-dependent; this does not invalidate the arguments).

One can now identify the maximally divergent graphs. If $\frac{\partial \omega(E,\lambda)}{\partial \lambda} \mid_{\lambda=0} \neq 0$, Eq. (51) implies that the degree in t at the most is equal to the order in λ; however, the number of propagators in tree graph of the order n with one external line unamputated is just n (remember we are considering tree graphs). Therefore, all propagators are then singular. Low order graphs of this type are shown in Fig. 1.

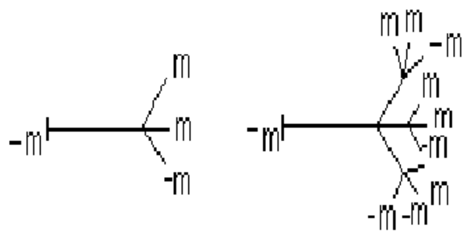

Fig. 1.

Let us now turn back to the threshold amplitudes. In order to produce the nonvanishing contribution, $\Phi(t \mid \Phi_0)$ should develop polynomial (in t) terms in λ expansion which, in turn, implies that the higher order tree amplitudes are divergent. In fact, the nonvanishing and nonsingular amplitude corresponds to the contribution to $\Phi(t \mid \Phi_0)$ with singular unamputated external line propagator and regular all other propagators. This tree graph can be then used as a building block in constructing higher order tree graphs. Therefore, the tree-graph expansion of $\Phi(t \mid \Phi_0)$ is divergent. On the other hand, if all terms in the tree-graphs expansion for $\Phi(t \mid \Phi_0)$ are well-defined, no polynomial terms in t develop which implies that $\omega(E,\lambda)$ is λ-independent so that $\omega(E,\lambda) \equiv m$. Then the recursive solution of Eq. (46) defines the Fourier series for the periodic (with the period m) function $\Phi(t \mid \Phi_0)$. All terms of λ-expansion of $\Phi(t \mid \Phi_0)$ are well defined and the relevant amplitudes must vanish.

Concluding, we are faced with the following dichotomy: either the tree-graph expansion of $\Phi(t \mid \Phi_0)$ is ill-defined or all threshold amplitudes vanish.

One can pose the question when it is possible to find the solutions to the classical equations of motion which are periodic in time with the basic frequency equal to m. Such solution will produce well defined term-by-term tree-graph expansion of $\Phi(t \mid \Phi_0)$. It is quite easy to see that for real motions one can hardly get a reasonable theory. For, if the period of motion, being equal to m, does not depend on energy, one is able to find the general class of potentials by the method explained in [26]; if one assumes the potential to be the polynomial the unique solution is harmonic oscillator of frequency m, *i.e.* free-field theory. For complex motions the situation is more complicated and such solutions are possible for nontrivial potentials [6, 18]. To see this consider once more the expansion for $\Phi(t)$,

$$\Phi(t) = \sum_{k=-\infty}^{\infty} \Phi_k(E, \lambda) e^{ik\omega(E,\lambda)t}; \tag{52}$$

due to $\frac{d^2V(\Phi)}{d\Phi^2} \mid_{\Phi=0} = m^2$ one has $\omega(E = 0, \lambda) = m$. With the reality condition $\overline{\Phi_k(E, \lambda)} = \Phi_{-k}(E, \lambda)$ the only solution with vanishing energy is $\Phi_k(E = 0, \lambda) = 0$. However, if the reality condition is abandoned, one can find nontrivial $\Phi_k(0, \lambda)$. The best examples are the solutions obeying [6]

$$\Phi_{\pm}(t) \mid_{\lambda=0} = \beta e^{\pm imt}. \tag{53}$$

Their tree-graph expansion produce well-defined amplitudes with threshold particles in final (initial) states only.

These results can be generalized to many-component fields. Assume that the reduced theory is an integrable system with r degrees of freedom, defined by the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{r}(\Pi_i^2 + m_i^2\Phi_i^2) + V(\underline{\Phi}; \underline{\lambda}), \tag{54}$$

$$V(\underline{\Phi}; \underline{0}) = 0, \quad V(0; \underline{\lambda}) = \frac{\partial V(\underline{\Phi}, \underline{\lambda})}{\partial \Phi_i}\Big|_{\underline{\Phi}=0} = \frac{\partial^2 V(\Phi; \lambda)}{\partial \Phi_i \partial \Phi_j}\Big|_{\underline{\Phi}=0} = 0.$$

Let (J_K, Θ_K), $k = 1, ..., r$, be the action-angle variables and assume that $\underline{J} = 0$ corresponds to the stationary point $\Pi_i = 0, \Phi_i = 0$, $i = 1, ..., r$. Any component Φ_i can be expanded in multiple Fourier serie

$$\Phi_i(t) = \sum_{n_1,..,n_r} \Phi_{i,n_1,...,n_r}(\underline{J}, \underline{\lambda}) e^{i\sum_{k=1}^{r} n_k\omega_k(\underline{J},\underline{\lambda})t}. \tag{55}$$

Our assumptions imply $\omega_k(\underline{0}; \underline{\lambda}) = m_k$. Again, skipping reality conditions one can often find nontrivial $\Phi_{i,n_1,...,n_r}(\underline{0}; \underline{\lambda})$. Generically, they correspond to the boundary conditions $\Phi_i(t) \mid_{\underline{\lambda}=0} = \beta_i e^{i\varepsilon_i m_i t}, \varepsilon_i = \pm 1$; these are simply solutions with $J_K = 0$ in the limit $\underline{\lambda} = 0$. In this way one obtains the tree approximation to the amplitudes with some kind of particles at the threshold in the initial state and other kind in the final one. According to the general arguments above, all amplitudes of these types vanish.

As an example consider O(2)-theory with the symmetry softly broken by a mass term [18],

$$L = \frac{1}{2}(\partial_\mu\Phi\partial^\mu\Phi + \partial_\mu\chi\partial^\mu\chi) - \frac{m_1^2}{2}\Phi^2 - \frac{m_2^2}{2}\chi^2 - \lambda(\Phi^2 + \chi^2)^2. \tag{56}$$

Assume the $\Phi \to -\Phi$ symmetry is spontaneously broken, $m_1^2 < 0, \;\; m_2^2 > 0$. The relevant solutions read

$$\begin{aligned}\Phi \;=\; \Phi_0 &\left(1+\frac{z_1}{2\Phi_0}+\frac{2\lambda}{4m_\chi^2-m_\Phi^2}z_2^2+\frac{\lambda}{\Phi_0}\frac{2m_\chi-m_\Phi}{(2m_\chi+m_\Phi)^3}z_1z_2^2\right)\\ &\times\left(1-\frac{z_1}{2\Phi_0}-\frac{2\lambda}{4m_\chi^2-m_\Phi^2}z_2^2+\frac{\lambda}{\Phi_0}\frac{2m_\chi-m_\Phi}{(2m_\chi+m_\Phi)^3}z_1z_2^2\right)^{-1}, \quad (57)\end{aligned}$$

$$\begin{aligned}\chi \;=\; z_2 &\left(1-(\frac{2m_\chi-m_\Phi}{2m_\chi+m_\Phi})\frac{z_1}{2\Phi_0}\right)\\ &\times\left(1-\frac{z_1}{2\Phi_0}-\frac{2\lambda}{4m_\chi^2-m_\Phi^2}z_2^2+\frac{\lambda}{\Phi_0}\frac{2m_\chi-m_\Phi}{(2m_\chi+m_\Phi)^3}z_1z_2^2\right)^{-1};\end{aligned}$$

here $z_1 = \beta_1 \mathrm{e}^{im_\Phi t}, z_2 = \beta_2 \mathrm{e}^{im_\chi t}, \Phi_0 \equiv <\Phi>$ and $m_\Phi \equiv \sqrt{2} \mid m_1 \mid, m_\chi \equiv \sqrt{\mid m_1^2 \mid + m_2^2}$ are physical masses. These solutions can serve to compute the amplitudes $\Phi \to n_1\Phi + n_2\chi$ or $\chi \to n_1\Phi + n_2\chi$ where the initial particle is off-shell while the final ones — at the threshold. Now, it is easy to check that the on-shell amplitude $\Phi \to 2\chi$ does not vanish provided $m_\Phi = 2m_\chi$. Then, according to Eq. (57) $\Phi(t)$ and $\chi(t)$ are divergent. On the other hand, if $m_\Phi \neq 2m_\chi$, all amplitudes with on-shell initial particle vanish while $\Phi(t)$ and $\chi(t)$ are regular.

Appendix A

The equation defining Green function reads [8]

$$[G_0^{-1} - V]G = 1 \tag{A.1}$$

with the formal solution

$$G = \frac{1}{1 - G_0 V} G_0 \,. \tag{A.2}$$

Eq. (36) defines the wave function

$$\Psi = G G_0^{-1} \Psi_0 \quad (\Psi_0 \equiv g) \tag{A.3}$$

which, together with $(A.2)$, gives

$$\Psi = \frac{1}{1 - G_0 V} \Psi_0 \tag{A.4}$$

or

$$\Psi = \Psi_0 + G_0 V \Psi \,. \tag{A.5}$$

Eq.(A.5) coincides with Eq. (2.22) of Ref. [8].

It is also easy to see that the recipe for calculating the amplitudes given below Eq. (36) coincides in turn with Eq. (2.24) from Ref. [8].

REFERENCES

[1] For a reviev see: M. Libanov, V. Rubakov, S. Troitsky, *Part. Nucl.* **28**, No3, 551 (1997), in Russian.

[2] O. Espinosa, *Nucl. Phys.* **B343**, 310 (1990); A. Ringwald, *Nucl. Phys.* **B330**, 1 (1990); M. Mattis, *Phys. Rep.* **214**, 159 (1992); P. Tinyakov, *Int. J. Mod. Phys.* **A8**, 1823 (1993); V. Rubakov, M. Shaposhnikov, *Usp. Fiz. Nauk* **166**, 493 (1996); J. Cornwall, *Phys. Lett.* **B243**, 271 (1990).

[3] H. Goldberg, *Phys. Lett.* **B246**, 445 (1990).

[4] M. Voloshin, *Nucl. Phys.* **B383**, 233 (1992).

[5] E. Argyres, R. Kleiss, C. Papadopoulos, *Nucl. Phys.* **B391**, 42 and 57 (1993).

[6] L. Brown, *Phys. Rev.* **D46**, 4125 (1992).

[7] B. Smith, *Phys. Rev.* **D47**, 3521 (1993).

[8] L. Brown, Cheng-Xing Zhai, *Phys. Rev.* **D47**, 5526 (1993).

[9] M. Libanov, V. Rubakov, S. Troitsky, *Nucl. Phys.* **B412**, 607 (1994).

[10] E. Argyres, R. Kleiss, C. Papadopoulos, *Nucl. Phys.* **B395**, 3 (1993).

[11] M. Voloshin, *Phys. Lett.* **B293**, 389 (1992); V. Zakharov, *Phys. Rev. Lett.* **67**, 3650 (1991); E. Argyres, R. Kleiss, C. Papadopoulos, *Phys. Lett.* **B296**, 139 (1992); A. Gorsky, M. Voloshin, *Phys. Rev.* **D48**, 3843 (1993); R. Mawhinney, R. Willey, *Phys. Rev. Lett.* **74**, 3728 (1995).

[12] M. Voloshin, *Phys. Rev.* **D47** (1993), 357.

[13] B. Smith, *Phys. Rev.* **D47**, 3518 (1993).

[14] M. Voloshin, *Phys. Rev.* **D47**, 2573 (1993); B. Smith, *Phys. Rev.* **D49**, 1081 (1994); E. Agyres, R. Kleiss, C. Papadopoulos, *Phys. Lett.* **B308**, 315 (1993).

[15] See first reference in [14].

[16] B. Smith, *Phys. Rev.* **D47**, 5531 (1993); M. Libanov, D. Son, S. Troitsky, *Phys. Rev.* **D52**, 3679 (1995).

[17] Yu. Makeenko, *Phys. Rev.* **D50**, 4137 (1994).

[18] M. Libanov, V. Rubakov, S. Troitsky, INR preprint INR 823/93; M. Libanov, V. Rubakov, S. Troitsky, *Phys. Lett.* **B318**, 134 (1993).

[19] A. Perelemov, *Integrable Systems of Classical Mechanics and Lie Algebras*, Nauka, Moscow 1980; R. Sahadevan, M. Lakshmanan, *Phys. Rev.* **A31**, 861 (1985); R. Sahadevan, M. Lakshmanan, *J. Phys.* **A19**, L949 (1986).

[20] A. Gorsky, K. Selivanov, *Mod. Phys. Lett.* **A11**, 1597 (1996).

[21] J. Gonera, *Phys. Lett.* **B508**, 191 (2001).

[22] Y. Nambu, *Phys. Lett.* **26B**, 626 (1968); D.G. Boulware, L.S. Brown, *Phys. Rev.* **172**, 1628 (1968).

[23] J. Arefeva, A. Slavnov, L. Faddeev, *Theor. Mat. Fiz.* **21**, 311 (1974).

[24] F. Brandt, J. Frenkel, *Phys. Rev.* **D53**, 2256 (1996).

[25] J. Białynicki-Birula, Z. Białynicka-Birula, *Quantum Electrodynamics*, PWN, Warsaw 1969 (in Polish).

[26] L. Landau, E. Lifshic, *Mechanics*, PWN, Warsaw 1966, in Polish.