# SCALAR VACUUM STRUCTURE IN GENERAL RELATIVITY AND ALTERNATIVE THEORIES CONFORMAL CONTINUATIONS*

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*(Received October 23, 2001)*

We discuss the global properties of static, spherically symmetric configurations of a self-gravitating real scalar field $\varphi$ in General Relativity (GR), scalar-tensor and high-order (curvature-nonlinear) theories of gravity in various dimensions. In GR, for scalar fields with an arbitrary potential $V(\varphi)$, not necessarily positive-definite, it is shown that the list of all possible types of space-time causal structure in the models under consideration is the same as the one for $\varphi = \text{const}$. In particular, there are no regular black holes with any asymptotics. These features are extended to scalar-tensor and curvature-nonlinear gravity, connected with GR by conformal mappings, unless there is a conformal continuation, *i.e.*, a case when a singularity in a solution of GR maps to a regular surface in an alternative theory, and the solution is continued through such a surface. Such an effect is exemplified by exact solutions in GR with a massless conformal scalar field, considered as a special scalar-tensor theory. Necessary conditions are found for the existence of a conformal continuation; they only hold for special choices of scalar-tensor and high-order theories of gravity.

PACS numbers: 04.20.Gz, 04.70.Bw, 04.50.+h, 04.40.Nr

* Presented at the XXV International School of Theoretical Physics "Particles and Astrophysics — Standard Models and Beyond", Ustroń, Poland, September 10–16, 2001.

## 1. Introduction

This paper describes and continues the study of global properties of scalar-vacuum configurations in general relativity, described by the action

$$S = \int d^D x \sqrt{|g|}\ [R + (\partial\varphi)^2 - 2V(\varphi)] \qquad (1)$$

and similar systems in some alternative theories of gravity, begun in Refs. [1,2]. Here $D$ is the number of space-time dimensions, $R$ is the scalar curvature, $g = \det(g_{\mu\nu})$, $\varphi$ is a real scalar field, $(\partial\varphi)^2 = g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$, and the function $V(\varphi)$ is a potential. This action in case $D = 4$, with many particular forms of $V(\varphi)$, is conventionally used to describe the vacuum (sometimes interpreted as a variable cosmological term) in inflationary cosmology, for the description of growing vacuum bubbles, *etc.*, to say nothing of diverse field-theoretical studies of scalar fields with different potentials. Where and when $\varphi = \text{const}$, the potential $V(\varphi)$ behaves as a cosmological constant.

In the latter case, spherically symmetric solutions to the Einstein equations (the Schwarzschild-(anti-)de Sitter metric and its multidimensional extension) and their global properties in different special cases are well-known [3], see also Sec. 2. They are static due to the extended Birkhoff theorem (see [4] and references therein), and all of them, except the solutions with zero mass parameter $m$, contain curvature singularities at the centre.

A wider set of space-times is connected with the so-called false vacuum, *i.e.*, the system with the action (1). One might expect that the inclusion of scalar fields with various potentials should considerably increase the choice of possible qualitative behaviours of static, spherically symmetric configurations. There are, however, very strong general restrictions that follow directly from the field equations due to (1). Thus, if $V \geq 0$, the only asymptotically flat BH solution in 4 dimensions is Schwarzschild, as follows from the well-known no-hair theorems (see Ref. [5] for a recent review). Another result concerns solitonic (particle-like) configurations with a regular centre and a flat asymptotic: if $V \geq 0$, then such a configuration cannot have a positive mass [6].

It is of interest what can happen if the asymptotic flatness and/or $V \geq 0$ assumptions are abandoned. Both assumptions are frequently violated in modern studies. Negative potential energy densities, in particular, the cosmological constant $V = \Lambda < 0$ giving rise to the Anti–de Sitter (AdS) solution or AdS asymptotic, do not lead to catastrophes (if restricted below), are often treated in various aspects and quite readily appear from quantum effects like vacuum polarization.

Our previous papers [1,2] have provided some essential restrictions on the possible behaviour of solutions of the theory (1) with arbitrary $V(\varphi)$ in $D$

dimensions. It has been shown that, whatever is the potential, the variable scalar field adds nothing to the list of causal structures known for $\varphi = \text{const}$. The possibility of regular configurations without a centre (wormholes and horns) was also ruled out. Extensions of these results to some more general field models were indicated. Considered were *(i)* generalized scalar field Lagrangians in GR, with an arbitrary dependence on the $\varphi$ field and its gradient squared; *(ii)* multiscalar theories of sigma-model type; *(iii)* Scalar-Tensor Theories (STT) of gravity; *(iv)* curvature-nonlinear (High-Order) Gravity (HOG) with the Lagrangian of the form $f(R)$ where $f$ is an arbitrary function. In items *(iii)* and *(iv)*, conformal mappings are used to reduce the original field equations to those following from (1).

This paper pays special attention to the nature of these conformal mappings. The point is that, when a manifold $\mathbb{M}[g_{\mu\nu}]$ is conformally mapped to another manifold $\overline{\mathbb{M}}[\overline{g}_{\mu\nu}]$ (so that $\overline{g}_{\mu\nu} = F(x)\overline{g}_{\mu\nu}$), the global properties of both manifolds are the same as long as the conformal factor $F$ is everywhere smooth and finite. It can happen, however, that a singular surface in $\overline{\mathbb{M}}$ maps to a regular surface in $\mathbb{M}$ due to a singularity in the conformal factor $F$. Then $\mathbb{M}$ can be continued in a regular manner through this surface, and the global properties if $\mathbb{M}$ can be considerably richer than those of $\overline{\mathbb{M}}$: in the new region, one can possibly find, *e.g.*, new horizons or another spatial infinity. A known example of this phenomenon, to be called *conformal continuation,* is provided by the properties of the static, spherically symmetric solution for a conformally coupled scalar field in GR [7,8] as compared with the corresponding solution for a minimally coupled scalar field — see Sec. 6.

It will be further shown that the mappings that connect STT and HOG (the so-called Jordan conformal frame) with GR with a minimally coupled scalar field described by the action (1) (the Einstein frame), provide conformal continuations only under certain special requirements upon the original theory. Under very general conditions, conformal continuations are absent, and the global structure restrictions obtained in GR are directly extended to STT and HOG.

I will not discuss the question of which conformal frame (Jordan, Einstein or some other) in the alternative theories should be regarded as a physical one, refering to our paper [9] and references therein.

The paper is organized as follows. Sec. 2 gives the field equations. Sec. 3 contains a brief description of purely vacuum structures in $D$ dimensions with a cosmological constant. Sec. 4 represents the results of Refs. [1,2] on scalar vacuum in GR. Some no-go theorems are mentioned without proofs, but the main theorem on the possible horizons dispositions is given a new proof. Two examples of configurations admitted by the no-go theorems, are mentioned: a black hole with a nontrivial scalar field and a particlelike solution, both with non-positive-definite potentials. In Sec. 5, the familiar

STT $\mapsto$ GR and HOG $\mapsto$ GR mappings are recalled and discussed, while in Sec. 6 possible conformal continuations are studied.

To conclude, with all theorems and examples at hand, we now have, even without solving the field equations, rather a clear picture of what can and what cannot be expected from static, spherically symmetric scalar-vacuum configurations in various theories of gravity with various scalar field potentials.

Throughout the paper all statements apply to static, spherically symmetric configurations, and all relevant functions are assumed to be sufficiently smooth, unless otherwise indicated.

## 2. Field equations

The field equations due to (1) are

$$\begin{aligned} \nabla^\alpha\nabla_\alpha\varphi + V_\varphi &= 0, && (2)\\ R_\mu^\nu - \tfrac{1}{2}\delta_\mu^\nu R + T_\mu^\nu &= 0, && (3) \end{aligned}$$

where $V_\varphi \equiv dV/d\varphi$, $R_\mu^\nu$ is the Ricci tensor and $T_\mu^\nu$ is the energy-momentum tensor of the $\varphi$ field:

$$T_\mu^\nu = \varphi_{,\mu}\varphi^{,\nu} - \tfrac{1}{2}\delta_\mu^\nu(\partial\varphi)^2 + \delta_\mu^\nu V(\varphi). \tag{4}$$

Consider a static, spherically symmetric configuration, with the space-time structure

$$\mathbb{M}^{\overline{d}+2} = \mathbb{R}_t \times \mathbb{R}_\rho \times \mathbb{S}^{\overline{d}}, \tag{5}$$

where $\mathbb{R}_t$ is the time axis, $\mathbb{R}_\rho \subset \mathbb{R}$ is the range of the radial coordinate $\rho$ and $\mathbb{S}^{\overline{d}}$ ($\overline{d} = D-2$) is a $\overline{d}$-dimensional sphere. The metric can be written in the form

$$ds^2 = A(\rho)dt^2 - \frac{d\rho^2}{A(\rho)} - r^2(\rho)d\Omega_{\overline{d}}^2, \tag{6}$$

where $d\Omega_{\overline{d}}^2$ is the linear element on $\mathbb{S}^{\overline{d}}$ of unit radius, and $\varphi = \varphi(\rho)$. (Without loss of generality, we suppose that large $\rho$ corresponds to large $r$.) Accordingly, Eq. (2) and certain combinations of Eqs. (3) lead to

$$\begin{aligned} \left(Ar^{\overline{d}}\varphi'\right)' &= r^{\overline{d}}V_\varphi\,, && (7)\\ \left(A'r^{\overline{d}}\right)' &= -\left(\frac{4}{\overline{d}}\right)r^{\overline{d}}V\,, && (8)\\ \overline{d}\,\frac{r''}{r} &= -\varphi'^2\,, && (9)\\ A(r^2)'' - r^2A'' + (\overline{d}-2)r'(2Ar' - A'r) &= 2(\overline{d}-1)\,, && (10)\\ \overline{d}(\overline{d}-1)(1-Ar'^2) - \overline{d}A'rr' &= -Ar^2\varphi'^2 + 2r^2V\,, && (11) \end{aligned}$$

where the prime denotes $d/d\rho$. Only three of these five equations are independent: the scalar equation (7) follows from the Einstein equations, while Eq. (11) is a first integral of the others. Given a potential $V(\varphi)$, this is a determined set of equations for the unknowns: $r$, $A$, $\varphi$.

The choice of the radial coordinate $\rho$ such that $g_{tt}g_{\rho\rho} = -1$ is convenient for a number of reasons. First, we are going to deal with horizons, which correspond to zeros of the function $A(\rho)$. One can notice that such zeros are regular points of Eqs. (7)–(11), therefore, one can jointly consider regions at both sides of a horizon. Second, in a close neighbourhood of a horizon $\rho$ varies (up to a positive constant factor) like manifestly well-behaved Kruskal-like coordinates used for an analytic continuation of the metric [10]. Third, with the same coordinate, horizons also correspond to regular points in geodesic equations [10]. Last but not least, this choice well simplifies the equations, in particular, (10) can be integrated, giving, for $\overline{d} \geq 2$,

$$B' \equiv \left(\frac{A}{r^2}\right)' = -\frac{2(\overline{d}-1)}{r^{\overline{d}+2}} \int r^{\overline{d}-2} d\rho \,. \tag{12}$$

Our interest will be in the generic global behaviour of the solutions and the existence of BHs and globally regular configurations.

In these issues, a crucial role belongs to *Killing horizons,* regular surfaces where the Killing vector $\partial_t$ is null. For the metric (6), a horizon $\rho = h$ is a sphere of nonzero radius $r = r_h$ where $A = 0$. The space-time regularity implies the finiteness of $T^\nu_\mu$, so that $V$ and $A\varphi'^2$ are finite at $\rho = h$. The $C^2$-smoothness requirement for $r(\rho)$ at $\rho = h$ means that $r''$ is finite, and (9) leads to $|\varphi'| < \infty$.

The horizon is *simple* or *multiple* (or *higher-order*) according to whether the zero of the function $A(\rho)$ is simple or multiple. Thus, the Schwarzschild horizon is simple while the extreme Reissner–Nordström one is double.

As usual, we shall call the space-time regions where $A > 0$ and $A < 0$ *static* (R) and *nonstatic* (T) regions, respectively. The T regions represent homogeneous cosmological models of Kantowski–Sachs type. A simple or odd-order horizon separates a static region from a nonstatic one, whereas an even-order horizon separates two regions of the same nature. On the construction of Carter–Penrose diagrams, characterizing the causal structure of arbitrary static 2-dimensional space-times [such as the $(t, \rho)$ section of (6)] see Refs. [11, 12] and more recent and more comprehensive papers [13, 14].

## 3. GR: vacuum with a cosmological constant

In case $\varphi = \mathrm{const}$, $V = \mathrm{const} = \Lambda$, one can without loss of generality take $r = \rho$, then Eq. (11) becomes a linear first-order equation with respect to $A(r)$ whose integration gives

$$A(r) = 1 - \frac{2m}{r^{\overline{d}-1}} - \frac{2\Lambda r^2}{\overline{d}(\overline{d}+1)}\,. \tag{13}$$

The metric has the form

$$ds^2 = A(r)dt^2 - \frac{dr^2}{A(r)} - r^2\, d\Omega_{\overline{d}}^{\,2}\,. \tag{14}$$

This is the multidimensional Schwarzschild–de Sitter solution. Its special cases correspond to the Schwarzschild ($\overline{d} = 2$, $\Lambda = 0$) and Tangherlini ($\overline{d}$ arbitrary, $\Lambda = 0$) solutions and the de Sitter solution in arbitrary dimension ($m = 0$). The latter is often called anti-de Sitter in case $\Lambda < 0$.

The different qualitative behaviours of $A(r)$ for different values of $\Lambda$ and $m$ correspond to the following structures:

1. $\Lambda = 0$, $m \le 0$: curves 1a and 1b in Fig. 1, diagram 1 in Fig. 2 (Minkowski and $m < 0$ Schwarzschild, respectively).
2. $\Lambda < 0$, $m \le 0$: curves 2a and 2b in Fig. 1, diagram 2 in Fig. 2 (AdS and $m < 0$ Schwarzschild-AdS).
3. $\Lambda < 0$, $m > 0$: curve 3 in Fig. 1, diagram 3 in Fig. 2 (Schwarzschild-AdS).
4. $\Lambda = 0$, $m > 0$: curve 4 in Fig. 1, diagram 4 in Fig. 2 (Schwarzschild).
5. $\Lambda > 0$, $m \le 0$: curves 5a and 5b in Fig. 1, diagram 5 in Fig. 2 ($m < 0$ Schwarzschild–de Sitter).
6. $\Lambda > 0$, $m > 0$: curves 6a, 6b and 6c in Fig. 1, and the corresponding diagrams in Fig. 2 (Schwarzschild–de Sitter in case 6a and Kantowski–Sachs cosmologies in cases 6b and 6c).

The centre $r = 0$ is regular for $m = 0$ and singular for $m \ne 0$.

In case 6, given a particular value of $\Lambda > 0$, the solution behaviour depends on the mass parameter $m$. When $m$ is smaller than the critical value

$$m_{\rm cr} = \frac{1}{\overline{d}+1}\left[\frac{\overline{d}(\overline{d}-1)}{2\Lambda}\right]^{(\overline{d}-1)/2}, \tag{15}$$

there are two horizons, the smaller one being interpreted as a black hole horizon and the greater one as a cosmological horizon. If $m = m_{\rm cr}$, these two horizons merge, and one has two homogeneous T regions separated by a double horizon. Lastly, the solution with $m > m_{\rm cr}$ is purely cosmological, having no Killing horizons.

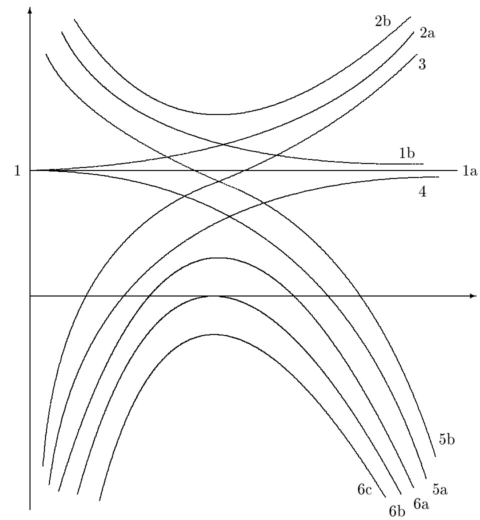


Fig. 1. The behaviour of $A(r)$, Eq. (13), for different values of $m$ and $\Lambda$.

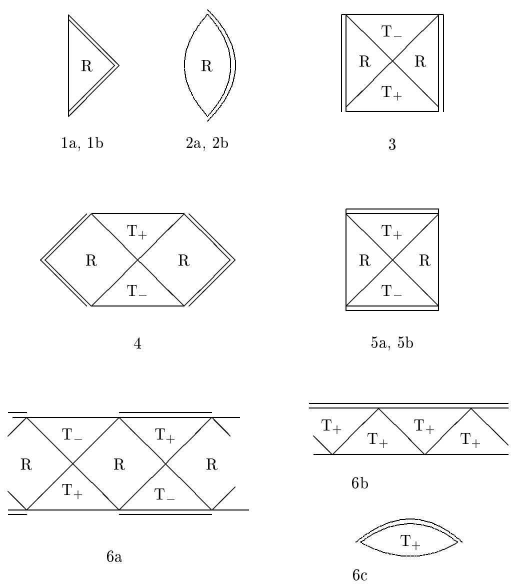


Fig. 2. Carter-Penrose diagrams for different cases of the metric (14), (13), labelled according to Fig. 1. The R and T letters correspond to R and T space-time regions; $T_+$ and $T_-$ denote expanding and contracting T region ( *i.e.*, with $r$ increasing and decreasing with time, respectively). Single lines on the border of the diagrams denote $r = 0$, double lines — $r = \infty$. Diagrams 6b and 6c are drawn for the case of expanding KS cosmologies; to obtain diagrams for contracting models, one should merely interchange $r = 0$ and $r = \infty$ and replace $T_+$ with $T_-$.

## 4. Scalar vacuum in GR. No-go theorems and global structures

### *4.1. Regular models without a centre?*

The first important restriction for the system (1) in the general case is that such configurations as wormholes, horns or flux tubes do not exist under our assumptions.

For the metric (6), a (traversable, Lorentzian) *wormhole* is, by definition, a configuration with two asymptotics at which $r \to \infty$, hence with $r(\rho)$ having at least one regular minimum. A *horn* is a region where, as $\rho$ tends to some value $\rho^*$, $r(\rho) \neq \mathrm{const}$ and $g_{tt} = A$ have finite limits while the length integral $l = \int d\rho/A$ diverges. In other words, a horn is an infinitely long $(\overline{d}+1)$-dimensional "tube" of finite radius, with the clock rate remaining finite everywhere. Such "horned particles" were, in particular, discussed as possible remnants of black hole evaporation [15]. Lastly, a *flux tube* is a configuration with $r = \mathrm{const}$.

**Theorem 1** *The field equations due to (1) do not admit (i) solutions where the function $r(\rho)$ has a regular minimum, (ii) solutions describing a horn, and (iii) flux-tube solutions with $\varphi \neq \mathrm{const}$.*

The formulation of the theorem and its proof [1,2] do not refer to any kind of asymptotic, therefore wormhole throats or horns are absent in solutions having any large $r$ behaviour — flat, de Sitter or any other, or having no large $r$ asymptotic at all.

It also follows that the full range of the $\rho$ coordinate covers all values of $r$, from the centre ($\rho = \rho_c$, $r(\rho_c) = 0$), regular or singular, to infinity, unless (which is not excluded) there is a singularity at finite $r$ due to a "pathological" choice of the potential.

The latter opportunity deserves attention since, being singular at zero or finite $r$, the space-time may in principle be still geodesically complete. In other words, any geodesic can only reach the singularity at an infinite value of its canonical parameter. No freely moving particle can then attain such a singularity (to be called a *remote singularity* in finite proper time. Examples of remote singularities are known in solutions of 2-dimensional gravity [16]).

We can, however, state the following [2]:

**Theorem 2** *If a solution to Eqs. (7)–(11) has a spatial asymptotic ($r \to \infty$), it cannot contain a remote singularity at $r < \infty$.*

Thus remote singularities can only exist in configurations like closed cosmological models, unable to describe isolated bodies observable from outside.

### *4.2. Global structures*

Now, taking into account Theorem 1, the global space-time structure corresponding to any particular solution is unambiguously determined (up to identification of isometric surfaces, if any) by the disposition of static ($A > 0$) and nonstatic ($A < 0$) regions. The following theorem severely restricts the choice of horizon dispositions in the theory under study.

**Theorem 3** *Consider solutions of the theory (1), $D \geq 4$, with the metric (6) and $\varphi = \varphi(\rho)$. Let there be a static region $a < \rho < b \leq \infty$. Then:*

*(i) all horizons are simple;*

*(ii) no horizons exist at $\rho < a$ and at $\rho > b$.*

**Proof** Eq. (10) may be rewritten as follows:

$$r^4 B'' + \left(\overline{d} + 2\right) r^3 r' B' = -2 \left(\overline{d} - 1\right) , \tag{16}$$

where $B(\rho) = A/r^2$. Evidently, zeros of $A(\rho)$ such that $r \neq 0$ are zeros of the same order of the function $B(\rho)$. By (16), $B(\rho)$ cannot have a regular minimum since $B' = 0$ implies $B''(h) = -2(\overline{d} - 1)/r^4 < 0$.

Therefore, if $\rho = h$ is a horizon of at least order, $B(h) = B'(h) = 0$, it is a maximum of $B(\rho)$, hence a double horizon separating two T regions. The absence of regular minima of $B$ then means that $B < 0$ for all $\rho \neq h$, *i.e.*, there is no static region — item *(i)* is proved.

Consider now the boundary $\rho = a$ of the static region. If $r(a) = 0$, it is the centre; be it regular or singular, it is then the left boundary of the range of $\rho$. If $r(a) \neq 0$, then it is a simple horizon: $B(a) = 0$, $B'(a) > 0$. Since $B$ has no minima, it is negative and non-decreasing for all $\rho < a$, *i.e.*, there is no horizon. In a similar way one obtains that horizons are absent to the right of $b$, thus completing the proof.

This theorem shows that the possible disposition of zeros of the function $A(\rho)$ (or $B(\rho)$) is the same as in the vacuum case described in Sec. 3. Therefore the list of possible global structures is also the same.

Theorem 3 shows, in particular, that the attractive idea of replacing the black hole singularity by a nonsingular vacuum core [16–20] cannot be realized in the theory (1). Indeed, such a BH, with any large $r$ behavior, must have static regions at small and large $r$, separated by at least two simple or one double horizon, *i.e.*, the function $B(\rho)$ must have at least one regular minimum. This is impossible due to Eq. (16).

More generally, one can conclude that if spatial infinity is static, there is at most one simple horizon; the same is true if the centre is in a static region.

*Special case: (2+1)-dimensional gravity*

In 3 dimensions we have $\overline{d} = 1$, and integration of (10) leads to an expression simpler than (12):

$$B' \equiv \left(\frac{A}{r^2}\right)' = \frac{C}{r^3}\,, \qquad C = \text{const.} \tag{17}$$

In Theorem 1, items *(i)* and *(iii)* hold due to Eq. (9), as before. Still, the proof of item *(ii)* does not work: a horn is possible if, in (17), $C = 0$. Though, due to $r'' < 0$, the horn radius $r^*$ is the maximum of $r(\rho)$, so that a horned configuration has no large $r$ asymptotic.

By virtue of (17), $B'$ has a constant sign coinciding with $\operatorname{sign} C$, and, instead of Theorem 3, we have a still more severe restriction:

**Theorem 3a** *A static, circularly symmetric configuration in the theory (1), $D = 3$, has either no horizon or one simple horizon.*

Accordingly, the list of possible global structures is even shorter than the previous one: the structures corresponding to the curves 6a and 6b are absent.

### *4.3. 4-dimensional GR: restrictions and examples*

The above theorems did not use any assumptions on the asymptotic behaviour of the solutions or the shape and even sign of the potential. Let us now mention some more specific but also very significant results for positive-semidefinite potentials.

Consider, for simplicity, $D = 4$. The field functions at a regular centre and at a flat asymptotic (if they exist) behave as follows.

A **regular centre**, where $r = 0$, implies a finite time rate and local spatial flatness. This means that at some finite $\rho = \rho_c$

$$Ar'^2 \to 1\,, \qquad A = A_c + O\left(r^2\right)\,, \tag{18}$$

where $A_c = A(\rho_c)$ and $r'(\rho_c)$ are finite and positive. Moreover, the values of $V$, $\varphi$ and $\varphi'$ should be finite there. Then from (9) and (12) one obtains:

$$r''(\rho_c) = 0\,, \qquad \rho_0 = \rho_c\,. \tag{19}$$

At a **flat asymptotic**, the metric should behave as the Schwarzschild one with a certain mass $M$, while $\varphi$ should tend to a finite value. Thus we have

$$\begin{aligned} &\rho \to \infty\,, \qquad r' \to 1\,, \qquad A(\rho) = 1 - \frac{2M}{\rho} + O\left(\rho^{-2}\right)\,, \\ &\varphi' = o\left(\rho^{-3/2}\right)\,, \quad V = o\left(\rho^{-3}\right)\,. \end{aligned} \tag{20}$$

One of the known restrictions is the no-hair theorem:

**Theorem 4 (no-hair)** *Suppose $V \geq 0$. Then the only asymptotically flat BH solution to Eqs. (7)–(11) in the range $(h, \infty)$ (where $\rho = h$ is the event horizon) comprises the Schwarzschild metric, $\varphi = \text{const}$ and $V \equiv 0$.*

This theorem was first proved by Bekenstein [22] for the case of $V(\varphi)$ without local maxima and was later refined for any $V \geq 0$ and for certain more general Lagrangians — see *e.g.* Ref. [5] for proofs and references.

Another restriction can be called the generalized Rosen theorem (Rosen [23] studied similar restrictions for flat-space nonlinear field configurations):

**Theorem 5 [6]** *An asymptotically flat solution with positive mass $M$ and a regular centre is impossible if $V(\varphi) \geq 0$.*

The above theorems leave some opportunities of interest, in particular:

1. BHs with $\varphi \neq \text{const}$, potentials $V(\varphi) \geq 0$ but non-flat large $r$ asymptotics;
2. asymptotically flat BHs with $\varphi \neq \text{const}$ but at least partly negative potentials $V(\varphi)$;
3. asymptotically flat particlelike solutions (solitons) with positive mass but at least partly negative potentials $V(\varphi)$.

That such solutions do exist, one can prove by presenting proper examples. For item 1, such examples have been given in Ref. [24], where, among other results, BHs with non-flat asymptotics were found for the Liouville ($V = 2\Lambda e^{2b\varphi}$) and double Liouville ($V = 2\Lambda_1 e^{2b_1\varphi} + 2\Lambda_2 e^{2b_2\varphi}$) potentials, where the $\Lambda$'s and $b$'s are positive constants.

Special analytical solutions to Eqs. (7)–(11) for $\overline{d} = 2$, exemplifying items 2 and 3 (Appendix B), were given in Ref. [2]. Unlike Ref. [24], where special solutions were sought for by making the ansatz $r(\rho) \propto \rho^N$, $N = \text{const}$ (in our notation), we have used in Ref. [2] the following approach. Suppose $V(\varphi)$ is one of the unknowns. Then our set of equations is underdeterminate, and we can choose one of the unknowns arbitrarily trying to provide the proper behaviour of the solution. Thus, one can choose a particular function $r(\rho)$: assigning it arbitrarily and substituting into (12), by single integration we obtain $A(\rho)$, after which $\varphi(\rho)$ and $V(\rho)$ are determined from (9) and (8), respectively. Thus $V(\varphi)$ is obtained in a parametric form; it can be made explicit if $\varphi(\rho)$ resolves with respect to $\rho$.

A black hole solution was obtained [2] by choosing

$$r(\rho) = \sqrt{\rho^2 - a^2}\,, \qquad a = \text{const} > 0\,, \tag{21}$$

whereas a solitonic solution with positive mass was found under the assumption

$$r^2(\rho) = \frac{a^2}{\rho^2}\frac{\tanh\left(\frac{a}{\rho}+c\right)}{\tanh c}\,, \tag{22}$$

with some positive constants $a$ and $c$. These assumptions lead to potentials $V(\varphi) < 0$ which do not seem quite realistic. However, the purpose of giving these examples was to merely demonstrate the existence of such kinds of solutions. After this demonstration, it makes sense to seek similar solutions for more plausible potentials using numerical methods.

### *4.4. More general Lagrangians in GR. Sigma models*

One can notice that Theorems 1–3 actually rest on two Einstein equations, (9) or (10), which in turn follow from the properties of the energy-momentum tensor. Namely, the property $T^t_t - T^\rho_\rho \geq 0$ expresses the validity of the null energy condition for systems with the metric (6). The corresponding Einstein equation then implies $r'' \leq 0$. Eq. (10), which leads to Theorem 3, follows from the property

$$T^t_t = T^\theta_\theta\,, \tag{23}$$

where $\theta$ is any of the coordinate angles that parametrize the sphere $\mathbb{S}^{\overline{d}}$.

Therefore these three theorems hold for all kinds of matter whose energy-momentum tensors satisfy these two conditions.

Consider, for instance, the following action, more general than (1):

$$S = \int d^D x\,\sqrt{-g}\,\left[R + F(I,\varphi)\right], \tag{24}$$

where $I = (\partial\varphi)^2$ and $F(I,\varphi)$ is an arbitrary function. The scalar field energy-momentum tensor is

$$T^\nu_\mu = \frac{\partial F}{\partial I}\varphi_{,\mu}\varphi^{,\nu} + \tfrac{1}{2}\delta^\nu_\mu F(\varphi). \tag{25}$$

In the static, spherically symmetric case, Eq. (23) holds automatically due to $\varphi = \varphi(\rho)$, while the null energy condition holds as long as $\partial F/\partial I \geq 0$, which actually means that the kinetic energy is nonnegative. Under this condition, all Theorems 1–3 are valid for the theory (24). Otherwise Theorem 3 alone holds; it correctly describes the $\rho$ dependence of $A$ and consequently the possible horizons disposition, but the situation is more complex due to possible non-monotonicity of $r(\rho)$.

Another important and frequently discussed class of theories are the so-called sigma models, where a set of $N$ scalar fields $\varphi = \{\varphi^a\}$, $a = \overline{1,N}$

are considered as coordinates of a target space with a certain metric $G_{ab} = G_{ab}(\varphi)$. The scalar vacuum action is then written in the form

$$S_\sigma = \int d^D x \sqrt{|g|}\ [R + G_{ab} g^{\mu\nu} \partial_\mu \varphi^a \partial_\nu \varphi^b - 2V(\varphi)]\,, \tag{26}$$

where, in general, $G_{ab}$ and $V$ are arbitrary functions of $N$ variables, but in practice they possess symmetries that follow from the nature of specific systems.

It is easily seen that, provided the metric $G_{ab}(\varphi)$ is positive-definite, Theorems 1–3 for static, spherically symmetric configurations are valid as before.

If $G_{ab}$ is not positive-definite, or if some of $\varphi^a$ are allowed to be imaginary, only Theorem 3 holds.

## 5. Scalar-tensor and higher-order gravity

Other extensions of the above results concern theories connected with (1) and (24) via $\varphi$-dependent conformal transformations, such as Scalar-Tensor Theories (STT) and the so-called High-Order Gravity (HOG) (*e.g.*, with the Lagrangian function $f(R)$).

Above all, it should be noted that if a space-time $\mathbb{M}[g]$ with the metric (6) is conformally mapped into another space-time $\overline{\mathbb{M}}[\overline{g}]$, equipped with the same coordinates, according to the law

$$g_{\mu\nu} = F(\rho)\overline{g}_{\mu\nu}\,, \tag{27}$$

then it is easily verified that a horizon $\rho = h$ in $\mathbb{M}$ passes into a horizon of the same order in $\overline{\mathbb{M}}$, *(ii)* a centre ($r = 0$), an asymptotic ($r \to \infty$) and a remote singularity in $\mathbb{M}$ passes into a center, an asymptotic and a remote singularity, respectively, in $\overline{\mathbb{M}}$ if the conformal factor $F(\rho)$ is regular (*i.e.*, finite, at least $C^2$-smooth and positive) at the corresponding values of $\rho$. A regular centre passes to a regular centre and a flat asymptotic to a flat asymptotic under evident additional requirements, but we will not concentrate on them here.

The general (Bergmann–Wagoner–Nordtvedt) STT action in $D$ dimensions can be written as follows:

$$S_{\rm STT} = \int d^D x \sqrt{|g|}\, [f(\phi)R + h(\phi)(\partial\phi)^2 - 2U(\phi) + L_m]\,, \tag{28}$$

where $f$, $h$ and $U$ are functions of the scalar field $\phi$ and $L_m$ is the matter Lagrangian. The metric $g_{\mu\nu}$ here corresponds to the so-called Jordan conformal frame. The standard transition to the Einstein frame [25],

$$g_{\mu\nu} \ = \ F(\varphi)\overline{g}_{\mu\nu}\,, \qquad F = |f|^{-2/(D-2)}\,, \tag{29}$$

$$\frac{d\varphi}{d\phi} = \frac{\sqrt{|l(\phi|}}{f(\phi)}, \qquad l(\phi) \stackrel{\rm def}{=} fh + \frac{D-1}{D-2}\left(\frac{df}{d\phi}\right)^2, \tag{30}$$

removes the nonminimal scalar-tensor coupling expressed in a $\phi$-dependent coefficient before $R$. Putting $L_m = 0$ (vacuum), one can write the action (28) in terms of the new metric $\overline{g}_{\mu\nu}$ and the new scalar field $\varphi$ as follows (up to a boundary term):

$$S_{\rm E} = \int d^D x \sqrt{|\overline{g}|}\, [R_{\rm E} + \eta_l (\partial\varphi)^2 - 2V(\varphi)]\,, \tag{31}$$

where $R_{\rm E}$ and $(\partial\varphi)^2$ are calculated using $\overline{g}_{\mu\nu}$,

$$V(\varphi) = \eta_f F^2(\varphi)\, U(\phi)\,, \tag{32}$$

and $\eta_{l,f}$ are sign factors:

$$\eta_l = {\rm sign}\, l(\phi)\,, \qquad \eta_f = {\rm sign}\, f(\phi)\,. \tag{33}$$

Note that $\eta_l = -1$ corresponds to the so-called anomalous STT, with a wrong sign of scalar field kinetic energy, while $\eta_f = -1$ means that the effective gravitational constant in the Jordan frame is negative. So the normal choice of signs is $\eta_{l,f} = 1$.

The action (31) obviously coincides with (1) up to the factor $\eta_l$. Thus Eq. (23) holds, and we can assert that, for static, spherically symmetric configurations, Theorem 3 is valid for the Einstein-frame metric $\overline{g}_{\mu\nu}$.

Theorems 1 and 2 hold for $\overline{g}_{\mu\nu}$ only in the "normal" case $\eta_l = 1$; let us adopt this restriction.

The validity of the theorems for the Jordan-frame metric $g_{\mu\nu}$ depends on the nature of the conformal mapping (29) between the space-times $\mathbb{M}[g]$ (Jordan) and $\overline{\mathbb{M}}[\overline{g}]$ (Einstein). There are four variants:

**I.** $\mathbb{M} \quad\longleftrightarrow\quad \overline{\mathbb{M}}\,,$
**II.** $\mathbb{M} \quad\longleftrightarrow\quad (\overline{\mathbb{M}}_1 \subset \overline{\mathbb{M}})\,,$
**III.** $(\mathbb{M}_1 \subset \mathbb{M}) \quad\longleftrightarrow\quad \overline{\mathbb{M}}\,,$
**IV.** $(\mathbb{M}_1 \subset \mathbb{M}) \quad\longleftrightarrow\quad (\overline{\mathbb{M}}_1 \subset \overline{\mathbb{M}})\,,$

where $\longleftrightarrow$ denotes a diffeomorphism preserving the metric signature. The last three variants are possible if the conformal factor $F$ vanishes or blows up at some values of $\rho$, which then mark the boundary of $\mathbb{M}_1$ or $\overline{\mathbb{M}}_1$.

Theorem 3 on horizon dispositions is obviously valid in $\overline{\mathbb{M}}$ in cases I and II. In case III or IV, the whole space-time $\overline{\mathbb{M}}$ or its part is put into correspondence to only a part $\mathbb{M}_1$ of $\mathbb{M}$, and, generally speaking, anything, including additional horizons, can appear in the remaining part $\mathbb{M}_2 = \mathbb{M}\setminus\mathbb{M}_1$ of the Jordan-frame space-time. The existence of such a region $\mathbb{M}_2$ will be refered to as a *conformal continuation* of $\overline{\mathbb{M}}$ in $\mathbb{M}$.

Theorem 1 cannot be directly transferred to $\mathbb{M}$ in any case except the trivial one, $F = \text{const}$. It is only possible to assert, without specifying $F(\varphi)$, that wormholes as global entities are impossible in $\mathbb{M}$ in cases I and II if the conformal factor $F$ is finite in the whole range of $\rho$, including the boundary values. Indeed, if we suppose that there is such a wormhole, it will immediately follow that there are two large $r$ asymptotics and a minimum of $r(\rho)$ between them even in $\overline{\mathbb{M}}$, in contrast to Theorem 1 which is valid there.

Theorem 2 also evidently holds in $\mathbb{M}$ in cases I and II if the conformal factor $F$ is regular in the whole range of $\rho$, including the boundary values.

Another class of theories conformally equivalent to (1) is the so-called Higher-Order Gravity (HOG) with the vacuum action

$$S_{\rm HOG} = \int d^D x \sqrt{|g|}\; f(R)\,, \tag{34}$$

where $f$ is a function of the scalar curvature $R$ calculated for the metric $g_{\mu\nu}$ of a space-time $\mathbb{M}$. In accord with the weak field limit $f \sim R$ ar small $R$, let us assume $f(R) > 0$ and $f_R \stackrel{\rm def}{=} df/dR > 0$, at least in a certain range of $R$ including $R = 0$. The conformal mapping $\mathbb{M}[g] \mapsto \overline{\mathbb{M}}[\overline{g}]$ with

$$g_{\mu\nu} = F(\varphi)\overline{g}_{\mu\nu}\,, \qquad F = f_R^{-2/(D-2)}, \tag{35}$$

transforms the "Jordan-frame" action (34) into the Einstein-frame action (1) where

$$\varphi \;=\; \sqrt{\frac{D-1}{D-2}}\log f_R\,, \tag{36}$$

$$2V(\varphi) \;=\; f_R^{-D/(D-2)}(Rf_R - f)\,. \tag{37}$$

The field equations due to (34) after this substitution turn into the field equations due to (1).

All the above observations on the validity of Theorems 1–3 in STT equally apply to higher-order gravity.

In what follows, we will first consider an exactly soluble example with a conformally coupled scalar field in GR, when the mapping follows variant III, and the conformal continuation creates a horizon or a wormhole throat outside $M_1$. Then we will obtain necessary conditions for the occurence of conformal continuations in 4-dimensional STT (28) and HOG (34), showing that this phenomenon is only possible under special requirements to the particular choice of these theories.

## 6. Conformal continuations

### *6.1. Conformal scalar field in GR: black holes and wormholes*

Conformal scalar field in GR can be viewed as a special case of STT, such that, in Eq. (28), $D = 4$ and

$$f(\phi) = 1 - \frac{\phi^2}{6}, \qquad h(\phi) = 1, \qquad U(\phi) = 0. \tag{38}$$

After the conformal mapping

$$g_{\mu\nu} = F(\varphi)\overline{g}_{\mu\nu}, \qquad F(\varphi) = \cosh^2\left(\frac{\varphi}{\sqrt{6}}\right), \tag{39}$$

$$\phi = \sqrt{6}\tanh\left(\frac{\varphi}{\sqrt{6}}\right), \tag{40}$$

we obtain the action (1) with $D = 4$ and $V \equiv 0$. The latter describes a minimally coupled massless scalar field in GR, and the corresponding static, spherically symmetric solution is well-known: it is the Fisher solution [26]. It is convenient to write it using the harmonic radial coordinate $u$ specified by the condition [8] $|g_{uu}| = g_{tt}g_{\theta\theta}^2$ ($u$ behaves as $1/r$ at large $r$):

$$\begin{aligned} ds_{\mathrm{E}}^2 &= \mathrm{e}^{-2mu}dt^2 - \frac{k^2\mathrm{e}^{2mu}}{\sinh^2(ku)}\left[\frac{k^2du^2}{\sinh^2(ku)} + d\Omega^2\right], \\ \varphi &= \sqrt{6}C(u+u_0), \end{aligned} \tag{41}$$

where the subscript "E" stands for the Einstein frame, $m$ (the mass), $C$ (the scalar charge), $k > 0$ and $u_0$ are integration constants, and $k$ is expressed in terms of $m$ and $C$:

$$k^2 = m^2 + 3C^2. \tag{42}$$

The previously used coordinate $\rho$, corresponding to the metric (6), $D = 4$, is $\rho = 2k/(1 - \mathrm{e}^{-2ku})$, and the metric in terms of $\rho$ has the form

$$ds_{\mathrm{E}}^2 = \left(1 - \frac{2k}{\rho}\right)^{m/k} dt^2 - \left(1 - \frac{2k}{\rho}\right)^{-m/k}\left[d\rho^2 + \rho^2\left(1 - \frac{2k}{\rho}\right)d\Omega^2\right]. \tag{43}$$

This solution is asymptotically flat at $u \to 0$ ($\rho \to \infty$), has no horizon when $C \neq 0$ (as should be the case according to the no-hair theorem) and is singular at the centre ($u \to \infty$, $\rho \to 2k$, $\varphi \to \infty$). It turns into the Schwarzschild solution when $C = 0$.

The "Jordan-frame" solution is described by the metric $ds^2 = F(\varphi)ds_{\mathrm{E}}^2$ and the $\phi$ field according to (40). It is the conformal scalar field solution

[7, 27], its properties are more diverse and can be presented as follows (putting, for definiteness, $m > 0$ and $C > 0$):

**1.** $C < m$. The metric behaves qualitatively as in the Fisher solution: it is flat at $u \to 0$, and both $g_{tt}$ and $r^2 = |g_{\theta\theta}|$ vanish at $u \to \infty$ — a singular attracting centre. A difference is that here the scalar field is finite: $\phi \to \sqrt{6}$.

**2.** $C > m$. Instead of a singular centre, at $u \to \infty$ one has a repulsive singularity of infinite radius: $g_{tt} \to \infty$ and $r^2 \to \infty$. Again $\phi \to \sqrt{6}$ as $u \to \infty$.

**3.** $C = m$. In this case the metric and $\phi$ are regular at $u = \infty$; a continuation across this regular sphere may be achieved using a new coordinate, *e.g.*,

$$y = \tanh(mu). \tag{44}$$

The solution acquires the form

$$\begin{aligned} ds^2 &= (1+yy_0)^2\left[\frac{dt^2}{(1+y)^2} - \frac{m^2(1+y)^2}{y^4(1-y_0)^2}(dy^2 + y^2 d\Omega^2)\right], \\ \phi &= \sqrt{6}\frac{y+y_0}{1+yy_0}, \end{aligned} \tag{45}$$

where $y_0 = \tanh(mu_0)$. The range $u \in \mathbb{R}_+$, describing the whole manifold $\overline{\mathbb{M}}$ in the Fisher solution, corresponds to the range $0 < y < 1$, describing only a region $\mathbb{M}_1$ of the manifold $\mathbb{M}$ of the solution (45). The properties of the latter depend on the sign of $y_0$ [8]. In all cases, $y = 0$ corresponds to a flat asymptotic, where $\phi \to \sqrt{6}y_0$, $|y_0| < 1$.

**3a:** $y_0 < 0$. The solution is defined in the range $0 < y < 1/|y_0|$. At $y = 1/|y_0|$, there is a naked attracting central singularity: $g_{tt} \to 0$, $r^2 \to 0$, $\phi \to \infty$.

**3b:** $y_0 > 0$. The solution is defined in the range $y \in \mathbb{R}_+$. At $y \to \infty$, we find another flat spatial infinity, where $\phi \to \sqrt{6}/y_0$, $r^2 \to \infty$ and $g_{tt}$ tends to a finite limit. This is a *wormhole solution*, found for the first time in Ref. [8] and recently discussed by Barcelo and Visser [28].

**3c:** $y_0 = 0$, $\phi = \sqrt{6}y$, $y \in \mathbb{R}_+$. In this case it is helpful to pass to the conventional coordinate $r$, substituting $y = m/(r-m)$. The solution

$$\begin{aligned} ds^2 &= \left(1 - \frac{m}{r}\right)^2 dt^2 - \frac{dr^2}{\left(1-\frac{m}{r}\right)^2} - r^2 d\Omega^2, \\ \phi &= \sqrt{6}\frac{m}{(r-m)} \end{aligned} \tag{46}$$

is the well-known BH with a conformal scalar field [7, 27], which seems to violate the no-hair theorem. The infinite value of $\phi$ at the horizon $r = m$ does not make the metric singular since, as is easily verified, the energy-momentum tensor remains finite there.

The whole case 3 belongs to variant III in the classification of Sec. 5, and the horizon in case 3c is situated in the region $\mathbb{M}_2 = \mathbb{M} \setminus \mathbb{M}_1$, where the action of the no-hair theorem cannot be extended.

In case 3b, the second spatial infinity and even the wormhole throat ($y = 1/\sqrt{y_0}$) are situated in $\mathbb{M}_2$, illustrating the inferences of Sec. 5.

An important lesson follows, however, from case 2, where the mapping is type I by the same classification ($\mathbb{M} \longleftrightarrow \overline{\mathbb{M}}$): there appears a minimum of $r(u)$ in the metric $g_{\mu\nu}$ (39), and $r$ even blows up at large $u$. This is connected with blowing up of the conformal factor $F$. Recall that, as mentioned in Sec. 5, the absence of another spatial infinity is only guaranteed under the finiteness condition for the conformal factor in the whole range of the radial coordinate, including its boundary values: we see that this condition is indeed essential.

The simple example of the conformal field thus illustrates the possible nontrivial consequences of conformal continuations. We shall see, however, that for most choices of STT and HOG one is guaranteed against such continuations.

### *6.2. Conformal continuation conditions in scalar-tensor and high-order gravity*

Let us put for simplicity $D = 4$ and consider possible conformal continuations of Einstein-frame solutions of STT and HOG due to transition to the Jordan frame.

In STT (28), such a continuation may occur at a zero of the function $f(\phi)$ in Eq. (28). If, at $\phi = \phi_0$, the function $f(\phi)$ has a simple zero, $f(\phi) = (\phi - \phi_0) \cdot O(1)$, then, in the transformation (29), (30) for $D = 4$ we have, without loss of generality,

$$|\phi - \phi_0| = \mathrm{e}^{-\sqrt{2/3}\varphi} \cdot O(1)\,, \tag{47}$$

$$F(\varphi) = \mathrm{e}^{\sqrt{2/3}\varphi} \cdot O(1) \tag{48}$$

as $\phi \to \phi_0$, so that $\varphi \to \infty$.

In HOG (34) a continuation is possible if $f_R \to 0$ at some $R$. Then in the transformation (35), (36) we obtain for $D = 4$, without loss of generality,

$$|f_R| \sim \mathrm{e}^{-\sqrt{2/3}\varphi} \tag{49}$$

and again the relation (48). Thus the conformal factor has the same leading order behaviour in both theories.

A conformal continuation of the metric (6) can obviously occur with the factor $F$ at some $\rho = \rho_0$ under the condition that the functions

$$F(\varphi)\, A(\rho)\,, \qquad F(\varphi)\, r^2(\rho)$$

have finite values at $\rho = \rho_0$. This means that $\rho = \rho_0$ is a generic regular sphere in the Jordan frame. Since this is a centre ($r \to 0$) in the Einstein frame, $\rho_0$ is finite, and we can put $\rho_0 = 0$ by a proper choice of the origin of $\rho$. Then Eqs. (9) and (10) show that, in the leading order of magnitude,

$$r^2(\rho) \sim A(\rho) \sim \sqrt{\rho} \sim \mathrm{e}^{-\sqrt{2/3}\varphi}\,. \tag{50}$$

Hence near $\rho = 0$ the functions $r(\rho)$ and $A_\rho)$ may be represented by the expansions

$$\begin{aligned} r &= \rho^{1/4}(r_0 + r_1\rho + \ldots)\,, \\ A &= \rho^{1/2}(A_0 + A_1\rho + \ldots)\,, \end{aligned} \tag{51}$$

where $r_0,\ r_1, \ldots,\ A_0,\ A_1, \ldots$ are constants. Substituting (51) into the field equations, in particular, (8), we find that generically the potential $V(\varphi)$ behaves as $1/\sqrt{\rho}$, but it may happen that the leading order (or orders) vanish due to special relations between the expansion constants in (51). One concludes, in general, that

$$V(\varphi) \sim \rho^{-1/2+n}\,, \qquad n = 0, 1, 2, \ldots\,. \tag{52}$$

Returning to Eqs. (47) and (48) and recalling that the potential function $U(\phi)$ in the STT (28) is expressed in terms of $V$ as $U = V/F^2$ [see Eq. (32)], we conclude that near $\phi_0$

$$\begin{aligned} F(\varphi) &\sim |\phi - \phi_0|^{-1} \sim 1/\sqrt{\rho}\,, \\ U(\phi) &\sim |\phi - \phi_0|^{1+2n}\,, \end{aligned} \tag{53}$$

where $n$ comes from (52). We conclude that *such a continuation is only possible when $U(\phi)$ has an odd-order zero at $\phi = \phi_0$.*

For HOG we have the expression (37) for the potential $V$, which, for $D = 4$, is rewritten as

$$2V(\varphi) = \frac{(Rf_R - f)}{f_R^2}\,. \tag{54}$$

In the case of interest, $f_R = 1/F(\varphi) \sim \sqrt{\rho}$, whereas $V(\varphi)$ either vanishes at $\rho = 0$ or, at most, blows up as $1/\sqrt{\rho}$. This is only possible if $f(\phi_0) = 0$. Thus *a necessary condition for a continuation is that $f = f_R = 0$ simultaneously at some value of $R$.* Moreover, the requirement that $f(R)$ should be smooth at $R = R_0$ leaves the only opportunity $V \sim 1/\sqrt{\rho} \sim 1/f_R$; Eq. (54) then shows that $R_0 \neq 0$.

Besides a generic sphere, a continuation may proceed through a horizon in the Jordan frame. In other words, in the metric

$$ds_{\rm J}^2 = F(\varphi)\left[A(\rho)dt^2 - \frac{d\rho^2}{A(\rho)} - r^2 d\Omega^2\right], \tag{55}$$

a certain value of $\rho$ (without loss of generality, $\rho = 0$) may correspond to a horizon of order $k \geq 1$. This means that $\rho = 0$ is a zero of order $k$ of the function $\overline{A}(q) = AF$, where $q(\rho)$ is a new coordinate satisfying the condition $g_{tt}g_{qq} = -1$ in (55) (see the comment on the choice of the $\rho$ coordinate in Sec. 2). As a result, we must have

$$\begin{aligned} \pm dq(\rho) &= F(\varphi)d\rho; \\ \overline{A}(q) &= AF \sim (q-q_0)^k, \qquad Fr^2 = O(1), \end{aligned} \tag{56}$$

where $q_0$ is the value of $q$ corresponding to $\rho = 0$. As before, let us suppose that $F(\varphi) \sim {\rm e}^{\sqrt{2/3}\varphi}$ and $\varphi \to \infty$ as $\rho \to 0$.

A substitution to Eq. (9) leads, as before, to $r^2 \sim 1/F \sim \sqrt{\rho}$. A further substitution to (10) then leaves two opportunities: *(i)* $A(\rho) \sim \sqrt{\rho}$ and *(ii)* $A(\rho) \sim \rho^{3/2}$.

In the first case $AF$ tends to a finite limit, contrary to what was assumed (we simply return to the case of a generic regular sphere).

In the second case, there can be a second-order horizon ($AF \sim \rho \sim (q-q_0)^2$). One can, however, show that, according to Eq. (10), $A(\rho) < 0$ as $\rho \to 0$, so this horizon is approached from a T region as $\rho \to +0$. If there is a static region at certain $\rho > 0$, this means that, as $\rho$ decreases, $A(\rho)$ changes its sign at some other horizon, say, $\rho = h > 0$. Recalling the proof of Theorem 3, one can assert that $B(\rho) = A/r^2$ is a nondecreasing function at $\rho < h$. On the other hand, in the case under consideration one has $B \sim -r^4 \sim -\rho$ near $\rho = 0$, *i.e.*, a decreasing function. This contradiction shows that *a continuation through a horizon in the Jordan frame is only possible when the whole region $\rho > 0$* (the whole space in the Einstein frame) *is a T region.*

In this case, as $\rho \to 0$, $V(\varphi) \sim 1/\sqrt{\rho}$. In STT this leads to $U(\phi) \sim \phi - \phi_0$. For HOG this is just the variant of $V$ admitted by (54), and the requirement to $F(R)$ is the same as before: at $R = R_0 \neq 0$, $f(R)$ should have at least a second-order zero.

Summing up, we have the following two theorems and comment:

**Theorem 6** *Consider static, spherically symmetric solutions in STT (28), $D = 4$. Suppose that (a) $f(\phi) > 0$ at $\varphi < \varphi_0$; (b) $f(\varphi_0) = 0$ but $df/d\phi(\phi_0) \neq 0$. Then the solution can be continued through the sphere where $\phi = \phi_0$ only if $U(\phi)$ has an odd-order zero at $\phi = \phi_0$.*

**Theorem 7** *Consider static, spherically symmetric solutions in HOG (34). Suppose that the function $f_R > 0$ at $R < R_0$ and $f_R(R_0) = 0$. Then the solution can be continued through the sphere where $R = R_0$ only if $R_0 \neq 0$ and $f(R)$ has an at least second order zero at $R = R_0$.*

**Comment** *The sphere $\phi = \phi_0$ or $R = R_0$, admitting a continuation, can be (but not necessarily is) a horizon, and it is then double, only if the whole Einstein-frame solution represents a T region. In STT, under the conditions of Theorem 6, this can only happen if $U(\phi)$ has a simple zero at $\phi = \phi_0$.*

One should stress that the conditions enumerated in Theorems 6 and 7 are only necessary for a possible continuation. It would be quite incorrect to think that any given solution to a theory satisfying these conditions may be continued in this way. This is perfectly well seen in the example of Sec. 6.1: the potential $U(\phi)$ is zero identically, so the restriction of Theorem 6 is avoided, but a continuation actually takes place only for a special subfamily of the solutions, selected by a certain relation between the integration constants.

On the other hand, Theorems 6 and 7 single out very narrow classes of theories among all STT and HOG. For all others, the Jordan-frame solutions obtained by conformal mappings from the Einstein frame are complete, and, in particular, Theorem 3 that determines the possible choice of global causal structures, is applicable.

This work was supported in part by the Russian Foundation for Basic Research. I am grateful to Oleg Zaslavskii, Mikhail Katanaev, Georgy Shikin and Marek Biesiada for helpful discussions, and to the Organizing Committee of the 25th School of Theoretical Physics at Ustroń for kind hospitality and perfect arrangement of the School.

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