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STRUCTURE FORMATION IN THE QUINTESSENTIAL UNIVERSE* **

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I review the main characteristics of structure formation in the quintessential Universe. Assuming equation of state $w = p/\varrho = \text{const.}$ I provide a brief description of the background cosmology and discuss the linear growth of density perturbations, the strongly nonlinear evolution, the power spectra and r.m.s. fluctuations as well as mass functions focusing on the three values $w = -1, -2/3$ and $-1/3$. Finally I describe the presently available and future constraints on w.

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1. Introduction

Our knowledge of background cosmology has recently improved dramatically due to new supernovae and cosmic microwave background data. Current observations favor a flat Universe with matter density $\Omega_0 = 0.3$ [1] and the remaining contribution in the form of cosmological constant [2, 3] or some other form of dark energy. The models with cosmological constant are known, however, to suffer from two major problems. One is related to the origin of the constant — it cannot be explained in terms of the vacuum energy since its energy is orders of magnitude smaller. The other is the lack of explanation why the present densities in matter and cosmological constant are comparable.

A new class of models that solve these problems and also satisfy present observational constraints has been proposed a few years ago [4]. In these models the cosmological constant is replaced with a new energy component,

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called quintessence, characterized by the equation of state $p/\varrho = w \neq -1$. The component can cluster on largest scales and, therefore, affect the mass power spectrum [5] and microwave background anisotropies [6, 7].

The investigations of the physical basis for the existence of such component are now more than a decade old [8]. One of the promising models is based on so-called "tracker fields" that display an attractor-like behavior causing the energy density of quintessence to follow the radiation density in the radiation dominated era but dominate over matter density after matter-radiation equality [9, 10]. It is still debated, however, how w should depend on time, and whether its redshift dependence can be reliably determined observationally [11–13].

A considerable effort has gone into attempts to put constraints on models with quintessence and presently the values of $-1 < w < -0.6$ seem most feasible observationally [14, 15]. Here I review the main characteristics of structure formation in the quintessential Universe which may provide constraints on the equation of state. In the last section I discuss the current status of observational limits on w and future perspectives.

2. Background cosmology

Quintessence obeys the following equation of state relating it's density ϱ_{Q} and pressure p_{Q}

$$p_{\mathrm{Q}} = w\varrho_{\mathrm{Q}}, \qquad \text{where} \quad -1 \leq w < 0\,. \tag{1}$$

The case of $w = -1$ corresponds to the usually defined cosmological constant.

The evolution of the scale factor $a = R/R_0 = 1/(1+z)$ (normalized to unity at present, z is the redshift) in the quintessential Universe is governed by the Friedmann equation

$$\frac{da}{dt} = \frac{H_0}{u(a)}\,, \tag{2}$$

where

$$u(a) = \left[1 + \Omega_0\left(\frac{1}{a} - 1\right) + q_0\left(\frac{1}{a^{1+3w}} - 1\right)\right]^{-1/2} \tag{3}$$

and H_0 is the present value of the Hubble parameter. The quantities with subscript 0 here and below denote the present values. The parameter Ω is the standard measure of the amount of matter in units of critical density and q measures the density of quintessence in the same units

$$q = \frac{\varrho_{\mathrm{Q}}}{\varrho_{\mathrm{crit}}}\,. \tag{4}$$

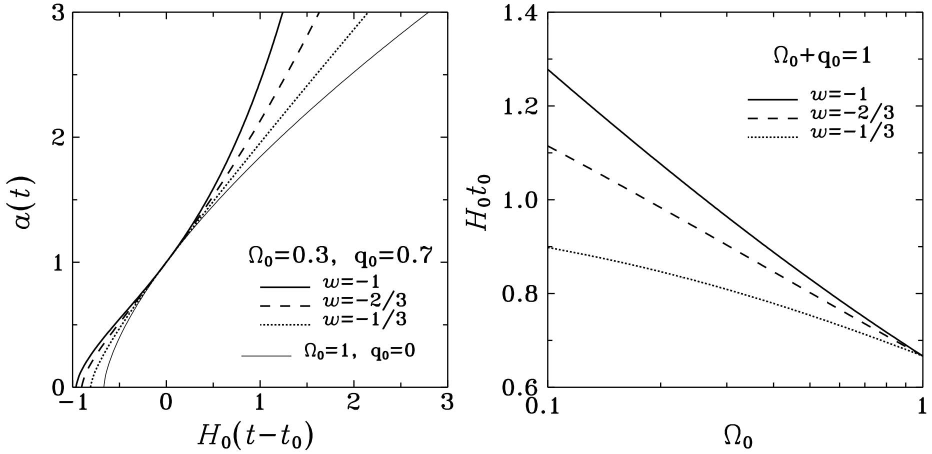

Fig. 1. Left panel: Time evolution of the scale factor in different models. Right panel: The present age of Universe in units of H_0^{-1} in flat models as a function of Ω_0 for different w.

The Einstein equation for acceleration $d^2a/dt^2 = -4\pi Ga(p + \varrho/3)$ shows that $w < -1/3$ is needed for the accelerated expansion to occur. The left panel of figure 1 shows the evolution of scale factor in different models. The right panel presents the dependence on w of the present age of Universe

$$t_0 = \frac{1}{H_0}\int\limits_0^1 u(a)da\,. \tag{5}$$

Solving the equation for the conservation of energy $d(\varrho_{\rm Q}a^3)/da = -3p_{\rm Q}a^2$ with condition (1) we get the evolution of the density of quintessence which for $w =$ const., the case considered in this paper, reduces to

$$\varrho_{\rm Q} = \varrho_{\rm Q,0}\ a^{-3(1+w)}\,. \tag{6}$$

The evolution of Ω and q with scale factor is given by

$$\Omega(a) = \frac{\Omega_0 u^2(a)}{a}, \qquad q(a) = \frac{q_0 u^2(a)}{a^{1+3w}}\,, \tag{7}$$

while the Hubble parameter itself evolves so that $H(a) = H_0/[a\ u(a)]$.

3. Linear growth of perturbations

The linear evolution of the matter density contrast $\delta = \delta\varrho/\varrho$ is governed by equation [16]

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\varrho\delta = 0\,, \tag{8}$$

where dots represent derivatives with respect to time. For flat models and arbitrary w an analytical expression for $D(a)$, the growing mode of the time-dependent part of δ, was found [17]. With our notation and the normalization of $D(a) = a$ for $\Omega = 1$ and $q = 0$ it becomes

$$D(a) = a\ {}_2F_1\left[-\frac{1}{3w}, \frac{w-1}{2w}, 1-\frac{5}{6w}, -a^{(-3w)}\frac{1-\Omega_0}{\Omega_0}\right]\,, \tag{9}$$

where ${}_2F_1$ is a hypergeometric function. The solutions (9) for different w and cosmological parameters $\Omega_0 = 0.3$ and $q_0 = 0.7$ are plotted in figure 2.

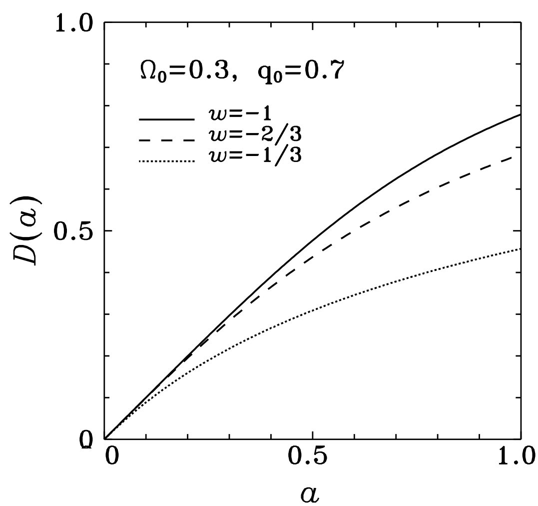

Fig. 2. The linear growth rate of density fluctuations for $\Omega_0 = 0.3$, $q_0 = 0.7$ in three cases of $w = -1, -2/3$ and $-1/3$.

The peculiar velocity field in linear perturbation theory is obtained from [16]

$$\boldsymbol{v} = \frac{2f}{3\Omega H}\boldsymbol{g}\,, \tag{10}$$

where $\boldsymbol{g} = -\nabla\phi/a$ is the peculiar gravitational acceleration and f is the dimensionless velocity factor

$$f = \frac{a}{\dot{a}}\frac{\dot{D}}{D}\,. \tag{11}$$

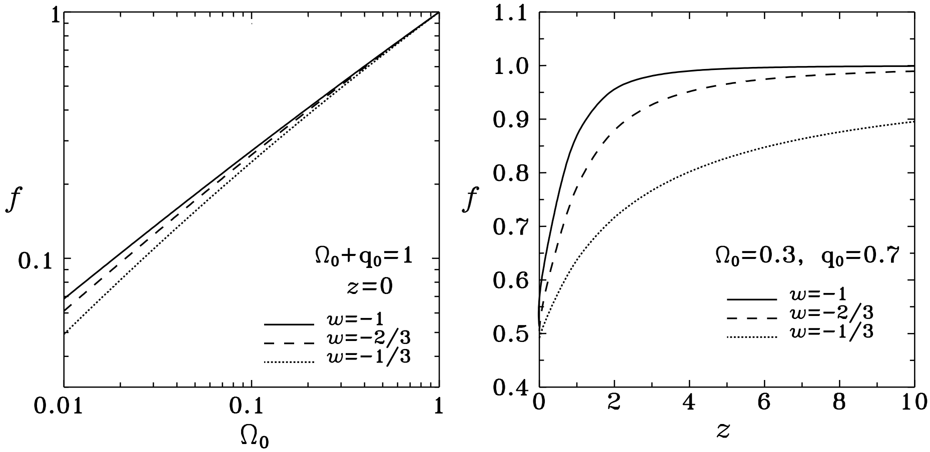

Fig. 3. Left panel: The velocity factor f at present as a function of Ω_0 for flat models with different w. Right panel: The redshift dependence of f for $\Omega_0 = 0.3$, $q_0 = 0.7$ and different w.

For flat models this formula can be evaluated analytically using Eq. (9). The dependence of f on Ω_0 at present ($z = 0$) for flat models with different w is shown in the left panel of figure 3. We see immediately that the dependence on w is very weak. However, as shown in the right panel of figure 3, when going to higher redshifts we find that the velocity factor is much more sensitive to w which gives some hope for applying it to determine w from local peculiar velocity field.

4. Strongly nonlinear evolution

The simplest model of formation of bound objects (called the spherical or top hat model) [18–21] describes the nonlinear evolution of spherical density perturbation of initial proper radius $r_{\rm i}$ and mass M

$$M = M(r_{\rm i}) = \frac{4\pi}{3}\rho_{\rm b,i}\, r_{\rm i}^3(1 + \Delta_{\rm i})\,, \tag{12}$$

where $\Delta_{\rm i}$ is the initial cumulative overdensity and $\rho_{\rm b,i}$ is the background density of matter at time $t_{\rm i}$. Evolution of this perturbation is governed by the energy equation

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} - \frac{H^2 q r^2}{2} = E\,. \tag{13}$$

If $\Delta_{\rm i}$ is larger than certain critical value [20, 21], the overdense region will eventually turn-around and collapse.

Conservation of energy leads to the following condition for the maximum expansion (turn-around) radius $r_{\rm ta}$ (or $s_{\rm ta} = r_{\rm ta}/r_{\rm i}$)

$$b_1 s_{\rm ta}^3 + b_2 s_{\rm ta} + b_3 = 0\,, \tag{14}$$

where $b_1 = c q_{\rm i}$, $b_2 = 1 - \Omega_{\rm i}(1+\Delta_{\rm i}) - q_{\rm i}$, $b_3 = \Omega_{\rm i}(1+\Delta_{\rm i})$ and $c = (a_{\rm i}/a_{\rm ta})^{3(1+w)}$ where $a_{\rm ta}$ is the scale factor at turn-around.

There are two real and positive solutions to equation (14)

$$s_{\rm ta} = \frac{2}{\sqrt{3}} \left(\frac{-b_2}{b_1} \right)^{1/2} \cos \left(\frac{\phi - 2\pi}{3} \right) \tag{15}$$

and

$$s_{\rm ta} = \frac{2}{\sqrt{3}} \left(\frac{-b_2}{b_1} \right)^{1/2} \cos \left(\frac{\phi}{3} \right)\,, \tag{16}$$

where

$$\phi = \arccos \frac{x}{(x^2 + y^2)^{1/2}} \tag{17}$$

and $x = -9 b_1^{1/2} b_3$, $y = [3(-4b_2^3 - 27 b_1 b_3^2)]^{1/2}$. For $q_0 = 0$ we simply get $s_{\rm ta} = -b_3/b_2$. The $q_0 = 0$ case is reproduced in the limit of small q_0 only by solution (15). However, although for $w = -1$ only (15) works, for higher values of w which solution is applicable depends on Ω_0.

Particularly useful quantities which can be derived from the model are the characteristic densities of the evolution usually denoted by $\delta_{\rm c}$ and $\Delta_{\rm c}$. $\delta_{\rm c}$ is the density contrast at the moment of collapse (with scale factor $a_{\rm coll}$) as predicted by linear theory

$$\delta_{\rm c} = h\,[\Omega_0,\, q_0,\, \Delta_{\rm i}(a_{\rm coll}),\, z_{\rm i}] D(a_{\rm coll})\,, \tag{18}$$

where

$$h(\Omega_0, q_0, \Delta_{\rm i}, z_{\rm i}) = \frac{3}{5} \left[\frac{1 - \Omega_0 - q_0}{\Omega_0} + \frac{\Omega_0[1 + \Delta_{\rm i}(1+z_{\rm i})] + q_0 - 1}{\Omega_0 (1 + \Delta_{\rm i})^{2/3}} \right]. \tag{19}$$

The values of $\delta_{\rm c}$ as a function of Ω_0 for flat models with different w are shown in the left panel of figure 4. They will be used in Section 6 to construct the mass functions of bound objects.

Another useful quantity is the ratio of the density of the object to the critical density at virialization

$$\Delta_{\rm c} = \frac{\rho_{\rm vir}}{\rho_{\rm crit}}\,(a_{\rm coll}) = \frac{\Omega(a_{\rm coll})}{s_{\rm coll}^3} \left(\frac{a_{\rm coll}}{a_{\rm i}} \right)^3 \Big[1 + \Delta_{\rm i}(a_{\rm coll}) \Big]\,, \tag{20}$$

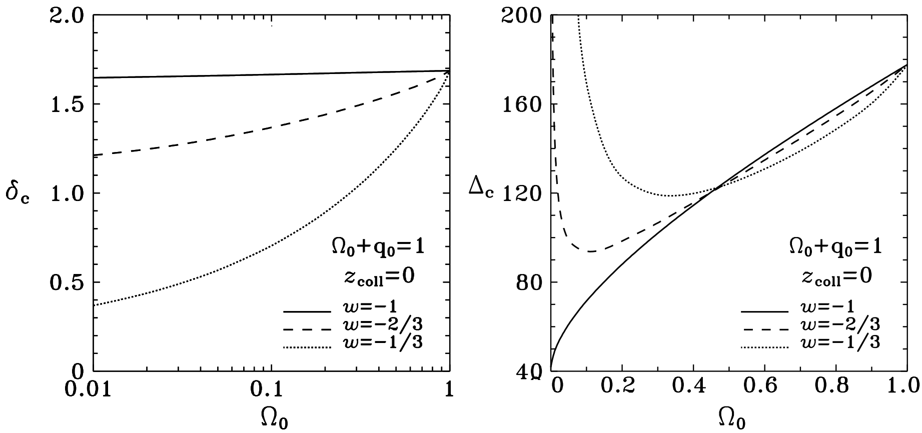

Fig. 4. Dependence of $\delta_{\rm c}$ (left panel) and $\Delta_{\rm c}$ (right panel) on Ω_0 and w for flat models and collapse time $z_{\rm coll} = 0$.

where $s_{\rm coll} = r_{\rm coll}/r_{\rm i}$ and $r_{\rm coll}$ is the effective final radius of the collapsed object. We assume that the object virializes at $t_{\rm coll}$, the time corresponding to $r \to 0$. The final radius $s_{\rm coll}$ needed in Eq. (20) is obtained by application of the virial theorem which leads to the following equation for the collapse factor $F = r_{\rm coll}/r_{\rm ta}$

$$2\eta F^3 - \left[2 + \eta\left(\frac{a_{\rm coll}}{a_{\rm ta}}\right)^{3(1+w)}\right] F + 1 = 0\,, \tag{21}$$

where

$$\eta = \frac{2 q_{\rm i} s_{\rm ta}^3}{\Omega_{\rm i}(1+\Delta_{\rm i})}\left(\frac{a_{\rm i}}{a_{\rm coll}}\right)^{3(1+w)}\,. \tag{22}$$

Numerical values of $\Delta_{\rm c}$ for the presently collapsing perturbations in different models are shown in the right panel of figure 4.

Using spherical model we can also estimate the redshift of a particular stage of evolution of the perturbation given its present overdensity as predicted by linear theory, δ_0. For the redshift of collapse we have

$$\delta_0 = \delta_{\rm c}(a_{\rm coll})\frac{D(a=1)}{D(a_{\rm coll})}\,. \tag{23}$$

With the previously obtained results for $\delta_{\rm c}$ and the formula for the linear growth of fluctuations (9), we can calculate the present linear density contrast of fluctuation that collapsed at $z_{\rm coll}$. This relation can only be inverted analytically in the case of $\Omega_0 = 1$, $q_0 = 0$ when we get $z_{\rm coll} = \delta_0/\delta_{\rm c} - 1$. For other cases the calculations have to be done numerically. Using equation

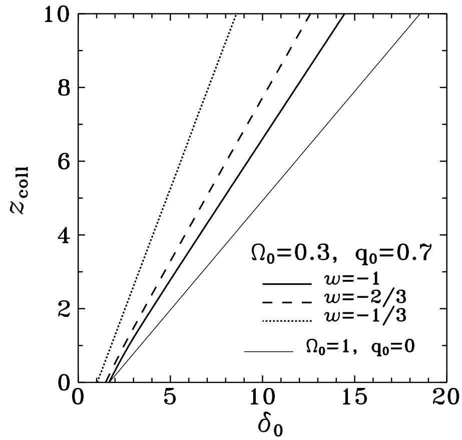

Fig. 5. Redshift of collapse (left panel) and turn-around (right panel) of density fluctuation with present linear density contrast δ_0 for flat models with different w.

analogous to (23) we can also calculate the redshift of turn-around, $z_{\rm ta}$. Parameter $\delta_{\rm c}(a_{\rm coll})$ has then to be replaced by the corresponding turn-around value $\delta_{\rm ta}(a_{\rm ta})$. Both redshifts are shown in figure 5 as functions of δ_0 for different models.

5. Power spectrum of density fluctuations

Power spectrum $P(k,a)$ is defined as the Fourier transform of the correlation function of density fluctuations

$$P(k,a) = \int \xi(r,a)\ {\rm e}^{-{\rm i}\boldsymbol{k}\cdot\boldsymbol{r}} d^3 r. \tag{24}$$

The spectra for Universe dominated by Cold Dark Matter (CDM) have been widely discussed in the literature, *e.g.* [22]. For the present time ($a=1$) the power spectrum is usually written in the form

$$P(k) = A\, k^n T^2(k)\,, \tag{25}$$

where n measures the slope of the primordial power spectrum (we will assume $n=1$), T is the transfer function and A is a normalization constant. In the presence of cosmological constant (ΛCDM) the transfer function T_Λ can be approximated by [22]

$$T_\Lambda^2 = \left\{1 + \left[c_1 k + (c_2 k)^{3/2} + (c_3 k)^2\right]\right\}^{-2/\nu}, \tag{26}$$

with $\nu = 1.13$, $c_1 = 6.4/\Gamma$, $c_2 = 3.0/\Gamma$, $c_3 = 1.7/\Gamma$ (c_1, c_2, c_3 in units of h^{-1}Mpc), $\Gamma = \Omega_0 h$ and $h = H_0/100$ [km/(s Mpc)] $= 0.7$.

For flat models with quintessence the modification of the transfer function has been proposed by Ma *et al.* [5]. The transfer function in equation (25) is then $T_{\rm Q} = T_{{\rm Q}\Lambda} T_\Lambda$, where $T_{{\rm Q}\Lambda} = T_{\rm Q}/T_\Lambda$ is approximated by fits given in [5]. Adopting COBE normalizations of the spectra we obtain the following constants for the three cases with $w = -1, -2/3$ and $-1/3$, respectively: $A = 3.74 \times 10^6, 2.62 \times 10^6$ and 8.53×10^5 $(h^{-1}{\rm Mpc})^4$. The present linear power spectra for these three cases are shown in the left panel of figure 6.

The r.m.s. density fluctuation, σ, at comoving smoothing scale R is given by

$$\sigma^2 = \frac{1}{(2\pi)^3} \int d^3k\, P(k)\, W^2_{\rm TH}(kR)\,, \tag{27}$$

where the smoothing is performed with the top hat filter

$$W^2_{\rm TH}(kR) = \frac{3}{(kR)^2} \left(\frac{\sin kR}{kR} - \cos kR \right). \tag{28}$$

The dependence of σ on smoothing scale for flat models with different w is shown in the right panel of figure 6. A particularly useful quantity, constrained by cluster abundance is the r.m.s. fluctuation at the scale of $8h^{-1}$ Mpc. Its values turn out to depend strongly on w and we get $\sigma_8 = 1.12, 0.935$ and 0.534 for $w = -1, -2/3$ and $-1/3$, respectively.

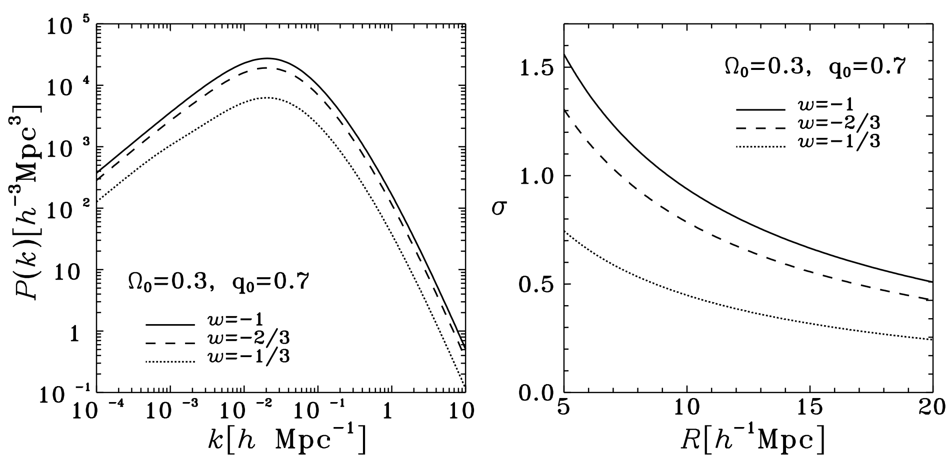

Fig. 6. The linear power spectra (left panel) and r.m.s. fluctuation as a function of smoothing scale (right panel) for flat models with different w.

6. Mass functions

One of the most important measures of structure formation is provided by the mass function of bound objects. Using the analytical prescription of Press and Schechter [23], we can estimate the cumulative mass function (the comoving number density of objects of mass grater than M)

$$N(>M) = \int_M^\infty n(M)\, dM \,, \tag{29}$$

where $n(M)$ is the number density of objects with mass between M and $M + dM$

$$n(M) = -\left(\frac{2}{\pi}\right)^{1/2} \frac{\varrho_{\rm b}}{M} \frac{\delta_{\rm c}}{\sigma^2} \frac{d\sigma}{dM} \exp\left(-\frac{\delta_{\rm c}^2}{2\sigma^2}\right) . \tag{30}$$

In the expression above, $\varrho_{\rm b}$ is the background density, $\delta_{\rm c}$ is the characteristic density discussed in Section 4 and given by equation (18), σ is the r.m.s. density fluctuation at comoving smoothing scale R described in Section 5. The mass is related to the smoothing scale by $M = 4\pi\varrho_{\rm b} R^3/3$.

Figure 7 shows the cumulative mass functions calculated from equation (30) with $\Omega_0 = 0.3$ and $q_0 = 0.7$ for three models with $w = -1, -2/3$ and $-1/3$. Comparison of the theoretical curves in figure 7 with data for rich clusters of galaxies [21] shows that for our choice of cosmological parameters the values of $w \approx -2/3$ are preferred. The Press–Schechter formulae are, however, known to underestimate the mass functions on the scale of clusters of galaxies when compared to N-body simulations so when more exact

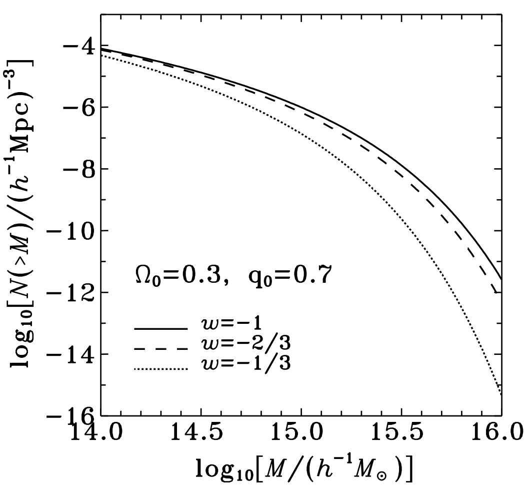

Fig. 7. The Press–Schechter cumulative mass functions for different w assuming $\Omega_0 = 0.3$ and $q_0 = 0.7$.

predictions are available this result is likely to become $w \geq -2/3$. The constraint on w from mass functions is potentially important since most other tests yield upper limits of w.

7. Constraints on w

The primary constraint on the value of w comes from the observations of accelerating expansion, which can only be obtained in models with $w < -1/3$. Age of Universe is quite sensitive to w and increases for lower w, however, the accuracy of our knowledge of both H_0 and t_0 is not good enough to put strong constraints on w. Current estimates are consistent with $w < -0.5$.

One of the strongest arguments for the existence of dark energy comes from the studies of Cosmic Microwave Background (CMB). Although its power spectrum is weakly sensitive to w (*e.g.* the height of the first acoustic peak is somewhat increased and its position is shifted to higher multipoles for lower w), it has already provided some limits. Combining the data from COBE and recent balloon experiments Balbi *et al.* [6] find $-1 < w < -0.6$ while Baccigalupi *et al.* [7] estimate the best-fitting value of w to be $w = -0.8$. The ongoing and future satellite experiments are expected to put even stronger limits on w.

Among the promising probes of dark energy are also supernovae Ia. The comoving distance they serve to measure is sensitive only to the interesting cosmological parameters and the errors related to supernova evolution or extinction are estimated to be small. The existing data restrict w only weakly [24], but future experiments like The Supernova Acceleration Probe are expected to measure the value of w with a few percent accuracy [25].

Structure formation also offers methods to constrain the cosmic equation of state. The suppression of linear growth of density fluctuations for higher w alone shows that only for $w < -1/2$ the structure observed today could have evolved from small initial perturbations deduced from CMB observations. The same range of acceptable w values follows from the behaviour of σ_8 which is a strongly decreasing function of w.

The most promising tests are based on the number counts of galaxy clusters. It turns out [26] that the slope of comoving abundance as a function of redshift depends sensitively on w and, therefore, can be used to break degeneracies between w and other cosmological parameters that appear *e.g.* in the analysis of CMB. Such measurements are expected to be performed using the proposed new X-ray and Sunyaev–Zeldovich effect surveys [27] and the ongoing DEEP Redshift Survey [28]. The constraints from structure formation appear to be complementary to those from supernovae and CMB measurements.

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