WHY AND HOW TO USE A DIFFERENTIAL EQUATION METHOD TO CALCULATE MULTI-LOOP INTEGRALS*

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A short pedagogical introduction to a differential method used to calculate multi-loop scalar integrals is presented. As an example it is shown how to obtain, using the method, large mass expansion of the two-loop sunrise master integrals.

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1. Introduction

Precision measurements have become one of the central issues in present particle physics allowing to test Standard Model and its extensions with unprecedented accuracy. To confront them with theoretical predictions it is necessary to know two- (or more) loop radiative corrections to the measured physical quantities. Despite enormous effort of the theoretical physics community in this field and existence of many partial results a universal method beyond first loop was not developed till now. While the planned linear colliders running in gigaZ mode will push the experimental accuracy even further.

One of the promising new directions in the field of multi-loop calculation is the differential equations method. A differential equations method, based on mass derivatives, has been proposed in [1]. In that approach, amplitudes with a single non vanishing mass m are expressed as a suitable integral of the

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corresponding massless amplitudes, which are taken as known. More systematic studies making use not only of masses, but also Lorentz invariants, as independent variables were initiated in [2] and then continued in [3–11]. In [3] small and large p^2 expansions were obtained for sunrise type master integrals. In [4] the same type of expansions was obtained for the two-loop two-point four denominator master integral. Analytical results for pseudothreshold [5] and threshold [6] expansions of the sunrise master integrals were obtained subsequently. Another application of the method is the calculation of the massless off shell double box contributing to $\gamma^* \to$ 3-jet process. The differential equations were presented in [7] and subsequently master integrals calculated for planar [9] and non-planar [10] topologies. In [11] and [12] efforts towards getting 4-loop corrections to $g-2$ and 2-loop corrections to Bhabha scattering correspondingly were started. The power of the method, besides being relatively simple and based mostly on algebraic manipulations, is that its mathematical basis was developed long ago. As it will be shown in Section 3 the master integrals ever satisfy a system of linear differential equations. Theory of such systems of differential equations was developed in XIX and at the beginning of XX century, so now it is a textbook knowledge (see for example [13]). All that helps a lot, making possible mathematical rigor without big effort.

This paper is organized as follows. In the next section some preliminary definitions are given and integration by part identities [14] shortly presented. In Section 3 it is shown how to get differential equations once the master integrals are identified. In Section 4 it is shown how to obtain large mass expansion of the sunrise master integrals. It is the first application of the differential equation method to the calculation of the mass expansions. In Section 5 a short summary is presented. All algebraic manipulations were performed using FORM [15].

2. Preliminaries

For a presentation how the differential equation method works in practice in a nontrivial case, but in the same time still not requiring large algebraic manipulations, we have chosen the two-point two-loop sunrise graph. A family of scalar integrals associated with that graph is defined by

$$\begin{aligned}&A(n,m_1^2,m_2^2,m_3^2,p^2,-\alpha_1,-\alpha_2,-\alpha_3,\beta_1,\beta_2)\\&=\int\frac{d^nk_1}{(2\pi)^{n-2}}\int\frac{d^nk_2}{(2\pi)^{n-2}}\,\frac{(p\cdot k_1)^{\beta_1}(p\cdot k_2)^{\beta_2}}{(k_1^2+m_1^2)^{\alpha_1}(k_2^2+m_2^2)^{\alpha_2}((p-k_1-k_2)^2+m_3^2)^{\alpha_3}}\,,\end{aligned} \tag{2.1}$$

where m_i $(i = 1,2,3\ )$ are the masses associated with internal lines, p is the external momentum, k_1, k_2 are loop momenta and α_i, $(i = 1,2,3)$,

$\beta_j,\quad (j=1,2)$ are integer numbers. The integrals are to be performed in n-dimensional Euclidean space. It implies we have used dimensional regularization and have performed Wick rotation. The scale parameter μ associated with dimensional regularization has been set to 1. Final results can be easily rewritten in Minkowski space by changing $p^2 \to -p^2$.

Not all of the integrals from the class (2.1) are independent. By means of a very simple but powerful method of integration by parts identities [14] one can find relations between them. In this particular case one uses the relations

$$\int d^n k_i \frac{\partial}{\partial(k_i)_\mu}\left[\frac{v_\mu \quad (p\cdot k_1)^{\beta_1}(p\cdot k_2)^{\beta_2}}{(k_1^2+m_1^2)^{\alpha_1}(k_2^2+m_2^2)^{\alpha_2}((p-k_1-k_2)^2+m_2^3)^{\alpha_3}}\right]=0\,, \tag{2.2}$$

where $i=1,2$, while v_μ denotes one of the momenta p, k_1 or k_2. It is crucial that in this way one gets a system of linear equations satisfied by the integrals (2.1) with a non-homogeneous terms given by integrals with lower number (in this case one) of denominators. This general property requires that one should start to solve a given problem from calculation of the integrals with smallest possible number of denominators (or if one is lucky enough, one can find them in the literature). The integrals to be calculated first for the sunrise problem can be expressed just by one integral

$$T\left(n,m^2\right)=\int\frac{d^n k}{(2\pi)^{n-2}}\frac{1}{k^2+m^2}=\frac{m^{n-2}}{(n-2)(n-4)}C(n)\,, \tag{2.3}$$

where

$$C(n)=\left(2\sqrt{\pi}\right)^{(4-n)}\Gamma\left(3-\frac{n}{2}\right)\qquad\text{and}\qquad C(4)=1\,. \tag{2.4}$$

With help of (2.2) one finds that only four independent integrals exist within the family (2.1) [16]. We choose them as

$$\begin{aligned}
F_0(n,m_1^2,m_2^2,m_3^2,p^2) &= A(n,m_1^2,m_2^2,m_3^2,p^2,-1,-1,-1,0,0)\,,\\
F_1(n,m_1^2,m_2^2,m_3^2,p^2) &= A(n,m_1^2,m_2^2,m_3^2,p^2,-2,-1,-1,0,0)\,,\\
F_2(n,m_1^2,m_2^2,m_3^2,p^2) &= A(n,m_1^2,m_2^2,m_3^2,p^2,-1,-2,-1,0,0)\,,\\
F_3(n,m_1^2,m_2^2,m_3^2,p^2) &= A(n,m_1^2,m_2^2,m_3^2,p^2,-1,-1,-2,0,0)\,.
\end{aligned} \tag{2.5}$$

Thereafter, we will not write explicitly the arguments of the functions $F_i(n,m_1^2,m_2^2,m_3^2,p^2)\equiv F_i$. The independence of the function F_i means that none of them can be expressed by a linear combination of the others and polynomials of the function T (2.3) with coefficients in the form of a ratio of polynomials in p^2, m_1^2, m_2^2, m_3^2. However, an obvious relation occurs

$$F_i = -\frac{\partial}{\partial m_i^2} F_0 \,, \qquad i = 1,2,3. \tag{2.6}$$

3. How to get differential equations for master integrals

Having a limited number of integrals to deal with, which we will call master integrals, we can write differential equations they obey. It is as simple as to differentiate the given integral and then by means of the integration by part identities express the result by the master integrals and the known function T. To illustrate as it works we write

$$\begin{aligned} & p^2 \frac{\partial}{\partial p^2} F_0 \\ & = \frac{1}{2} p_\mu \frac{\partial}{\partial p_\mu} \int \frac{d^n k_1 d^n k_2}{(2\pi)^{2n-4}} \frac{1}{(k_1^2+m_1^2)(k_2^2+m_2^2)((p-k_1-k_2)^2+m_3^2)} \\ & = \int \frac{d^n k_1 d^n k_2}{(2\pi)^{2n-4}} \frac{-p^2 + p\cdot k_1 + p\cdot k_2}{(k_1^2+m_1^2)(k_2^2+m_2^2)((p-k_1-k_2)^2+m_3^2)^2} \,. \end{aligned} \tag{3.1}$$

The last expression is nothing but a linear combination of three integrals from the family (2.1), so one can express each of them as a linear combination of the master integrals. This gives after a short algebraic calculation [3]

$$p^2 \frac{\partial}{\partial p^2} F_0 = (n-3) F_0 + m_1^2 F_1 + m_2^2 F_2 + m_3^2 F_3 \,. \tag{3.2}$$

Similarly one finds [3]

$$p^2 D(m_1^2, m_2^2, m_3^2, p^2) \frac{\partial}{\partial p^2} F_i = \sum_{j=0}^{3} M_{i,j} F_j + T_i \,, \quad i = 1,2,3 \tag{3.3}$$

where explicit form of functions T_i (which can be expressed by the function T (2.3)) and $M_{i,j}$ (polynomials of p^2, m_1^2, m_2^2, m_3^2) can be found in [3]. The function D is defined by

$$\begin{aligned} D(m_1^2, m_2^2, m_3^2, p^2) \; = \; & \left(p^2 + (m_1+m_2+m_3)^2\right) \left(p^2 + (m_1+m_2-m_3)^2\right) \\ & \times \left(p^2 + (m_1-m_2+m_3)^2\right) \left(p^2 + (m_1-m_2-m_3)^2\right) \,, \end{aligned} \tag{3.4}$$

and is equal to zero at the three pseudothresholds and at the threshold of the master integrals.

Differential equations with m_i as an independent variables can be found in a similar way. They are presented in the Appendix of [3] or can be obtained from the formulae presented there by appropriate permutations of the masses $m_i, \quad i = 1, 2, 3$.

The property that the derivative of a given master integral is a linear combination of the master integrals plus terms with smaller number of denominators is obviously a general property of all possible multi-loop scalar integrals. It is valid due to linearity of the integration by parts identities and the differentiation operation itself, and also due to the form of the integrands, which are ever ratios of polynomials of masses and Lorentz invariants.

4. An application: Large mass expansion of the master integrals

Let us assume that one of the square of the masses, say m_3^2, is much larger then m_1^2, m_2^2 and $|p^2|$. The general form of the expansion in that region can be written [13] as

$$F_0 = \sum_{\alpha \in A} (m_3^2)^\alpha \sum_{k=0}^{\infty} F_k^{(\alpha)} \frac{1}{(m_3^2)^k} \,, \tag{4.1}$$

where A is a finite set of numbers, whose differences are not equal to an integer number. Other master integrals are related to F_0 by (2.6). The allowed values of α can be found from the system of equations (with m_3^2 as an independent variable) itself. One substitutes the F_i in the system with its expansions and by examining the coefficients of the highest powers in m_3^2 (they have to be equal to zero) one finds allowed values of α's. In this particular case

$$A = \left\{ -1, \ \ \frac{n-4}{2}, \ \ n-3, \ \ \frac{3}{2}n - 4 \right\} \equiv \left\{ r, \ \ s_1, \ \ s_2, \ \ s_3 \right\}, \tag{4.2}$$

where a shorthand notation was introduced for different values of α's. The series with an integer power of α is called the regular series, while the other are called singular series. The singular parts are sources of logarithmic terms when expanded around $n = 4$, the value of n we are interested in.

Not always all parts of the expansion corresponding to allowed values of α's are actually present in the solution. That depends on the initial conditions and the regularity of the F_i at $p^2 = 0$ is crucial for their properties [3]. We will see that also in the presented below example. It reflects the fact that the differential equations can be satisfied by a wider class of functions, not only by the master integrals. It means also that usually one has to calculate the master integrals for special values of the parameters by other means then the differential equations to fix constants of integration. That, however, is always simpler then the general case.

Having the allowed values of α one can try to calculate coefficients in the expansion (4.1). The crucial point is to find the first coefficient in each of the series as the others can be found by solving a system of linear algebraic equations (in this case system of 3 linear equations). Two of the coefficients are fixed by non-homogeneous terms in the differential equations and read

$$F_0^r = \frac{C^2(n)}{(n-4)^2(n-2)^2}(m_1^2 m_2^2)^{\frac{n-2}{2}} \quad , \tag{4.3}$$

$$F_0^{s_1} = -\frac{C^2(n)}{((n-4)(n-2))^2}\left((m_1^2)^{\frac{n-2}{2}} + (m_2^2)^{\frac{n-2}{2}}\right). \tag{4.4}$$

The other two cannot be fixed this way as in the non-homogeneous part of the equations there is no term $\sim (m_3^2)^{s_2}$ or $\sim (m_3^2)^{s_3}$. One can find, however, the following relations between the next to leading and the leading terms in the expansion

$$\begin{aligned} F_1^{s_2} &= -(n-3)\left[(m_1^2+m_2^2+p^2) - \frac{4}{n}p^2\right]F_0^{s_2} \\ &\quad + \frac{4}{n}(n-3)m_1^2(m_1^2+p^2)\frac{\partial}{\partial m_1^2}F_0^{s_2} \\ &\quad + \frac{4}{n}(n-3)m_2^2(m_2^2+p^2)\frac{\partial}{\partial m_2^2}F_0^{s_2} \\ \frac{\partial}{\partial m_1^2}F_1^{s_2} &= (n-3)\left[(p^2-m_1^2+m_2^2)\frac{\partial}{\partial m_1^2}F_0^{s_2} - F_0^{s_2}\right] \\ \frac{\partial}{\partial m_2^2}F_1^{s_2} &= (n-3)\left[(p^2+m_1^2-m_2^2)\frac{\partial}{\partial m_2^2}F_0^{s_2} - F_0^{s_2}\right]. \end{aligned} \tag{4.5}$$

Using differential equations (3.2) and (3.3), the expression (4.1) and the above relations one finds that $F_0^{s_2}$ satisfies the following system of differential equations

$$-m_1^2\frac{\partial}{\partial m_1^2}F_0^{s_2} - m_2^2\frac{\partial}{\partial m_2^2}F_0^{s_2} - p^2\frac{\partial}{\partial p^2}F_0^{s_2} = 0\,, \tag{4.6}$$

$$\begin{aligned} \frac{n-2}{2}\frac{\partial}{\partial m_1^2}F_0^{s_2} + p^2\frac{\partial}{\partial m_1^2}\frac{\partial}{\partial p^2}F_0^{s_2} &= 0\,, \\ \frac{n-2}{2}\frac{\partial}{\partial m_2^2}F_0^{s_2} + p^2\frac{\partial}{\partial m_2^2}\frac{\partial}{\partial p^2}F_0^{s_2} &= 0\,. \end{aligned} \tag{4.7}$$

From this system one can deduce, eliminating m_i^2 derivatives, that $F_0^{s_2}$ fulfills a very simple differential equation

$$-p^2\frac{\partial^2}{\partial(p^2)^2}F_0^{s_2} - \frac{1}{2}n\frac{\partial}{\partial p^2}F_0^{s_2} = 0\,. \tag{4.8}$$

Its solution has the following form

$$F_0^{s_2} = \frac{2S_2(n,m_1^2,m_2^2)}{n-2}(p^2)^{-\frac{n-2}{2}} + S_1(n,m_1^2,m_2^2)\,, \tag{4.9}$$

where $S_i(n,m_1^2,m_2^2),\ i=1,2$ are still unknown functions. Using (4.7) and (4.9) one finds

$$\frac{\partial}{\partial m_1^2}S_1(n,m_1^2,m_2^2) = \frac{\partial}{\partial m_2^2}S_1(n,m_1^2,m_2^2) = 0\,, \tag{4.10}$$

so the function S_1 does not depend on masses: $S_1(n,m_1^2,m_2^2) = S_1(n)$. This information together with (4.5) and (4.6) gives $S_2(n,m_1^2,m_2^2) = 0$. It means that $F_0^{s_2}$ is a function of n only. Its value will be found later on.

Similar, but even simpler, analysis can be done for $\alpha = s_3$. From the differential equations with m_3 as an independent variable one finds that $F_0^{s_3}$ does not depend on the masses. Using that information, from differential equations (3.2) and (3.3) one finds

$$p^2\frac{\partial}{\partial p^2}F_0^{s_3} = -\frac{1}{2}(n-2)F_0^{s_3}\,, \tag{4.11}$$

which gives

$$F_0^{s_3} = S(n)(p^2)^{-\frac{n-2}{2}}\,, \tag{4.12}$$

where $S(n)$ is a function depending only on n. As the master integrals for $n=4$ are analytic functions at $p^2=0$, the function $S(n)$ and consequently $F_0^{s_3}$ have to be identically equal to zero. As all higher order coefficients $F_i^{s_3} \sim F_0^{s_3},\ i=1,\cdots,$ the whole series with $\alpha = s_3$ vanishes.

The only unknown function $F_0^{s_2}(n)$ can be found using known analytical result for $F_0(n,0,0,m^2,p^2)$ [3]. Performing its expansion for large m^2 and comparing the appropriate terms with (4.1),where two masses were set to zero, one finds

$$F_0^{s_2}(n) = \frac{C^2(n)}{16}\left[-\frac{2}{(n-4)^2} + \frac{3}{(n-4)} - \left(\frac{7}{2}+\zeta_2\right) + O(n-4)\right]. \tag{4.13}$$

As $F_0(n,0,0,m^2,p^2)$ was given [3] in a form of an expansion around $n=4$ the function $F_0^{s_2}(n)$ is given only in the form of the $n=4$ expansion. As this was the last missing part of the expansion one can find now the complete formula. We report here only leading terms of the expansion, but higher order terms can be easily found by algebraic means, if necessary.

$$\begin{aligned}
F_0(n,m_1^2,m_2^2,m_3^2,p^2) = &\quad \text{pole terms in } (n-4) \\
&+C^2(n)\Bigg\{ -\frac{1}{16}m_3^2\log^2(m_3^2) + \frac{3}{16}m_3^2\log(m_3^2) - \frac{1}{16}m_3^2\left(\frac{7}{2}+\zeta(2)\right) \\
&+\frac{1}{32}\left(m_1^2+m_2^2\right)\log^2(m_3^2) \\
&+\frac{1}{32}\left(2m_1^2+2m_2^2+p^2-2m_1^2\log(m_1^2)-2m_2^2\log(m_2^2)\right)\log(m_3^2) \\
&+\frac{1}{128}\Big(-3p^2+16m_1^2\log(m_1^2)+16m_2^2\log(m_2^2)-4m_1^2\log^2(m_1^2) \\
&-4m_2^2\log^2(m_2^2)+(8\zeta(2)-20)(m_1^2+m_2^2)\Big)+O\left(\frac{1}{m_3^2},n-4\right)\Bigg\}. \quad (4.14)
\end{aligned}$$

The pole terms being identical to the exact pole terms [3] are not reported here. As a cross check we have compared the above result with the small p^2 expansion of $F_0(n,m_1^2,m_2^2,m_3^2,p^2)$. We have expanded the first two coefficients of the expansion in p^2 [3], dependent on m_1,m_2,m_3, assuming $m_3^2 \gg m_1^2,m_2^2$ and found complete agreement between the two results. The result (4.14), which is valid for arbitrary p^2,m_1^2 and m_2^2 provided they are much smaller then m_3^2, cannot be, however, deduced from the small p^2 expansion itself.

5. Summary

A short introduction to the differential equations method used in calculation of the scalar multi-loop integrals was presented. A nontrivial large mass expansion of the master two-loop sunrise integrals was obtained almost completely by algebraic means. The only 'difficult' task, besides solving systems of linear algebraic equations, was to solve two simple differential equations (4.8) and (4.11). It shows that the method is extremely powerful and opens new possibilities in the field of multi-loop calculations.

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