INFLUENCE OF THE GRAVITATIONAL LENSING EFFECT ON DISTANCE DETERMINATION

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In this paper we have estimated correction to the distance modulus, derived from the generalized Dyer–Roeder equation, due to a single clump of matter inducing gravitational lensing effect. To describe the influence of the gravitational lensing, we have used the "Swiss cheese" model and the Jacobi equation. In the case, when the source is at redshift $z = 1$ and the lensing object is a galaxy modeled by the Singular Isothermal Sphere (SIS) with mass $M = 10^{11} \times M_{\odot}$, we have found that the correction is about 0.01 mag. We have also show relations between the described approach and the convenient approach using the gravitational lens equation. In particular, we have derived the lens equation from the geodesic deviation equation and showed that the obtained dependence of the magnification factor on the impact parameter is well approximated (with accuracy $\sim 0.2\%$) by a function obtained from the lens equation for the SIS.

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1. Introduction

To determine cosmological parameters and evolution of the universe, it has been customary to compare observations of extragalactic objects with the predictions of the spatially homogeneous and isotropic Friedman–Robertson–Walker (FRW) cosmological models. However, the assumption, that the matter distribution is homogeneous and isotropic, seems to be reasonable only for the largest distance scale (larger than 300 Mpc), on smaller scale we observe many inhomogeneities such as clusters of galaxies and galaxies. It is interesting to study the effect of a locally inhomogeneous mass distribution on the optical properties of distant objects, in a universe which is homogeneous and isotropic on sufficiently large scale.

A beam of light propagating outside the mass clumps in such a universe has less mass within the beam, so it diverges faster than it would in the homogeneous universe of the same mean density. A discussion of this effect was first published by Zel'dovich [1] and later taken up by Dashevskii and Zel'dovich [2] and Dashevskii and Slysh [3]. If the beam passes close to any clumps, it may be intensified and sheared by the gravitational lens effect of the clumps. These two effects were discussed by Gunn [4]. All of them used models of the inhomogeneous universe, which are approximate solutions of the field equations. The first application of an exact solution of the field equations to this problem was done by Kantowski [5]. He considered the "Swiss cheese" inhomogeneous model.

Recent results [6–8] of two projects: the Supernova Cosmology Project (SCP) and High-Z supernova search team (HIZ), which main goals are observations of high-redshift supernovae and determination of cosmological parameters through the distance-redshift relation for type Ia Supernova (SN Ia), indicate that the major energy density in the universe must be of the "vacuum" type related to a non-zero value of the cosmological constant. To check these results and determine cosmological parameters with higher precision we should study changes in the distance measurements due to the effect of the gravitational lensing.

In this paper we estimate correction to the distance modulus obtained from the Dyer–Roeder equation caused by a single gravitational lens. To describe the spacetime containing a gravitational lens we use the "Swiss cheese" model. All interesting us properties of the light beam we derive from the geodesic deviation equation. This approach is applied to the observed source at redshift $z = 1$ lensed by a galaxy modeled by the SIS with mass $M = 10^{11} \times M_{\odot}$. Obtained dependence of the magnification factor on the impact parameter is compared with the same dependence derived from the lens equation for the SIS. We also show how our approach relates with the gravitational lens equation used in the conventional approach.

The paper is organized as follow. Sec. 2 contains basic definitions and relations between different kinds of distances used in cosmology. A brief review of the FRW model, the distance-redshift relation in this model and the Dyer–Roeder equation are presented in Sec. 3. The "Swiss cheese" model is described in Sec. 4 and the SIS in Sec. 5. The procedure of calculating the correction is presented in Sec. 6. In Sec. 7 the derivation of the lens equation from the deviation equation is showed. Results of calculations are in Sec. 8 and 9.

2. Luminosity and angular-diameter distances

In cosmology distance is not directly measurable quantity. Thus there is no unique definition of the distance. Two typical ones which are frequently used are the luminosity and angular-diameter distances.

If one knows the source luminosity L and observes the flux S, one can determine the luminosity distance D_L using relation

$$S = \frac{L}{4\pi D_L^2} \ . \tag{1}$$

The angular-diameter distance is defined as

$$D_A := \sqrt{\frac{dA_{\mathrm{S}}}{d\Omega_{\mathrm{O}}}} \ , \tag{2}$$

where $d\Omega_{\mathrm{O}}$ is the solid angle at the observation point subtended by the surface element dA_{S}.

Similarly, interchanging the roles of source and observer, one defines, so called, the "corrected" luminosity distance D'_L

$$D'_L := \sqrt{\frac{dA_{\mathrm{O}}}{d\Omega_{\mathrm{S}}}} \ , \tag{3}$$

which connects with the luminosity distance by relation $D_L = (1+z)D'_L$, where z is the redshift of the source.

It is known that the "corrected" luminosity distance and the angular-diameter distance are not mutually independent but they are related by the Etherington's reciprocity relation [9] $D'_L = (1+z)D_A$, so one has

$$D_L = (1+z)^2 D_A \ . \tag{4}$$

The reciprocity relation means that study of a future-directed null geodesic congruence, which starts expanding from the source S to the observer O (its Jacobi vector vanish at S), is equivalent to study of a past-directed null geodesic congruence, which starts expanding from the observer O to the source S (its Jacobi vector vanish at O). The latter approach is more conveniently for observer, so we will use it.

3. The distance-redshift relations in homogeneous and inhomogeneous models of universe

3.1. The distance-redshift relation in the FRW models

Geometry of a homogeneous and isotropic universe is described by the FRW metric

$$ds^2 = c^2dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right), \tag{5}$$

where c is the speed of light, t is a time coordinate, (r,θ,φ) — spatial coordinates, $R(t)$ — a scale factor, $k = 0, +1, -1$ depend on, respectively, zero, positive or negative curvature of space.

After inserting this metric in the Einstein's equation one obtains the Friedman's equation

$$\left(\frac{\dot{R}(t)}{R(t)}\right)^2 = \frac{8\pi G}{3}\varrho - \frac{kc^2}{R(t)^2} + \frac{1}{3}\Lambda c^2\ , \tag{6}$$

where $\dot{} \equiv \frac{d}{dt}$, G is the gravitational constant, $\varrho = \varrho(t)$ is density of matter, Λ is the cosmological constant.

The above equation can be written in terms of four parameters[1]: the Hubble constant $H(t) := \frac{\dot{R}(t)}{R(t)}$ and density parameters $\Omega_m := \frac{\varrho_{m0}}{\varrho_c}$, $\Omega_k := \frac{-kc^2}{R_0^2H_0^2}$ and $\Omega_\Lambda := \frac{\varrho_\Lambda}{\varrho_c}$, where ϱ_{m0} is matter density at the present time[2], $\varrho_\Lambda := \frac{\Lambda c^2}{8\pi G}$ density of "matter" connected with the cosmological constant, $\varrho_c = \frac{3H_0^2}{8\pi G}$ the critical density, then

$$H^2(z) = H_0^2\left(\Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda\right), \tag{7}$$

where $z = \frac{R_0}{R} - 1$ is the redshift. Using the relation $\Omega_m + \Omega_k + \Omega_\Lambda = 1$ we obtain

$$H^2(z) = H_0^2\left((1+z)^2(1+\Omega_m z) - z(z+2)\Omega_\Lambda\right). \tag{8}$$

As we see the three parameters H_0, Ω_m, Ω_Λ uniquely determine evolution of the universe.

Values of the cosmological parameters may be determined through the distance-redshift relation. In the FRW dust models the luminosity distance D_L is a function of redshift and cosmological parameters and is given by [10]:

[1] We neglect a radiation energy density.

[2] Index 0 means evaluated at the present age of the universe.

$$D_L(z; H_0, \Omega_m, \Omega_\Lambda) = \frac{(1+z)c}{H_0\sqrt{|\Omega_k|}} S\left(\sqrt{|\Omega_k|}\int_0^z \frac{dz'}{\sqrt{(1+z')^2(1+\Omega_m z') - z'(z'+2)\Omega_\Lambda}}\right), \quad (9)$$

where $S(x)$ denotes $\sin(x)$ for $\Omega_k < 0$ ($k = +1$), x for $\Omega_k = 0$ ($k = 0$), or $\sinh(x)$ for $\Omega_k > 0$ ($k = -1$). When $\Omega_\Lambda = 0$ it simplifies to the Mattig's formula [11]

$$D_L(z; H_0, \Omega_m) = \frac{2c}{H_0\Omega_m^2}\left[\Omega_m z + (\Omega_m - 2)\left(\sqrt{1+\Omega_m z} - 1\right)\right]. \quad (10)$$

3.2. The Dyer–Roeder equation

Although the universe may be approximated by the FRW model on average, observed distribution of matter is far from isotropic and homogeneous. A large part of the matter is concentrated in galaxies, galaxies tend to form groups and clusters, clusters form super-clusters. In such an universe the light may travel far away from all clumps through the intergalactic space, which is nearly vacuum. In this situation it is said, that the light propagates through an *empty cone* in contrast to a *filled cone*, when the light propagates through homogeneous and isotropic background with the uniform matter density ρ. For these two cases the distance measures may be completely different, because for *empty cone* the light beam diverges faster than for *filled cone*.

To take into account a non homogeneous distribution of matter, one can assume, following Dyer–Roeder [12], that a mass-fraction $\tilde{\alpha}$, called the *smoothness parameter* or *clumpiness parameter*, of matter in the universe is smoothly distributed, while a fraction $1 - \tilde{\alpha}$ is bound in galaxies. The case $\tilde{\alpha} = 0$ corresponds to the vacuum in the intergalactic space, while $\tilde{\alpha} = 1$ to the uniform FRW model. In such an universe the largest possible (for given redshift) angular-diameter distance D_A is determined from the generalized Dyer–Roeder equation [12]

$$(1+z)(1+\Omega_m z)\frac{d^2 D_A}{dz^2} + \left(3 + \frac{1}{2}\Omega_m + \frac{7}{2}\Omega_m z\right)\frac{dD_A}{dz} + \frac{3}{2}\tilde{\alpha}\Omega_m D_A = 0\,, \quad (11)$$

with the initial conditions $D_A(z)_{|z=0} = 0$, $dD_A(z)/dz_{|z=0} = c/H_0$, so it is often called the *Dyer–Roeder distance*. As we see the above equation does not take into account the cosmological constant Λ. The generalized Dyer–Roeder equation with Λ takes the form [13]:

$$(1+z)\left(\Omega_m(1+z)^3+\Omega_k(1+z)^2+\Omega_\Lambda\right)\frac{d^2D_A}{dz^2}$$
$$+\left(\frac{7}{2}\Omega_m(1+z)^3+3\Omega_k(1+z)^2+2\Omega_\Lambda\right)\frac{dD_A}{dz}+\frac{3}{2}\tilde{\alpha}\Omega_m(1+z)^2D_A=0\,. \quad (12)$$

The Dyer–Roeder distance will be our reference distance in the calculation of the magnification factor (Sec. 6.2).

4. The "Swiss cheese" model

The "Swiss cheese" model, also called the Einstein–Strauss model [14], describes an inhomogeneous universe. The name "Swiss cheese" refers to the fact that in this model static spherical voids are created within a large, time-dependent spacetime. A void is constructed by removing homogeneously distributed matter from a sphere and replacing it by a compact object with the same mass, placed at the centre of the sphere.

Mathematically, the "Swiss cheese" model is realized by matching of a FRW metric describing the exterior spacetime

$$ds_f^2=c^2dt^2-R^2(t)\left(\frac{dr^2}{1-kr^2}+r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right)\right), \quad (13)$$

to a Schwarzschild metric with the cosmological constant describing the void

$$ds_s^2=c^2\left(1-\frac{r_g}{r_s}-\frac{\Lambda}{3}r_s^2\right)dT^2-\frac{dr_s^2}{1-\frac{r_g}{r_s}-\frac{\Lambda}{3}r_s^2}-r_s^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right), \quad (14)$$

where[3] r_g denotes the Schwarzschild's radius ($r_g=2GM/c^2$, M mass of the central object), across a spherical boundary. The spherical boundary Σ stays at a fixed coordinate radius in the FRW frame (r_Σ =const.), but changes with time in the Schwarzschild frame. The smooth matching of the two spacetimes across a boundary is guaranteed if junction conditions are satisfied: the first fundamental forms and the second fundamental forms are identical on both sides of the hypersurface. From these conditions we obtain

$$R(t)r_\Sigma=r_{s|_{\mathrm{at}\ \Sigma}}, \quad (15)$$

$$\frac{dT}{dt}=\sqrt{1-kr_\Sigma^2}\left(1-\frac{r_g}{r_s}-\frac{\Lambda}{3}r_s^2\right)^{-1}\Bigg|_{\mathrm{at}\ \Sigma}. \quad (16)$$

[3] Indices f and s refer to, respectively, quantities in the FRW metric and the Schwarzschild metric.

The above equation in terms of the cosmological parameters and the redshift z takes on the form

$$\frac{dT}{dz} = -\frac{(1+z)c^2\sqrt{1-kr_\Sigma^2}}{H(z)\left[(1+z)^2c^2 - H_0^2(R_0\,r_\Sigma)^2(\Omega_m(1+z)^3+\Omega_\Lambda)\right]}\,, \tag{17}$$

where $H(z) = H_0\sqrt{\Omega_m(1+z)^3+\Omega_k(1+z)^2+\Omega_\Lambda}$.

Radius $R_s = R_0\,r_\Sigma$ of the void described by the Schwarzschild metric can be calculated from the condition that the average matter density in the void should be equal to the average matter density of the universe $\Omega_m\varrho_c$. It means that $M = (4\pi/3)\,\Omega_m\varrho_c R_s^3$, what could be rewritten in the form

$$MG = \tfrac{1}{2}\Omega_m H_0^2 R_s^3\,. \tag{18}$$

In contrast to the "pure" "Swiss cheese" model, where outside the sphere there are homogeneously and isotropicly distributed matter with density ϱ_m, we assume that outside the sphere there are matter with uniform density $\tilde{\alpha}\varrho_m$ as in the Dyer–Roeder approach presented in the previous section.

5. Model of the singular isothermal sphere

In our case a compact object in the center of the sphere models a galaxy. On the other hand observations of galaxies indicate that they are surrounded by dark matter halo in ranges much larger than a visible part of galaxy, so treating them as a point mass is too crude. We must consider more realistic model of the matter distribution.

A simple model for the density profile ρ of a galaxy is the Singular Isothermal Sphere (SIS)

$$\rho_{\mathrm{SIS}}(r) = \frac{\sigma_\nu^2}{2\pi G r^2}\,, \tag{19}$$

where σ_ν is the line-of-sight velocity dispersion of the mass particles. Mass conservation implies that the velocity dispersion is related to the mass of the halo M and to the redshift by [15]

$$\sigma_\nu = M^{1/3}\left[H^2(z)\Delta(z)G^2/16\right]^{1/6}, \tag{20}$$

where $\Delta(z)$ is the mean density of the halo in units of the critical density at that redshift and, for a flat universe ($\Omega_k = 0$), it may be approximated by [15] $\Delta(z) = 18\pi^2 + 82x - 39x^2$ where $x = H_0^2\Omega_m(1+z)^3/H^2(z) - 1$.

A SIS halo is truncated at radius d given by relation

$$M = \int_0^d \rho_{\rm SIS}(r) dV = 2\frac{\sigma_\nu^2 d}{G}\,, \tag{21}$$

where M is mass of the halo.

The gravitational field inside the SIS is static and spherically symmetric so it can be described by the line element

$$ds^2_{\rm SIS} = c^2 {\rm e}^{2A(r)} dT^2 - {\rm e}^{2B(r)} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{22}$$

where coordinates T, r, θ, φ are the same as in the Schwarzschild metric introduced in previous section and functions $A(r), B(r)$ are given by equations

$$\begin{aligned} {\rm e}^{-2B(r)} &= 1 - \frac{1}{3}\Lambda r^2 - \frac{8\pi G}{c^2 r}\int_0^r \rho_{\rm SIS}(r') r'^2 dr'\,, & (23)\\ 2A(r) &= \int_0^r \left(\left(\frac{8\pi G p}{c^4} - \Lambda\right) r' {\rm e}^{2B(r')} + \frac{{\rm e}^{2B(r')} - 1}{r'} \right) dr'\,. & (24) \end{aligned}$$

Taking the equation of state for the dust $p = 0$ and the density profile of the SIS we have ${\rm e}^{-2B(r)} = 1 - \frac{1}{3}\Lambda r^2 - \frac{4\sigma_\nu^2}{c^2}$, so

$$2A(r) = \int_0^r \frac{-\frac{2}{3}\Lambda r'^2 + \frac{4\sigma_\nu^2}{c^2}}{r'\left(1 - \frac{1}{3}\Lambda r'^2 - \frac{4\sigma_\nu^2}{c^2}\right)} dr'\,. \tag{25}$$

Because above integral is divergent at zero, we introduce some cutoff radius $r_{\rm cutoff} \ll d$ below which we assume the constant density profile $\rho_{\rm cutoff}$, such that $M(r_{\rm cutoff}) = 4\pi \int_0^{r_{\rm cutoff}} \rho_{\rm SIS}(r) r^2 dr = 4\pi \int_0^{r_{\rm cutoff}} \rho_{\rm cutoff} r^2 dr$. Then we have $\rho_{\rm cutoff} = \frac{3\sigma_\nu^2}{2\pi G r^2_{\rm cutoff}}$ and convergent integral (24) at zero.

6. A procedure for calculating the correction

6.1. The Jacobi equation

Let us consider an infinitesimal beam of light propagating from the source to the observer, described by a congruence of null geodesics

$$k_\mu k^\mu = 0\,, \quad \frac{Dk^\mu}{d\lambda} \equiv k^\mu_{;\nu} k^\nu = 0\,, \tag{26}$$

where[4] $D/d\lambda$ denotes covariant differentiation, λ is affine parameter, and k^μ is a wave vector tangent to the ray. A vector $X^\mu(\lambda)$, called the deviation vector or the Jacobi vector, connects "central" light ray to one of its neighbors, which belongs to the same congruence. It has the property that

$$\frac{DX^\mu(\lambda)}{d\lambda} \equiv X^\mu_{;\nu}k^\nu = k^\mu_{;\nu}X^\nu \ . \tag{27}$$

After differentiation of the above equation we obtain the geodesic deviation equation also called the Jacobi equation

$$\frac{D^2X^\mu(\lambda)}{d\lambda^2} = -R^\mu{}_{\nu\rho\sigma}k^\nu X^\rho(\lambda)k^\sigma \ , \tag{28}$$

where $R^\mu{}_{\nu\rho\sigma}$ is the Riemann tensor[5]. As we see, the Jacobi equation describes all gravitational focusing and shearing effects on an infinitesimal beam of light rays, so we will use it to study the influence of inhomogeneities and curvature of spacetime on the observed properties of light.

We choose the affine parameter λ so that:

- $\lambda = 0$ at the point of observation,
- λ increases from the observer to the source,
- for the observer with the 4-velocity u^μ, $k_\mu u^\mu = c\,(1+z)$.

With this choice, the affine parameter is related to the redshift by

$$\frac{dz}{d\lambda} = H_0(1+z)^2\sqrt{(1+z)^2(1+\Omega_m z) - z(z+2)\Omega_\Lambda} \ . \tag{29}$$

In the case of null geodesics the Jacobi equation can be rewritten in a more convenient form for calculations. One can always choose a deviation vector X^μ such that, besides $k_\mu X^\mu = 0$, also $X_\mu u^\mu = 0$ at the point of observation. The vector X^μ chosen in this manner spans the two-dimensional, space-like, subspace orthogonal to the ray, which we call a screen. The Jacobi vector can then be written as

$$X^\mu = X_1 e_1^\mu + X_2 e_2^\mu + X_3 k^\mu \ , \tag{30}$$

where e_1^μ, e_2^μ are vectors spanning an orthonormal basis on the screen

$$k_\mu e_1^\mu = k_\mu e_2^\mu = u_\mu e_1^\mu = u_\mu e_2^\mu = e_1^\mu e_{2\,\mu} = 0 \ , \tag{31a}$$

$$e_1^\mu e_{1\,\mu} = e_2^\mu e_{2\,\mu} = -1 \ . \tag{31b}$$

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[4] Greek indices run from 0 to 3.

[5] $R^\mu{}_{\nu\rho\sigma} \equiv \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma}$.

e_1^μ and e_2^μ are parallel transported along the ray

$$e^{\mu}_{1\,;\nu}k^\nu = 0\ , \quad e^{\mu}_{2\,;\nu}k^\nu = 0\ . \tag{32}$$

Using the fact, that deviation vectors differing by a constant multiple of k^μ represent displacements to the same nearby ray, we choose $X_3 = 0$ so that

$$X^\mu = X_1 e_1^\mu + X_2 e_2^\mu\ . \tag{33}$$

The equation

$$\frac{DX^\mu(\lambda)}{d\lambda} = k^{\mu}_{;\,\nu} X^\nu(\lambda)\ , \tag{34}$$

in terms of the screen components X_1 and X_2 takes on the form

$$\dot{X}_i(\lambda) = S_{ij}X_j(\lambda), \qquad S_{ij} = -e_i^\mu\, k_{\mu;\nu}\, e_j^\nu\ , \qquad i,j = 1,2 \tag{35}$$

where $\dot{} \equiv \frac{d}{d\lambda}$. In the matrix notation we have

$$\dot{\boldsymbol{X}}(\lambda) = \mathcal{S}\boldsymbol{X}(\lambda)\ , \tag{36}$$

where the matrix $\mathcal{S}$ after introducing quantities, which more conveniently describe changes in the cross-section of the beam, such as: the expansion parameter $\theta \equiv \frac{1}{2}\, k^{\mu}_{;\,\mu}$, which describes the rate of expansion, the shear $\sigma \equiv \frac{1}{2}\, k_{\mu;\nu}\varepsilon^{*\mu}\varepsilon^{*\nu}$, where $\varepsilon^\mu = e_1^\mu + ie_2^\mu$, describing the rate of distortion the shape of the cross-section, takes on the form

$$\mathcal{S} = \begin{pmatrix} \theta - \mathrm{Re}\,\sigma & \mathrm{Im}\,\sigma \\ \mathrm{Im}\,\sigma & \theta + \mathrm{Re}\,\sigma \end{pmatrix}. \tag{37}$$

Differentiation of (36) gives

$$\ddot{\boldsymbol{X}}(\lambda) = \mathcal{T}\boldsymbol{X}(\lambda)\ , \tag{38}$$

where

$$\mathcal{T} = \dot{\mathcal{S}} + \mathcal{S}^2\ , \tag{39}$$

and after combining the last equation with Sachs' equations for θ and σ [16]

$$\dot\theta + \theta^2 + |\sigma|^2 = -\tfrac{1}{2}R_{\mu\nu}k^\mu k^\nu\ , \tag{40a}$$

$$\dot\sigma + 2\theta\sigma = -\tfrac{1}{2}C_{\mu\nu\rho\sigma}\varepsilon^{*\mu}k^\nu\varepsilon^{*\rho}k^\sigma\ , \tag{40b}$$

we obtain

$$\mathcal{T} = \begin{pmatrix} \mathcal{R} - \mathrm{Re}\,\mathcal{F} & \mathrm{Im}\,\mathcal{F} \\ \mathrm{Im}\,\mathcal{F} & \mathcal{R} + \mathrm{Re}\,\mathcal{F} \end{pmatrix}\ , \tag{41}$$

where $\mathcal{R} \equiv -\frac{1}{2}R_{\mu\nu}k^{\mu}k^{\nu}$, $\mathcal{F} \equiv -\frac{1}{2}C_{\mu\nu\rho\sigma}\varepsilon^{*\mu}k^{\nu}\varepsilon^{*\rho}k^{\sigma} = F\mathrm{e}^{i\beta}$ ($R_{\mu\nu}$ — the Ricci tensor, $C_{\mu\nu\rho\sigma}$ — the Weyl tensor).

In this manner we obtain an equation, which is equivalent to the geodesic deviation equation (28). As we see one may distinguish two forms of matter influence on the beam: the Ricci focusing (described by the quantity $\mathcal{R}$), which is due to the matter contained in the beam and the Weyl focusing (described by the quantity $\mathcal{F}$), which is due to tidal effects produced by distant clumps of matter.

Because the phase of the Weyl term β, is constant along the beam in a spherically symmetric field, we can set it equal to zero. Thus we have $\mathrm{Im}\,\mathcal{F} = 0$. In the FRW metric $\mathcal{R} = -4\pi G\varrho_{\mathrm{m}0}(1+z_l)^2$ and $C_{\mu\nu\rho\sigma} = 0$, so $\mathcal{F} = 0$. In the Schwarzschild metric $\mathcal{R} = 0$ ($R_{\mu\nu} = 0$) and $\mathcal{F} = \frac{3b^2c^2(1+z_l)^2r_g}{2r^5}$. In the SIS

$$\mathcal{R} = -4\pi G(1+z_l)^2\mathrm{e}^{-2A(r)}\left(\rho_{\,\mathrm{SIS}}(r) + \frac{p}{c^2}\right),$$

and

$$\mathcal{F} = \frac{4\pi Gb^2(1+z_l)^2}{r^2}\left(\frac{3}{r^3}\int\limits_0^r \rho_{\,\mathrm{SIS}}(r')r'^2dr' - \rho_{\,\mathrm{SIS}}(r)\right) = \frac{4\sigma_\nu^2 b^2(1+z_l)^2}{r^4},$$

where b is the impact parameter and the last equation is valid for $r > r_{\mathrm{cutoff}}$ — the cutoff radius introduced in Sec. 5.

To solve equation (38) it is necessary to specify initial conditions $\boldsymbol{X}(0)$ and $\dot{\boldsymbol{X}}(0)$. If we assume that we knew the Jacobi vectors at the observation point $\boldsymbol{X}(0)$ and at the point of the source $\boldsymbol{X}(\lambda_s)$, we may estimate $\dot{\boldsymbol{X}}(0)$ from the integrated form of equation (38) [17]

$$\begin{aligned}\dot{\boldsymbol{X}}(0) &= \frac{1}{\lambda_s}\left(\boldsymbol{X}(\lambda_s) - \boldsymbol{X}(0)\right) - \frac{1}{\lambda_s}\int\limits_0^{\lambda_s}(\lambda_s - \lambda)\mathcal{T}(\lambda)\boldsymbol{X}(\lambda)d\lambda \qquad (42)\\ &\approx \frac{1}{\lambda_s}\left(\boldsymbol{X}(\lambda_s) - \boldsymbol{X}(0)\right)\\ &\quad -\frac{1}{\lambda_s^2}\int\limits_0^{\lambda_s}(\lambda_s - \lambda)\mathcal{T}(\lambda)\left((\lambda_s - \lambda)\boldsymbol{X}(0) + \lambda\boldsymbol{X}(\lambda_s)\right)d\lambda\,,\end{aligned}$$

where λ_s is the value of affine parameter at the point of the source.

6.2. The magnification factor

Gravitational lenses can magnify images of distant objects. This effect can be described by the so-called magnification factor μ [18], such that

$$\mu = \frac{S_l}{S} \,, \tag{43}$$

where S_l is the observed flux changed by gravitational lensing, and S is the flux that would be received if the same source was observed through an empty cone. Using the relation between the luminosity distance and the observed flux (1) and equation (4), we obtain

$$\mu = \frac{\bar{D}_A^2}{D_A^2} \,, \tag{44}$$

where $\bar{D}_A$ is the Dyer–Roeder distance, relative to which is determined the magnification, and D_A is the angular-diameter distance measured on the basis of flux S_l. The distance D_A could be computed from the definition (2), where one takes $dA_S = X_1(\lambda_s)X_2(\lambda_s)$ and $d\Omega_{\mathrm{O}} = \frac{dX_1(0)}{cd\lambda}\,\frac{dX_2(0)}{cd\lambda}$, then one has

$$D_A^2 = c^2 \frac{X_1(\lambda_s)X_2(\lambda_s)}{\dot{X}_1(0)\dot{X}_2(0)} \,, \tag{45}$$

where $X_1(\lambda_s), X_2(\lambda_s), \dot{X}_1(0), \dot{X}_2(0)$ are coordinates of the Jacobi vector in the universe described by the Einstein–Strauss model.

The magnification μ_e of an extended source, considered as an assembly of radiating point sources with surface brightness profile $I(\boldsymbol{y})$, is given generally as

$$\mu_e = \frac{\iint I(\boldsymbol{y})\mu_p(\boldsymbol{y})d^2y}{\iint I(\boldsymbol{y})d^2y} \,, \tag{46}$$

where $\boldsymbol{y}$ is a vector of position at the coordinate frame with the origin at the center of the source, $\mu_p(\boldsymbol{y})$ is the magnification of a point source at position $\boldsymbol{y}$. Let us assume that cross-section of the source is circular and $I(\boldsymbol{y})$ =const., then in the polar coordinates we obtain

$$\mu_e = \frac{\int\limits_0^{2\pi}\int\limits_0^{R_s} \mu_p(r,\varphi) r dr d\varphi}{\pi R_s^2} \,, \tag{47}$$

where R_s is the radius of the source.

7. Relation between the geodesic deviation equation and the gravitational lens equation

Let us see how presented above approach to the gravitational lensing relates with the gravitational lens equation used in the conventional approach. We will restrict our consideration to the "pure" Einstein–Strauss model without the SIS inside the void described by the Schwarzschild metric and to the plane containing the source, lens and observer. Then the gravitational lens equation is given by

$$\beta = \theta - 2r_g \frac{D_{ls}}{D_s D_l \, \theta} \,, \tag{48}$$

where r_g is the Schwarzschild's radius for a body deflecting light, D_{ls}, D_s, D_l are the angular-diameter distances[6] of, respectively, the source from the lens, the source from the observer and the lens from the observer, β is the angular separation of the source from the lens, which would be observed in the absence of lensing, θ is the observed angular separation between the lens and the deflected ray.

Under the thin-lens approximation, we assume that the change in[7] $\dot{X}_1(\lambda)$ in the vicinity of the deflection point λ_l may be approximated by a step function

$$\dot{X}_{1|_{l_+}} - \dot{X}_{1|_{l_-}} = -\int\limits_{\lambda_l-\varepsilon}^{\lambda_l+\varepsilon} X_1(\lambda) \mathrm{Re}\,\mathcal{F}\; d\lambda \tag{49a}$$

$$\approx 2 \int\limits_{r_{\min}}^{R} \frac{3b^2 c(1+z_l) r_g X_{1|_l} dr}{2r^5 \sqrt{\left(1-kr_\Sigma^2\right)\left(1-\frac{b^2}{(1-kr_\Sigma^2)r^2}\right)}} \tag{49b}$$

$$\approx 2\, \frac{c(1+z_l) r_g \sqrt{1-kr_\Sigma^2}}{r_{\min}^2} X_{1|_l} \,, \tag{49c}$$

where b is the impact parameter and in equation (49b) we use relation between the radius and the affine parameter

$$\frac{d\lambda}{dr} = c(1+z)\sqrt{(1-kr_\Sigma^2)\left(1-\frac{b^2}{(1-kr_\Sigma^2)r^2}\left(1-\frac{r_g}{r}-\frac{\Lambda}{3}r^2\right)\right)} \,,$$

where as regards assumptions $r \gg r_g$ and $\Omega_\Lambda \lesssim 1$, we neglect terms $\frac{r_g}{r}, \frac{\Lambda}{3}r^2 \ll 1$. Inserting this into relation between the deviation vector at

[6] Which one may identify with the generalized Dyer–Roeder distances.

[7] As was mentioned in Sec. 6.1 $\beta = 0$, then the vector e_1^μ lies in the considered plane source-lens-observer.

the source $X_{1|_s}$ and the deviation vector which would be in the absence of lens $\bar{X}_{1|_s}$ [19]

$$X_{1|_s} = \bar{X}_{1|_s} + \frac{D_{ls}}{c(1+z_l)} \left(\dot{X}_{1|_{l_+}} - \dot{X}_{1|_{l_-}}\right) \bar{X}_{1|_s} + 2\frac{D_{ls} r_g \sqrt{1-k r_\Sigma^2}}{r_{\min}^2} X_{1|_l} \ , \quad (50)$$

and using relations between the infinitesimal angles and the Jacobi vectors

$$X_{1|_s} = D_s d\beta \ , \quad (51a)$$

$$\bar{X}_{1|_s} = D_s d\theta \ , \quad (51b)$$

$$X_{1|_l} = D_l d\theta \ , \quad (51c)$$

one has

$$d\beta = d\theta + 2\frac{D_l D_{ls}}{D_s} \frac{r_g \sqrt{1 - k r_\Sigma^2}}{r_{\min}^2}\, d\theta \ .$$

Finally, because $r_{\min} = D_l \theta$ and $\sqrt{1 - k r_\Sigma^2} \approx 1$ we have

$$d\beta = d\left(\theta - 2 r_g \frac{D_{ls}}{D_s D_l}\, \theta\right) \ . \quad (52)$$

We see that this is just the differential form of the lens equation (48).

8. Results

Now we will use the presented above procedure of calculating the magnification factor to compute difference between the distance modulus based on the generalized Dyer–Roeder equation (12) and the distance modulus obtained from the presented approach using the "Swiss cheese" model. All calculations were done numerically due to a lack of analytical solutions of equation (38).

Let us consider the beam of light propagating from a source with redshift $z_s = 1$, and passing a galaxy with mass $M = 10^{11} \times M_\odot$ and redshift $z_G = 0.4$. On the basis of the measurements of the cosmic background radiation and prediction of the theory of inflation in the Big Bang model, we assume a flat universe $\Omega_k = 0$, and from the measurements based on the motions of clusters of galaxies, we have $\Omega_m = 0.3$. Then, from the relation $\Omega_k + \Omega_m + \Omega_\Lambda = 1$, we obtain $\Omega_\Lambda = 0.7$. We take also the most favorable at the present time value of the Hubble constant $H_0 = 65\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. The difference between distances $\Delta(m - M)$ depends very weak on the clumpiness parameter $\tilde{\alpha}$, so we can choose it arbitrary. We assume that

$\tilde{\alpha} = 0.5$. For above parameters of the model the radius of the hole is about 900 kpc and the radius of the SIS halo is about 100 kpc.

On Fig. 1 we could see that with these assumptions the magnification factor decreases from value 1.01, for the impact parameter $b = 15$ kpc corresponding to the radius of the galaxy disk, with increasing impact parameter b. The dependence $\mu(b)$ could be well approximated with accuracy $\sim 0.2\%$ in the range from 15 kpc to 100 kpc (the radius of the SIS) by a function obtained for the SIS in the conventional approach using the gravitational lens equation [20] $\mu(b) = 1 + \mathrm{const.}/b$.

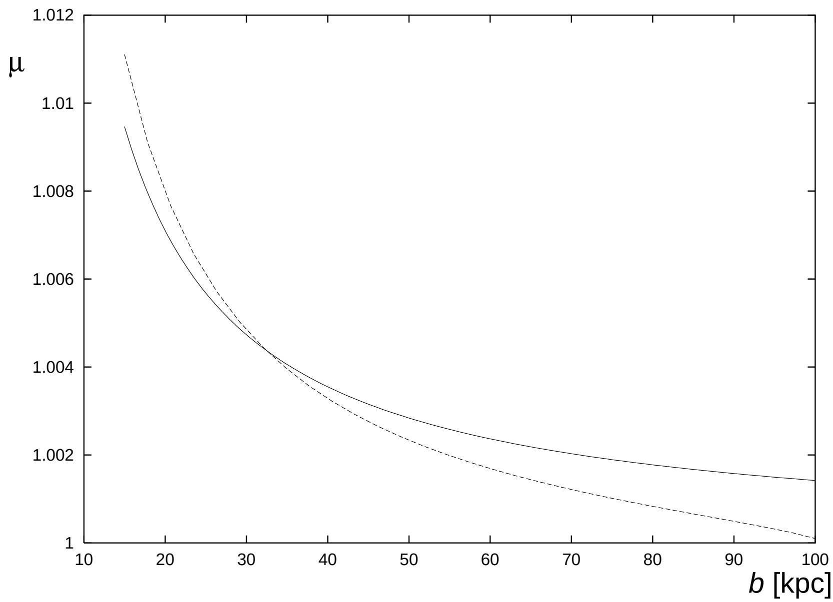

Fig. 1. Relation between the magnification factor μ and the impact parameter b. The continuous line corresponds to the dependence $\mu(b)$ derived from the lens equation for the SIS, and the dashed line describes the dependence obtained from the presented approach.

The difference $\Delta(m-M)$ between the Dyer–Roeder distance modulus

$$(m-M)_{\mathrm{DR}} = 5\log(1+z)^2 D_A + 25\ , \tag{53}$$

and the distance modulus changed by gravitational lensing magnification

$$(m-M)_{\mathrm{lensing}} = 5\log((1+z)^2 D_A/\sqrt{\mu}) + 25\ , \tag{54}$$

is equal

$$\Delta(m-M) = 5\log\sqrt{\mu}\ , \tag{55}$$

where D_A is the Dyer–Roeder distance derived from (12) and is expressed in Mpc. Taking the largest obtained values of the magnification factor $\mu = 1.01$, we estimate the correction as equal 0.01 mag.

9. Summary and conclusions

Using the "Swiss cheese" model to describe the inhomogeneous universe and the geodesic deviation equation, we have investigated the influence of a single clump of matter on the distance modulus for the source observed through an empty cone. The advantage of this approach is the use of a known exact solution of the field equations and the general relativity formalism. Because in practice there are used more realistic models of the gravitational lens based on the weak field approximation and the lens equation, we have also showed how the presented approach relates with conventional approach. In particular we have derived the lens equation from the Jacobi equation and showed that the dependence of the magnification factor on the impact parameter, $\mu(b)$, for the source at redshift $z = 1$ and a galaxy modeled by the SIS with mass $M = 10^{11} \times M_{\odot}$ as the object caused the gravitational lensing effect, is well approximated (with accuracy $\sim 0.2\%$) by the relation $\mu(b) = 1 + \mathrm{const.}/b$ obtained from the lens equation for the SIS.

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