QUANTUM-LIKE APPROACH TO FINANCIAL RISK: QUANTUM ANTHROPIC PRINCIPLE*

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(Received October 12, 2001)

We continue the analysis of quantum-like description of market phenomena and economics. We show that it is possible to define a risk inclination operator acting in some Hilbert space that has a lot of common with quantum description of the harmonic oscillator. The approach has roots in the recently developed quantum game theory and quantum computing. A quantum anthropic principle is formulated.

PACS numbers: 02.50.Le, 03.67.–a, 03.65.Bz

1. Noncommutative quantum mechanics

The term *noncommutative quantum mechanics* is misleading as it suggests that there might exist something like commutative quantum mechanics. In fact quantum theory possesses various types of noncommutativity built in since its infancy. Nowadays the adjective "noncommutative" reflects the possibility that the space variables $\hat{x}^i$ might not commute. This sort of generalizations has its roots in M-theory (branes), deformation quantization, Weyl quantization and curiosity [1]. This is, of course, connected with

* Presented at the XXV International School of Theoretical Physics "Particles and Astrophysics — Standard Models and Beyond", Ustroń, Poland, September 10–16, 2001.

the Connes program [2–4]. Among various possible generalizations the most popular one are

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}\,,\ \theta^{ij} \in \mathbb{C}, \tag{1a}$$
$$[\hat{x}^i, \hat{x}^j] = iC^{ij}{}_k \hat{x}^k\,,\ C^{ij}{}_k \in \mathbb{C}, \tag{1b}$$
$$\hat{x}^i \hat{x}^j = q^{-1} \hat{R}^{ij}{}_{kl} \hat{x}^k \hat{x}^l\,,\ \hat{R}^{ij}{}_{kl} \in \mathbb{C}. \tag{1c}$$

In all these cases the index i takes values from 1 to N. We shall suppose that the appropriate algebra A_x has a unit element [2,3]. Eqs. (1b) and (1c) describe the so called Lie algebraic and quantum group generalizations, respectively. Here we would be interested in a generalization of the form given by Eq. (1a) which may be traced back to works of Weyl [5] and Moyal [6].

2. Quantum market games

It was a complete surprise for us to find the University of Bologna preprint entitled *Quantum Mechanics and Mathematical Economics are Isomorphic* and written by Lambertini [7]. When the preprint appeared we were actually working on the direct application of recently developed quantum game theory [8–10] in economics [11–16]. The Lambertini's paper was a further incentive to work harder. In the "standard" quantum game theory one tries in some sense to quantize an operational description of "classical" versions of the game being analyzed. It usually enlarges the set of admissible strategies in a nontrivial way. We follow a different route. The market players strategies are described in terms of state vectors $|\psi\rangle$ belonging to some Hilbert space $\mathcal{H}$ [12,13,15]. The probability densities of revealing the players', say Alice and Bob, intentions are described in terms of random variables p and q:

$$\frac{|\langle q|\psi\rangle_{\mathrm{A}}|^2}{{}_{\mathrm{A}}\langle\psi|\psi\rangle_{\mathrm{A}}} \frac{|\langle p|\psi\rangle_{\mathrm{B}}|^2}{{}_{\mathrm{B}}\langle\psi|\psi\rangle_{\mathrm{B}}}\, dq dp\,, \tag{2}$$

where $\langle q|\psi\rangle_{\mathrm{A}}$ is the probability amplitude of offering the price q by Alice who wants to buy and the demand component of her state is given by $|\psi\rangle_{\mathrm{A}} \in \mathcal{H}_A$. Bob's amplitude $\langle p|\psi\rangle_{\mathrm{B}}$ is interpreted in an analogous way (opposite position). Of course, the "intentions" q and p usually do not result in the accomplishment of the transaction [13]. If one considers the following facts [11,17,18]:

- error theory: second moments of a random variable describe errors,
- M. Markowitz's portfolio theory (Nobel Prize 1990),
- L. Bachelier's theory of options: the random variable $q^2 + p^2$ measures joint risk for a buying-selling stock transaction (Merton & Scholes won Nobel Prize in 1997)

then it seems reasonable to define an observable *the risk inclination operator*:

$$H(\mathcal{P}_k, \mathcal{Q}_k) := \frac{(\mathcal{P}_k - p_{k0})^2}{2\,m} + \frac{m\,\omega^2(\mathcal{Q}_k - q_{k0})^2}{2}\,, \tag{3}$$

where $p_{k0} := \frac{{}_k\langle\psi|\mathcal{P}_k|\psi\rangle_k}{{}_k\langle\psi|\psi\rangle_k}$, $q_{k0} := \frac{{}_k\langle\psi|\mathcal{Q}_k|\psi\rangle_k}{{}_k\langle\psi|\psi\rangle_k}$, $\omega := \frac{2\pi}{\theta}$. The variable θ denotes characteristic time of transaction [11] which is, roughly speaking, an average time span between two opposite moves of a player (*e.g.* buying and selling the same asset). The parameter $m > 0$ measures the risk asymmetry between buying and selling positions. Analogies with quantum harmonic oscillator allow for the following characterization of quantum market games. The constant h_E describes the minimal inclination of the player to risk. It is equal to the product of the lowest eigenvalue of $H(\mathcal{P}_k, \mathcal{Q}_k)$ and 2θ. 2θ is in fact the minimal interval during which it makes sense to measure the profit. Except the ground state all the adiabatic strategies $H(\mathcal{P}_k, \mathcal{Q}_k)|\psi\rangle =$ const.$|\psi\rangle$ are giffens [11,13] that is goods that do not obey the law of demand and supply. It should be noted here that in a general case the operators $\mathcal{Q}_k$ do not commute because traders observe moves of other players and often act accordingly. One big bid can influence the market at least in a limited time spread. Therefore, it is natural to apply the formalism of noncommutative quantum mechanics where one considers

$$[x^i, x^k] = i\Theta^{ik} := i\Theta\,\varepsilon^{ik}\,. \tag{4}$$

The analysis of harmonic oscillator in more than one dimension [18] imply that the parameter Θ modifies the constant $\hbar_E \to \sqrt{\hbar_E^2 + \Theta^2}$ and, accordingly, the eigenvalues of $H(\mathcal{P}_k, \mathcal{Q}_k)$. This has the natural interpretation that moves performed by other players can diminish or increase one's inclination to taking risk.

3. Market as a measuring apparatus

When game allows a great number of players it is useful to consider it as a two-players game: the trader $|\psi\rangle_k$ against the Rest of the World (RW). The concrete algorithm $\mathcal{A}$ may allow for an effective strategy of RW (for a sufficiently large number of players the single player strategy should not influence the form of the RW strategy). If one considers the RW strategy it makes sense to declare its simultaneous demand and supply states because for one player RW is a buyer and for another it is a seller. To describe such situation it is convenient to use the Wigner formalism. The following subsection describe shortly various aspects of quantum markets.

3.1. Quantum Zeno effect

If the market continuously measures the same strategy of the player, say the demand $\langle q|\psi\rangle$, and the process is repeated sufficiently often for the whole market, then the prices given by some algorithm do not result from the supplying strategy $\langle p|\psi\rangle$ of the player. The necessary condition for determining the profit of the game is the transition of the player to the state $\langle p|\psi\rangle$. If many of the players simultaneously change their strategies then the quotation process may collapse due to the lack of opposite moves. In this way the quantum Zeno [20] effect explains stock exchange crashes. Another example of the quantum market Zeno effect is the stabilization of prices of an asset provided by a monopolist.

3.2. Eigenstates of $\mathcal{Q}$ and $\mathcal{P}$

Let us suppose that the amplitudes for the strategies $\langle q|\psi\rangle_k$ or $\langle p|\psi\rangle_k$ that have infinite integrals of squares of their modules, ($\langle q|\psi\rangle_k \notin L^2$) have the natural interpretation as the will of the k-th player of buying (selling) the amount d_k (s_k) of the asset $\mathfrak{G}$. So the strategy $\langle q|\psi\rangle_k = \langle q|a\rangle = \delta(q,a)$ means that in the case of classifying the player to the set $\{k_d\}$, refusal of buying cheaper than at $c = e^a$ and the will of buying at any price equal or above e^a. In the case of "measurement" in the set $\{k_d\}$ the player declares the will of selling at any price. The above interpretation is consistent with the Heisenberg uncertainty relation. The strategies $\langle q|\psi\rangle_2 = \langle q|a\rangle$ (or $\langle p|\psi\rangle_2 = \langle p|a\rangle$) do not correspond to the RW behavior because the conditions $d_2, s_2 > 0$, if always fulfilled, allow for unlimited profits (the readiness to buy or sell $\mathfrak{G}$ at any price). The demand and supply functions give probabilities of coming off transactions in the game when the player uses the strategy $\langle p|\mathrm{const.}\rangle$ or $\langle q|\mathrm{const.}\rangle$ and RW, proposing the price, uses the strategy ρ.

3.3. Correlated coherent strategies

We will define correlated coherent strategies as the eigenvectors of the annihilation operator $\mathcal{C}_k$

$$\mathcal{C}_k(r,\eta) := \frac{1}{2\eta}\Big(1 + \frac{ir}{\sqrt{1-r^2}}\Big)\mathcal{Q}_k + i\eta\mathcal{P}_k\,, \tag{5}$$

where r is the correlation coefficient $r \in [-1,1]$, $\eta > 0$. In these strategies buying and selling transactions are correlated and the product of dispersions fulfills the Heisenberg-like uncertainty relation $\Delta_p\Delta_q\sqrt{1-r^2} \geq \frac{\hbar_E}{2}$ and is minimal. The annihilation operators $\mathcal{C}_k$ and their eigenvectors may be parameterized by $\Delta_p = \frac{\hbar_E}{2\eta}$, $\Delta_q = \frac{\eta}{\sqrt{1-r^2}}$ and r.

3.4. Mixed states and thermal strategies

According to classics of game theory the biggest choice of strategies is provided by the mixed states $\rho(p,q)$. The most interesting among them are the thermal ones. They are characterized by constant inclination to risk, $E(H(\mathcal{P},\mathcal{Q}))$ = const. and maximal entropy. The Wigner measure for the n-th exited state of harmonic oscillator have the form

$$W_n(p,q)dpdq = \frac{(-1)^n}{\pi\hbar_E}\,\mathrm{e}^{-\frac{2H(p,q)}{\hbar_E\omega}}\,L_n(\frac{4H(p,q)}{\hbar_E\omega})\,dpdq\,, \tag{6}$$

where L_n is the n-th Laguerre polynomial. The mixed state ρ_β determined by the Wigner measures $W_n dpdq$ weighted by the Gibbs distribution $w_n(\beta) := \frac{\mathrm{e}^{-\beta n\hbar_E\omega}}{\sum_{k=0}^{\infty}\mathrm{e}^{-\beta k\hbar_E\omega}}$ have the form

$$\begin{aligned}\rho_\beta(p,q)dpdq :&= \sum_{n=0}^{\infty} w_n(\beta)W_n(p,q)dpdq\\ &= \frac{\omega}{2\pi}\,x\,\mathrm{e}^{-xH(p,q)}\Big|_{x=\frac{2}{\hbar_E\omega}\tanh(\beta\frac{\hbar_E\omega}{2})}dpdq.\end{aligned}$$

3.5. Market cleared by quantum computer

When the algorithm $\mathcal{A}$ calculating in a separable Hilbert space H_k does not know the players strategies it must choose the basis in an arbitrary way. This may result in arbitrary long representations of the amplitudes of strategies. Therefore, the algorithm $\mathcal{A}$ should be looked for in the NP (Non-Polynomial) class and quantum markets may be formed provided the quantum computation technology is possible. Is the quantum arbitrage possible only if there is a unique correspondence between $\hbar$ and h_E? Was the hypothetical evidence given by Robert Giffen in the British Parliament the first ever description of quantum phenomenon? The commonly accepted universality of quantum theory should encourage physicist in looking for traces quantum world in social phenomena.

4. Conclusions

We think that the formalism of quantum theory may provide us with tools of unexpected power that combined with methods of game theory may allow for much deeper understanding of financial phenomena than it is usually expected. Let us quote the Editor's Note to Comlexity Digest 2001.27(4) (`http://www.comdig.org`) "It might be that while observing the due ceremonial of everyday market transaction we are in fact observing capital flows resulting from quantum games eluding classical description. If human decisions can be traced to microscopic quantum events one would expect that nature would have taken advantage of quantum computation in evolving complex brains. In that sense one could indeed say that quantum computers are playing their market games according to quantum rules". During the past decade options gained a significant position in capital turnover all over the world. It may mean that simple methods of maximization of profit became unsatisfactory. At present minimization of financial risk is playing the key role. Lower risk means better prognoses for industrial and market development and this results in higher profits. Methods of minimazing the financial risk will become a focus of investors attention. All this tempts us to formulate the *quantum anthropic principle* of the following form. *At earlier civilization stages markets are governed by classical laws (as classical logic prevailed in reasoning) but the incomparable efficacy of quantum algorithms in multiplying profits*[1] *will result in quantum behavior prevailing over the classical one.* We envisage markets cleared by quantum algorithms (computers) [11, 21].

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