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THE LUND FRAGMENTATION OF A MULTIGLUON STRING STATE*

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I will present a new fragmentation model for a multigluon string state that will exactly fulfil the Area Law that is at the basis of the original (1+1)-dimensional Lund Model. This means that I will have to briefly discuss string motion, in particular the description of the general string. As the string surface is a minimal surface it is mathematically completely determined by its boundary curve and I will show how to use the symmetries of string dynamics to devise a process along this boundary curve. I will also show that the new model is closely related to the T functional and the λ measure that we have repeatedly used in investigations in the Lund Group.

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1. Introduction

In this talk I will present a model that we have recently developed [1], for the fragmentation of a multigluon string state into final state hadrons. The original Lund String Fragmentation Model was developed many years ago, [2, 3], and as implemented in the well-known Monte Carlo simulation program JETSET, [4], it has been very successful in reproducing experimental data from all high energy multi-particle processes. Our reason to come back to the model and to extend it is to be able to implement the Area Law in a more precise manner.

The Lund Model Area Law stems from a few general assumptions: the final state particles are produced in the breakup of a string-like force field spanned between the colored constituents, there is causality and Lorentz

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invariance and the production of the particles can be described in terms of a stochastic process obeying a saturation assumption. The results of [2] (*cf.* also [5]) are derived for events with a quark (q, color-3) and an antiquark ($\bar{q}$, color-$\bar{3}$) at the endpoints of the string and when there are no interior gluonic (g, color-8) excitations in the centre. The Area Law then describes the (non-normalised) probability to produce an n-particle final state of hadrons with energy momenta $\{p_j\}$ and masses $\{m_j\}$:

$$dP_n(\{p_j\};P_{\rm tot}) = \prod_{j=1}^{n} N_j d^2p_j \delta\left(p_j^2 - m_j^2\right) \delta\left(\sum_{j=1}^{n} p_j - P_{\rm tot}\right) \exp(-bA)\,. \quad (1)$$

Here A is the area spanned by the string "before" the breakup, $P_{\rm tot}$ the total energy momentum of the state and $\{N_j\}$ and b parameters related to the density of hadronic states and the breakup properties of the string field, respectively.

Shortly after the original derivation of the Area Law [2], Sjöstrand [6], provided an implementation of the model applicable also for multi-gluon states, *i.e.* when the string surface is no longer flat but geometrically bent due to the internal excitations. The method of Sjöstrand is to project the positions of the breakup points (the vertices) from the (flat) (1+1)-dimensional model as given by Eq. (1) onto the surface of the bent string. The projection is done so that the proper times of the vertices and the energy-momentum in the string between them are the same. Unfortunately this method does not fulfil the Area Law on the bent surface because it is a geometrical fact that the areas "below the vertices" are not invariant under such a projection from a flat to a bent surface. Although the Area Law is not fulfilled on an event to event basis by the method in Ref. [6] it seems to be fulfilled in an average sense, *i.e. the predicted inclusive distributions are little affected by the differences* as we show in [1].

In the new model [1], we present another method for particle production in multi-gluon states which does fulfil the Area Law at every single step in the production process. It is then necessary to face a set of problems in the definition of the states that we apply the process to. We note that the states defined by perturbation theory are resolved only to the scale of some virtuality cutoff. We found that our method provides a set of excitations on the scale of the hadronic mass in the string field but time and space will not allow me to go into the details. At the time of writing this report we know a lot more about the properties of both the perturbative cascades and of these "soft Coulomb gluons", [7] but that will be reported at future meetings.

The states of the massless relativistic string fulfils a minimum principle, *i.e.* the surface spanned by the string during a period of motion is a minimal surface. This means on the one hand that the states should be stable against small-scale variations and on the other hand that the surface is fully determined by the boundary curve. In this case the boundary curve corresponds to the orbit of one of the endpoints, conventionally the q-endpoint. Therefore *the process we are going to define is a process along this curve*, to be called the directrix curve, $\mathcal{A}_\mu$.

The directrix curve is completely defined by the multigluon state and can be obtained by laying out the energy momentum vectors of the partons in color order, *i.e.* $k_1, k_2 \ldots k_n$ with index 1 the original q and index n the $\bar{q}$ vectors. In this talk I will treat the partons as massless particles (implying that the directrix curve has an everywhere light-like tangent) although both the process and the directrix curve can be defined for a general case with massive quarks.

The results of the new fragmentation process can be described by means of a hadronic curve, the X-curve, made up by laying out the hadronic energy momentum vectors $p_1, p_2 \ldots p_N$ in rank order (rank is defined in such a way that the first rank particle contains the original q and a $\bar{q}$ from the first production place, the second rank from the q from the first and a $\bar{q}$ from the second production place, *etc.*).

The relationship between the directrix curve $\mathcal{A}$ and the hadronic curve X can in an intuitive way be described as the building of four-sided plaquettes in between the curves. Such a plaquette will be bordered by one hadron vector p_j from the X-curve, one "original" and one "final" vertex vector, x_{j-1} and x_j, stretching from the beginning and the end of the p_j to the directrix curve and one piece $\delta\mathcal{A}_j$ from the directrix in such a way that:

$$x_{j-1} + \delta\mathcal{A}_j = p_j + x_j \,. \tag{2}$$

The interpretation is that the hadron obtains its energy momentum p_j from a "new" part of the directrix, $\delta\mathcal{A}_j$ and from the original vertex vector x_{j-1} that contains "the memory" of the earlier directrix parts and then the "new" remainder is brought forward through the final vertex vector x_j to the next plaquette, *etc.*

The sum of the areas of the plaquettes corresponds to the area A in the Area Law, Eq. (1) and the plaquette building process *per se* can be made in a one-to-one correspondence to the factorisation into the transfer operators as it is presented in [5]. The production process of a particle p_j in between the two vertex vectors x_{j-1} and x_j can, in the same way, be described in terms of harmonic oscillator wave functions.

The Area Law in Eq. (1) is derived by means of semi-classical considerations but the result is nevertheless similar to the one expected for a quantum mechanical transition probability, *i.e.* it is the final state phase space multiplied by a possible squared matrix element, in this case the negative area exponential. I have presented reasons for such a transition matrix in earlier talks here in Zakopane and also shown that the parameters obtained in the model can be derived from an assumption of gauge invariance and the corresponding Wilson loop integrals, *cf.* [3]. The above-mentioned diagonalisation process in Ref. [5] of the transfer operators would in a quantum mechanical language correspond to a description in terms of density operators.

One property that can be derived from the Area Law in Eq. (1) is that the average decay region is bordered by a typical hyperbola. The final state hadrons in our process will on the average be produced in the same way albeit this time along a set of connected hyperbolas. In Ref. [8] we have defined such an average curve and I will call it the $\mathcal{X}$-curve. Just as a simple hyperbola has a length proportional to the hyperbolical angle that it spans (in practical terms this corresponds to the available rapidity range in a two-jet of hadrons stemming from the simple (1+1)-dimensional $(q\bar{q})$-jets discussed in Section 2) the $\mathcal{X}$-curve has a length corresponding to a generalised rapidity variable, usually called λ, [3, 8]. The $\mathcal{X}$-curve is defined in terms of differential equations and I will show the close relationship between the $\mathcal{X}$-curve and the new fragmentation process, in this case defined in the limit of a vanishing mass.

There are several reasons to build the new fragmentation model. One is to compare the precise implementation of the Area Law to the approximate process in Ref. [6] and this we did in [1] and will also do in future publications.

Another reason is to get a handle on the general structure of fragmentation, in particular to be able to treat also the multigluon fragmentation states by the analytical methods introduced in Ref. [5]. This is of particular interest for the transition region, *i.e.* the region in between where we expect perturbation theory to work and where we know that the nonperturbative fragmentation sets in. Results of this kind is on the way to publication, [7].

A final reason is to investigate the stability of the states in QCD under fragmentation, *i.e.* given a multigluon state defined according to the rules of perturbation theory (with virtuality cutoffs as mentioned above) to find out to what extent it can be modified so that the observable results after fragmentation still are in agreement with the experiments. In the Lund interpretation of fragmentation where the particles stem from the energy of the force field it is evident that modifications of the perturbative state below and up to the scale of the hadronic masses should have no effects.

I will be satisfied to discuss e^+e^--annihilation events with a connected final state string (*i.e.* the gluon splitting process is neglected into $(q\bar{q})$ states) and only consider a single kind of hadron with the mass m. I will also neglect the transverse momentum fluctuations stemming from the tunnelling of the $(q\bar{q})$ states in the force field.

In the next section I will briefly recall some necessary formulas and exhibit the factorisation properties obtained in [5]. After that I will (after a brief introduction to string motion) show the plaquette building process in more detail. In [1] we introduced the "the corner crossing process" that corresponds to the introduction of the local deviations that I called Coulomb gluons above but I will not have time to cover this process. I will, however, show the relationship between the hadronic X-curve and the $\mathcal{X}$-curve mentioned above and then I will show that it is possible to find the directrix curve $\mathcal{A}$ from a knowledge of the hadronic X-curve (this is a realisation of the Hadron–Parton Duality introduced by the Leningrad Group). I will end with a few comments on future and ongoing work. I apologise for not providing any pictures in this description of the talk but I have instead provided many references where you can find extensive pictorial descriptions of what I have said.

2. The (1+1)-dimensional Lund Model and how to diagonalise it

The Lund Model contains a non-trivial interpretation of the QCD force-field in terms of the massless relativistic string with the quarks (q) and antiquarks ($\bar{q}$) at the endpoints and the gluons (g) as internal excitations on the string field. It is assumed that the force field can break up into smaller parts in the fragmentation process by the production of new ($q\bar{q}$) -states (*i.e.* new endpoints). A q from one such breakup point ("vertex") can together with a $\bar{q}$ from an adjacent vertex along the string field form a hadron composed of the pair and the field in between (all hadrons are in the model taken to be on the mass-shell).

For the simple case when there are no gluons the string field only corresponds to a constant force field (with a phenomenological size $\kappa \simeq 1$ GeV/fm) spanned between the original $q\bar{q}$-pair. In a semi-classical picture energy-momentum conservation allows that a new massless pair may be produced in a vertex-point along the field. The pair will then go apart along opposite lightcones, thereby using up the energy in the field in between (in this way the confined fields always will end on the charges). In order that the hadron produced between two adjacent vertices should have a positive mass it is necessary that the vertices are placed in a spacelike manner with respect to each other. Time-ordering will consequently be a frame-dependent statement (a little thought tells us that in any Lorentz frame the slowest particles will be the first to be produced, thereby fulfilling the necessary requirements in a Landau–Pomeranchuk formation time scenarium). It is possible to order the production process instead along the lightcones and introduce the notion of rank in the way I discussed it above. It is, of course, possible to introduce a rank-ordering also from the end containing the original $\bar{q}$.

One obtains, [2,3,5], the unique process described by Eq. (1) from these observations and an assumption that the breakup process obeys a saturation assumption, *i.e.* that after very many steps, when we are far from the endpoints, the proper times of the vertices will be distributed according to an energy-independent distribution.

A particular feature is that if a particle with energy momentum $p = (p_+, p_-)$ and with squared mass $m^2 = p^2 = p_+p_-$ is produced in between the two vertices with $x = (x_+, x_-)$ and $x' = (x'_+, x'_-)$ then we have

$$\begin{aligned} p_+ &= \kappa(x_+ - x'_+) \equiv q_+ - q'_+ \ , \\ p_- &= \kappa(x'_- - x_-) \equiv q_- - q'_- \ . \end{aligned} \tag{3}$$

Thus we find that on a flat string surface the difference between the vertex points will fulfil

$$(x - x')^2 = -m^2/\kappa^2 \ . \tag{4}$$

Eq. (3) implies that the (1+1)-dimensional Lund Fragmentation Model may also be described by means of a multiperipheral chain diagram with the particles emitted along the chain with propagators carrying the momentum transfers.

This is used in [5] (where there are also some useful pictures) in order to subdivide the Area Law process into steps in between the vertices. The energy momentum conserving δ-distribution in Eq. (1) can be "solved" by introducing the momentum transfers $\{q_j\}$ instead of the hadron momenta $\{p_j\}$. Then the mass-shell conditions means that the hyperbolic angle between the vertices is fixed by the squared sizes $q^2 = -\Gamma$, $(q')^2 = -\Gamma'$ and $(q - q')^2 = m^2$. The result is that Eq. (1) can be rewritten as a product of steps between the $\{\Gamma_j\}$ (where the area A in the exponent in every step is subdivided into triangular "slits" between the origin and the relevant two vertices):

$$\begin{aligned} dP_n\left(\{p_j\}, P_{\text{tot}}\right) &= \prod K\left(\Gamma_j, \Gamma_{j-1}, m^2\right) d\Gamma_j \ , \\ K\left(\Gamma, \Gamma', m^2\right) &= N \frac{\exp -b/2 \sqrt{\lambda\left(\Gamma, \Gamma', -m^2\right)}}{\sqrt{\lambda\left(\Gamma, \Gamma', -m^2\right)}} \ , \\ \lambda(a,b,c) &= a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \, . \end{aligned} \tag{5}$$

It is remarkable fact that the transfer operators K can be diagonalised in terms of the eigenfunctions of the harmonic oscillator (those that are boost-invariant in a (1+1)-dimensional space-like Minkowski space, in a two-dimensional euclidian space they correspond to a vanishing angular momentum) $g_n(\Gamma)$ with the eigenvalues solely determined by the squared mass of the hadrons produced in between

$$K(\Gamma,\Gamma',m^2)=\sum_{n=0}^{\infty}g_n(\Gamma)\lambda_n(m^2)g_n(\Gamma')\,. \tag{6}$$

The eigenvalues λ_n are analytic continuation of the harmonic oscillator eigenfunctions to time-like values of the argument [5]. A useful representation of K and the eigenvalues λ_n are

$$\begin{aligned} K\left(\Gamma,\Gamma',m^2\right) &= \int_0^1 \frac{dz}{z}\exp -\frac{b}{2}\left(z\Gamma+\frac{m^2}{z}\right)\delta\left(\Gamma'-(1-z)\left(\Gamma+\frac{m^2}{z}\right)\right),\\ \lambda_n\left(m^2\right) &= N\exp\left(\frac{bm^2}{2}\right)\int_0^1 \frac{dz}{z}(1-z)^n\exp-\left(\frac{bm^2}{z}\right). \end{aligned} \tag{7}$$

We have then introduced the positive lightcone fraction of the produced hadron z by $(x_+ - x'_+) = zx_+$. It is straightforward algebra to prove that the area slit between the vertices is given by the exponent $(z\Gamma + m^2/z)/2$ in the representation of the kernel K. There is also a simple relationship between the two adjacent values of Γ in the representation of K

$$\Gamma' = (1-z)\left(\Gamma+\frac{m^2}{z}\right). \tag{8}$$

Eq. (8) is a particular consequence of the fact that it is impossible to use up all the lightcone energy-momentum in a typical central step of the process.

Such a requirement also comes out of the following argument. Suppose that we would integrate dP_n in Eq. (1) over all possible energy-momenta and then sum over all multiplicities. Due to the Lorentz invariance we will obtain a function $R(s)$ that only can depend upon the total squared energy-momentum $s = P_{\rm tot}^2$. If we pick out the dependence on the first particle and sum and integrate over all the rest we obtain an integral equation for the function R

$$\begin{aligned} R(s) &= ({\rm BT}) + \int_0^1 N\frac{dz}{z}\exp\left(\frac{-bm^2}{z}\right)R(s')\,,\\ s' &= (1-z)\left(s-\frac{m^2}{z}\right), \end{aligned} \tag{9}$$

where (BT) stands for "boundary condition term" and where the variable s' is equal to the squared mass of all the remaining particles if the first hadron takes the lightcone fraction z (we note the similarity to the Eq. (8) and also the difference between the occurrence of the spacelike momentum transfer variables Γ and the timelike mass variables s). The integral equation Eq. (9) will have an asymptotic solution $R \propto s^a$ (with the parameter a a function of N and bm^2, *cf.* [3, 7]) with the requirement

$$\int_0^1 N \frac{dz}{z}(1-z)^a \exp\left(\frac{-bm^2}{z}\right) = 1\,. \tag{10}$$

Consequently while *the exclusive formula* for the production of a particular hadron with the lightcone fraction z is given by the Area Law, *the inclusive probability to produce this hadron* (irrespectively of what comes after it in the process) must be weighted with $g(s')/g(s) \simeq (1-z)^a$. Therefore, the well-known Lund fragmentation formula is given by the integrand in Eq. (10) and there is a power suppression for large values of the fragmentation variable.

The formulas presented above correspond to an ordering along the positive lightcone. It is, of course, possible to redefine everything in terms of an ordering along the negative lightcone, *i.e.* to introduce the corresponding negative lightcone component ζ by $(x'_- - x_-) = \zeta x'_-$. It is straightforward to prove that

$$\zeta = \frac{m^2}{m^2 + z\Gamma} \qquad \text{and} \qquad z = \frac{m^2}{m^2 + \zeta\Gamma'}\,, \tag{11}$$

and from this we find that the integrand in the representation for the kernel K can be reformulated from $(z, \Gamma) \to (\zeta, \Gamma')$ to exhibit the symmetry between the positive and the negative lightcone directions

$$\begin{aligned} & \frac{dz}{z}\delta\left(\Gamma' - (1-z)\left(\Gamma + \frac{m^2}{z}\right)\right)\exp -\frac{b}{2}\left(z\Gamma + \frac{m^2}{z}\right) \\ \to\quad & \frac{d\zeta}{\zeta}\delta\left(\Gamma - (1-\zeta)\left(\Gamma' + \frac{m^2}{\zeta}\right)\right)\exp -\frac{b}{2}\left(\zeta\Gamma' + \frac{m^2}{\zeta}\right). \end{aligned} \tag{12}$$

3. The description of a multi-gluon string state

The dynamics of the massless relativistic string is based upon the requirement that *the surface spanned by the string during one period of motion should be a minimal surface.* This means (as always in mathematics) that *the surface is completely determined by its boundary.* In the Lund Model the string is used as a model for the confined color force field in QCD and the above property then has the further important implication that the dynamics will be infrared stable, *i.e. all predictable features from the decay of the force field should be stable against minor deformations.*

For an open string there is a single wave moving to and through across the space-time surface and it is bouncing at the endpoints. The wave motion is determined by a (four-)vector-valued shape function, that we will call the directrix, $\mathcal{A}$. Thus a point on the string, parametrised by the amount of energy, σ, between the point and (for definiteness) the q-endpoint is at the time t at the position

$$x(\sigma, t) = \frac{\mathcal{A}\left(t + \frac{\sigma}{\kappa}\right) + \mathcal{A}\left(t - \frac{\sigma}{\kappa}\right)}{2}\,. \tag{13}$$

We will from now on put the string constant κ equal to unity in order to simplify the formulas and we note that *any point on the string surface in this way can be described as the average of two points on the directrix.*

While the tension $\vec{T} = \partial\vec{x}/\partial\sigma$ is directed along the string, the velocity $\vec{v} = \partial\vec{x}/\partial t$ is directed transversely so that $\vec{T}\cdot\vec{v} = 0$. The definition of σ also implies that $\vec{T}^2 + \vec{v}^2 = 1$ (all the three-vector relations are valid in the local restframe). Together this means that *the directrix function everywhere must have a lightlike tangent*

$$\left(\frac{d\vec{\mathcal{A}}}{d\xi}\right)^2 = 1\,. \tag{14}$$

The tension must vanish at the endpoints ($\sigma = 0$ and $\sigma = E_{\rm tot}$) and this implies that the directrix must be a periodic function with the property that

$$\mathcal{A}(\xi + 2E_{\rm tot}) = \mathcal{A}(\xi) + 2P_{\rm tot}\,, \tag{15}$$

where $P_{\rm tot}$ ($E_{\rm tot}$) is the total energy momentum (energy) of the state. While the directrix $\mathcal{A}(t)$ according to Eq. (13) describes the motion of the q-end it is from Eq. (15) evident that

$$\mathcal{A}_{\bar{q}}(t) = \mathcal{A}(t + E_{\rm tot}) - P_{\rm tot}\,, \tag{16}$$

will describe the motion of the $\bar{q}$-end. Finally, if the string starts out from a point (at the time $t = 0$) then due to symmetry we must have

$$\mathcal{A}(\xi) = -\mathcal{A}(-\xi)\,. \tag{17}$$

Using the Lund interpretation of the gluons as internal excitations on the string it is easy to construct the first half period of the directrix: it will start with the quark energy momentum k_1 and then the gluon energy momenta $\{k_j\}$ are laid out in color order and it ends with the $\bar{q}$. In this way the q-endpoint will be acted upon by the color-ordered excitations as they arrive in turn.

From Eqs. (15) and (17) it follows that we obtain the directrix of the second half period by reversing the order, starting with the $\bar{q}$ and ending with the q energy momentum (besides the translation this is the way the $\bar{q}$-endpoint will move according to Eq. (16)).

With respect to the energy momentum content in the string we obtain that in between the point σ and the q-end it is given by

$$\int_0^\sigma d\sigma' \frac{\partial x}{\partial t} = \frac{\mathcal{A}(t+\sigma) - \mathcal{A}(t-\sigma)}{2} \,. \tag{18}$$

4. The Lund Fragmentation as a process along the directrix

In this section I will go directly to the process we have devised in [1]. I will skip the considerations that we presented in that paper on the possibility to devise a more general method for string fragmentation (partly because they did not lead to a consistent and viable process). In order to exhibit the ideas that leads to a consistent process I will start with the (1+1)-dimensional model described above and rewrite it in a useful way and after that extend the method to the general case. I will also describe the $\mathcal{X}$-curve and the λ measure in order to introduce a simple and intuitive way to see the multi-dimensional process that we are discussing.

4.1. The directrix process for the (1+1)-dimensional case

I will start to show how we can use the symmetries of the string dynamics to rewrite the process from a process "across" the string surface to a process along the directrix.

In the (1+1)-dimensional case the directrix only contains two directions, given by the $\bar{q}$ energy momentum vector (to be called $\mathcal{A}_+$) and the q energy momentum ($\mathcal{A}_-$). A vertex point x_j, obtained after the production of j hadrons from the q-side, $p_1, \ldots, p_j$ is then described (with respect to the origin) by two points on the directrix (*cf.* Eq. (13))

$$x_j = \frac{(\mathcal{A}_{+j} + \mathcal{A}_{-j})}{2} \,. \tag{19}$$

We can also deduce from Eq. (18) that

$$\sum_1^j p_\ell = \frac{(\mathcal{A}_{+j} - \mathcal{A}_{-j})}{2} \,. \tag{20}$$

Using the symmetry of a directrix passing through a single point (Eq. (17)) we may find another point on the directrix with the property that

$$\mathcal{A}_{-j} \equiv \mathcal{A}_-(\xi_j) = -\mathcal{A}_-(-\xi_j) \equiv -\mathcal{B}_{-j} \,. \tag{21}$$

We will from now on drop the indices $\pm$ on $\mathcal{A}_+$ and $\mathcal{B}_-$ but we note that they do describe points on the same directrix. While $\mathcal{A}_-$ goes "backward" for increasing j-values, $\mathcal{B}$ follows the q-direction.

We may now consider the hadron energy momenta to define a curve from the origin "along the directrix" such that the curve after j steps has reached the point, *cf.* Eq. (20)

$$X_j = \frac{(\mathcal{A}_j + \mathcal{B}_j)}{2}\,, \tag{22}$$

while the difference between the point $\mathcal{A}_j$ on the directrix and X_j is given by x_j in Eq. (19). The production of a new particle p_{j+1} then corresponds to choosing two new points X_{j+1} and (along the directrix) $\mathcal{A}_{j+1}$ such that

$$X_{j+1} - X_j = p_{j+1} \quad \text{and} \quad \mathcal{A}_{j+1} - \mathcal{A}_j \equiv k_{j+1}\,. \tag{23}$$

We also obtain a new "vertex" vector x_{j+1} by the evident identity:

$$p_{j+1} + x_{j+1} = x_j + k_{j+1}\,. \tag{24}$$

In this way the vertex vector fulfils $x_{j+1} = \mathcal{A}_{j+1} - X_{j+1}$ just as $x_j = \mathcal{A}_j - X_j$. We have then arranged it so that the hadrons are produced along a curve, the X-curve, from the origin and the vertex vectors are the connectors for this curve going from from the produced particle to the directrix. Before we consider the Area Law in this situation we note the symmetry between a reversed process and the process described above, *i.e.* when we go from X_j to X_{j+1} thereby producing p_{j+1} by the use of a part k_{j+1} of the directrix along $\mathcal{A}$.

To see the reverse process we note that the vector x_j can just as well be reached by taking the difference between the point X_j on the hadron curve, Eq. (22), and "the backward point" on the directrix $\mathcal{B}_j$

$$x_j = \frac{(\mathcal{A}_j - \mathcal{B}_j)}{2} = \mathcal{A}_j - X_j = X_j - \mathcal{B}_j\,. \tag{25}$$

Using this we could evidently consider the production of the particle p_{j+1} as a step from $\mathcal{B}_j$ to $\mathcal{B}_{j+1} = \mathcal{B}_j + \ell_j$ (*cf.* Eq. (23)) such that we have in correspondence to Eq. (24)

$$p_{j+1} + x_j = x_{j+1} + \ell_{j+1}\,. \tag{26}$$

In order to formalise the determination of the particle energy momentum p, we may then in "the k-process" (along $\mathcal{A}$) assume that we know the starting vertex vector x, connected to the point $\mathcal{A}_P$. We may then chose a piece

k from $\mathcal{A}_P$ along $\mathcal{A}$ (of a size to be determined) and then define the other lightcone direction in the plane determined by (x, k) by

$$\hat{\ell} = x - k\frac{x^2}{2xk} . \tag{27}$$

The vector p will be described in terms of $(k, \hat{\ell})$ as

$$p = z\hat{\ell} + \frac{k}{2} = zq + \frac{k}{2}\left(1 - \frac{zx^2}{xk}\right) \tag{28}$$

with the requirement that the particle should be on the mass-shell

$$p^2 = m^2 = zkx , \qquad i.e. \qquad kx = \frac{m^2}{z} . \tag{29}$$

From Eq. (24) we obtain the new vertex vector x' by

$$\begin{aligned} x' &= (1-z)x + \frac{k}{2}\left(1 + \frac{zx^2}{xk}\right) , \\ (x')^2 &= (1-z)\left(x^2 + xk\right) , \end{aligned} \tag{30}$$

and we recognise the results corresponding to the Lund Model formulas given in Section 2. The area "slit" that was defined in connection with the definition of the transition operators (*cf.* Eqs. (5) and (6)) is now placed in the region in between the hadronic vector p, the directrix vector k and bordered from below by the "original" vertex vector x and from above by the "new" vertex vector x'. This corresponds to the plaquette building that I mentioned in the Introduction.

It is also obvious that we may define an ℓ-process similar to the k-process we have discussed above. We just write $z\hat{\ell} = \ell/2$ and introduce the variable ζ such that $k/2 = \zeta\hat{k}$ with ζ chosen such that $m^2/\zeta = \ell x'$. Actually we obtain the same process (although "in the opposite order") under the assumption that we start at x' and chose ℓ along the $\mathcal{B}$-part of the directrix with the variable ζ in accordance with Eq. (11). In this way the "backward" variable ζ evidently obeys the same distribution as the "forward" variable z and the Area Law is fulfilled.

In conclusion in the process along the directrix a particle production step starts from a knowledge of a vector x connected to a lightcone-direction. Then we chose a lightcone vector k such that Eq. (29) is fulfilled with a z-value stochastically chosen from the fragmentation function in Eq. (10). After that we construct the particle energy momentum and a new vector x' according to Eqs. (28) and (30). We may start out choosing the "first" x-vector equal to the q (lightcone) energy momentum. The process can evidently be generalised to an arbitrary directrix (although there is a need to discuss how to pass around a gluon "corner", *cf.* [1]).

4.2. The $\mathcal{X}$-curve and its properties

The Area Law distribution in Eq. (1) contains two terms, the phase space and the exponential area suppression. In order to obtain a large probability it is necessary for a given total energy-momentum on the one hand, to make many particles on the other hand, to make them in such a way that the area is small. The obvious compromise is that the decay region is around a typical hyperbola with an average squared distance to the origin $\langle \Gamma \rangle \equiv \Gamma_0$. The length of the hyperbola is proportional to the available rapidity range for the final state particles, *i.e.* $\Delta y = \log(s/\Gamma_0)$.

For a string with a single gluon excitation there will be two parts of the string, one spanned between the q and the g and one between the g and the $\bar{q}$. Each of them should break up in a similar way as the single string region described by Eq. (1) and then there will be one or a few particles produced in the connected region around the gluon "tip". If the energy-momenta of the partons is k_j, $j = 1, 2, 3$ there will then be two hyperbolic angular ranges; $(\Delta y)_{12}$ and $(\Delta y)_{23}$. The total region will be

$$\begin{aligned} \lambda = (\Delta y)_{12} + (\Delta y)_{23} &= \log\left(\frac{s_{12}}{2}\Gamma_0\right) + \log\left(\frac{s_{23}}{2}\Gamma_0\right) \\ &= \log\left(\frac{s}{\Gamma_0}\right) + \log\left(\frac{s_{12}s_{23}}{4\Gamma_0 s}\right). \end{aligned} \tag{31}$$

Here $s_{j\ell} = (k_j + k_\ell)^2$ and $s = s_{12} + s_{23} + s_{13}$ and the factor 2 is introduced because only half of the gluon energy-momentum goes into each string region.

The quantity $k_\perp^2 \equiv s_{12}\, s_{23}/s$ is a convenient (and Lorentz invariant) approximation for the transverse momentum of the emitted gluon. We conclude that the phase space after the emission of a single gluon is increased from the single hyperbola result above by an amount corresponding to a "sticking-out tip" of logarithmic length given by the emitted transverse momentum. In conventional notions this is known as the "anomalous dimensions" of QCD, *i.e.* the emission of a gluon increases the region of color flow inside which one can emit further gluons and, finally, hadronise. The whole scenarium is easily visualised and used in the Lund Dipole cascade model with the corresponding Monte Carlo simulation program ARIADNE [9].

It is straightforward to see that if there are many gluons then there is a corresponding quantity, a generalised rapidity $\lambda \simeq \log(\prod s_{jj+1})$ stemming from the hyperbolas spanned between the color-connected gluons. We note that this is not an infrared stable definition. We will now provide a convenient generalisation.

A closer examination of the region around the tip of a gluon tells us that there is a correction corresponding to a connected hyperbola in the region (k_1, k_3) between the "endpoint" of the hyperbola in the region spanned between $(k_1, k_2/2)$ and the one spanned between $(k_2/2, k_3)$. In formulas we obtain for the average hyperbolas

$$\left(\alpha_1 k_1 + \frac{\beta_1 k_2}{2}\right)^2 = \Gamma_0 \qquad \text{and} \qquad (\gamma_3 k_3 + \beta_3 k_2)^2 = \Gamma_0 \,,$$
$$\left(\alpha_2 k_1 + \gamma_2 k_3 + \frac{k_2}{2}\right)^2 = \Gamma_0 \,, \tag{32}$$

with the ranges $1 \geq \alpha_1 \geq 2\Gamma_0/s_{12}$, $2\Gamma_0/s_{12} \geq \alpha_2 \geq 0$, $2\Gamma_0/s_{12} \leq \beta_1 \leq 1$ and similarly for the other variables. The length of the two hyperbolas in the segments $(k_1, k_2/2)$ and $(k_2/2, k_3)$ are then given by Eq. (31) but the third hyperbola provides an extra contribution (in the appropriate limit $s_{13} \simeq s$) equal to $\log(1 + 4\Gamma_0 s/s_{12}s_{23})$. Then the total (generalised) rapidity length becomes

$$\lambda_{123} = \log\left(\frac{s}{\Gamma_0} + \frac{s_{12}\, s_{23}}{(2\Gamma_0)^2}\right) . \tag{33}$$

This is evidently a nice interpolation between the situations with and without a gluon on the string and it is also an infrared stable definition of the notion of rapidity. Eq. (33) is noted in Ref. [8] and led us to introduce a functional defined on a multigluon string directrix.

We may firstly define the set of connected integrals

$$I_n = \int ds_{01} ds_{12} \cdots ds_{nE} \,, \tag{34}$$

with the easily understood notation (*cf.* Eq. (18)) $s_{jj+1} = (\mathcal{A}(\xi_j) - \mathcal{A}(\xi_{j+1}))^2$, *i.e.* it is proportional to the squared mass between the points ξ_j and ξ_{j+1} along the directrix. It is then obvious that the argument in the logarithm in Eq. (33) is given by the sum $I_1/\Gamma_0 + I_2/2\Gamma_0^2$ and that we may in general define the functional T by

$$T = 1 + \sum_{n=1}^{\infty} \frac{I_n}{\left(2m_0^2\right)^n} \,, \tag{35}$$

as a suitable generalisation for any string state. For a finite number of partons N the terms with $n > N$ will all vanish and we also note that the highest degree term will always have the generic form

$$2\frac{s_{12}}{4m_0^2}\,\frac{s_{23}}{4m_0^2} \cdots \frac{s_{N-1N}}{4m_0^2} \,. \tag{36}$$

We also note that for a finite total energy E the contributions romr the large degree polynomial terms will become smaller and smaller compared to the scale m_0.

In order to study the functional T it is suitable to introduce a varying value ξ instead of the total energy E in the connected integrals. The corresponding functional $T(\xi)$ will fulfil the integral equation

$$T(\xi) = 1 + \int_0^\xi \frac{ds(\xi,\xi')}{2m_0^2} T(\xi')\,. \tag{37}$$

We will also introduce the vector-valued function $q_T(\xi)$ together with T so that we have

$$\begin{aligned} q_{T\mu}(\xi) &= \frac{\int_0^\xi d\mathcal{A}_\mu(\xi')T(\xi')}{T(\xi)}\,, \\ T(\xi) &= 1 + \int_0^\xi \frac{q_T(\xi')d\mathcal{A}(\xi')}{m_0^2} T(\xi')\,. \end{aligned} \tag{38}$$

By differentiation and integration we obtain the results

$$\begin{aligned} T &= \exp\left(\int_0^\xi \frac{q_T(\xi')d\mathcal{A}(\xi')}{m_0^2} \right) \equiv \exp(\lambda(\xi))\,, \\ q_T^2(\xi) &= m_0^2 \left(1 - T^{-2}(\xi)\right)\,, \end{aligned} \tag{39}$$

which implie that the functional T is the exponential of an area (note that $d\mathcal{A}$ is everywhere lightlike and, therefore, the area spanned between the vector q_T and $d\mathcal{A}$ is $\sqrt{(q_T d\mathcal{A})^2 - q_T^2 d\mathcal{A}^2} = q_T d\mathcal{A}$) scaled by m_0^2. This quantity is equal to the generalised rapidity λ for the simple case described above and it provides an infrared stable definition for any multigluon state. Further, the vector q_T is time-like and will quickly approach the finite length m_0. The interpretation (as it is worked out in Ref. [8], *cf.* also [3]) is that there is a vector valued function $\mathcal{X}_\mu(\lambda)$ conveniently labelled by λ such that

$$\begin{aligned} \mathcal{X} + q_T &= \mathcal{A}\,, \\ \frac{d\mathcal{X}}{d\lambda} &= q_T\,, \\ \frac{dq_T}{d\lambda} &= -q_T + \frac{d\mathcal{A}}{d\lambda}\,, \end{aligned} \tag{40}$$

i.e. the vector q_T is the tangent to the curve defined by $\mathcal{X}$ such that it reaches to the directrix. We will need two further properties of the $\mathcal{X}$-curve, on the one hand the change of q_T and $\mathcal{X}$ for a finite parton energy-momentum k_j on the other hand an interpretation of the differential equation for q_T in Eq. (40).

By direct integration we find that if we have the vector q_{Tj} then "after" application of the parton energy momentum k_j we obtain the vector q_{Tj+1} and will take a step along the $\mathcal{X}$-curve equal to $\delta\mathcal{X}_j$

$$\begin{aligned} q_{Tj+1} &= \gamma_j q_{Tj} + \frac{(1+\gamma_j)k_j}{2}, \\ \delta\mathcal{X}_j &= (q_{Tj} + \frac{k_j}{2})(1-\gamma_j), \\ \gamma_j &= \frac{1}{1+\frac{q_{Tj}k_j}{m_0^2}}, \end{aligned} \tag{41}$$

(we also note that the products of the γ_j is equal to T^{-1}). Further, if we define the (1+4)-dimensional vector ($Q_\mu \equiv Tq_{T\mu}/m_0, T$) (which has a length in the (1+4)-dimensional Minkowski metric equal to $Q^2 - T^2 = -1$) then the differential equation for q_T can be rewritten as

$$dT = Qd\mathcal{A} \qquad \text{and} \qquad dQ = TdA, \tag{42}$$

i.e. as a group of special rotations in this space (corresponding to a subgroup of SO(1,4)) that are defined by the incremental changes along the directrix curve.

It is obvious that our particle fragmentation process, defined in the earlier subsection to produce the hadronic X-curve is similar to the production of the $\mathcal{X}$-curve devised above. Actually there is a limiting case where the fragmentation process and the differential equations discussed above coincide. I will briefly consider this case before I go over to some further properties of the fragmentation process.

4.3. The relationship of the process to the $\mathcal{X}$-curve

There is a direct connection between a differential version of our hadronisation process and the $\mathcal{X}$-curve that was referred to in Section 4.2. In order to see that we consider the limiting situation when the mass parameter is vanishing. Under those circumstances the distribution function will develop a pole for $z \to 0$. We will assume that the model is defined by the step size dz with the ratio $m \to m_0 dz$. The corresponding incremental k vector will be called $d\mathcal{A}$ and it will fulfil the mass-shell condition (we will use the notation q_P instead of x for the vertex vector)

$$d\mathcal{A}q_P = \frac{m^2}{dz} \to dz m_0^2 \,. \tag{43}$$

From the model formulas for the change in $x \to q_P$ and the particle energy momentum p (Eqs. (28) and (30)) we obtain the following differential equations defining a curve to be called the $\mathcal{P}$-curve

$$\begin{aligned} d\mathcal{P} &= dz\, q_P + \frac{d\mathcal{A}}{2}\left(1 - \frac{q_P^2}{m_0^2}\right) , \\ dq_P &= -dz\, q_P + \frac{d\mathcal{A}}{2}\left(1 + \frac{q^2}{m_0^2}\right) . \end{aligned} \tag{44}$$

The equations are similar but not identical to the defining differential equations for the $\mathcal{X}$-curve (*cf.* Eqs. (40)). We firstly note that from the sum and differences of the Eqs. (44) we obtain

$$\begin{aligned} \mathcal{P} + q &= \mathcal{A} \,, \\ \mathcal{P} - q &= \mathcal{L} \,, \end{aligned} \tag{45}$$

and that the vector $\mathcal{L}$ has a lightlike tangent just as the directrix $\mathcal{A}$:

$$d\mathcal{L} = 2dz\, q_P - d\mathcal{A}\frac{q^2}{m_0^2} \,. \tag{46}$$

In that way the $\mathcal{P}$-curve goes in between two curves with everywhere lightlike tangents and the vector q_P connects to both of them.

Just as the vector q_T in Section 4.2 the vector q_P is timelike and is quickly reaching the length m_0:

$$q_P^2 = m_0^2\left(1 - T_P^{-1}\right) \qquad \text{with} \qquad T_P = \exp\left(\int (q_P d\mathcal{A})\right) , \tag{47}$$

although we note the change in power with respect to T^{-1}. For a finite length k vector it is easy to integrate the two equations in (44) and we obtain similarly to Eq. (41)

$$\begin{aligned} q_P' &= \gamma q_P + \frac{\left(1 + \frac{q_P^2\gamma}{m_0^2}\right)}{2} k \,, \\ \delta\mathcal{P} &= (1-\gamma) q_P + \frac{\left(1 - \frac{q_P^2\gamma}{m_0^2}\right)}{2} k \,, \\ \delta\mathcal{L} &\equiv \ell = 2(1-\gamma) q_P - \frac{q_P^2\gamma}{m_0^2} \,, \\ \gamma &= \frac{1}{1 + \frac{q_P k}{m_0^2}} \qquad \text{and} \qquad (T_P)^{-1} = \prod \gamma_j \,. \end{aligned} \tag{48}$$

I will end this short digression by showing that although there are small deviations between the differential version of our fragmentation process and the corresponding properties for the $\mathcal{X}$-curve the two processes are actually identiacal when we consider them in terms of the rotations in the $(1+4)$ -dimensional space discussed at the end of the last subsection. We obtain from the Eqs. (44) and (47) that

$$d(q_P T_P) = d\mathcal{A}\left(T_P - \tfrac{1}{2}\right) \qquad \text{and} \qquad dT_P = q_P d\mathcal{A} T_P\,, \tag{49}$$

so that the $(1+4)$-dimensional vector $(2q_P T_P, 2T_P - 1)$ has both the same length and fulfils the same differential equations with respect to incremental changes along the directrix as (Q, T) do in Eqs. (42). The changes above correspond to different boundary values for the rotational equations.

We have — at the time when this is written — been able to show that *the $\mathcal{P}$-curve that is defined above corresponds to the average hadronic X-curve for the multigluon fragmentation process* in the same way as the hyperbola corresponds to the average fragmentation region for the earlier (1+1)-dimensional model. In particular the considerations for the behaviour of the sum and the integral over the original Area Law formulas in Eq. (1) that lead to Eqs. (9) and (10) will be obtained in the general model but this time with the functional T_P instead of the squared cms energy s.

4.4. The reverse problem: how to find the directrix from the hadronic curve

I will very briefly discuss the reverse problem to the hadronisation process, *i.e.* to what extent we can trace the directrix from a knowledge of the hadronic curve, that we will call the X-curve in accordance with the notation introduced in Section 4.1.

We will then assume that the X-curve is defined by the hadronic energy momenta $\{p_j\}$, ordered and laid out according to rank. We will concentrate on the hadron p_j, produced in between the vertex vectors x_{j+1} and x_j with the directrix vector k_j. According to Eq. (24) it is in order to construct k_j sufficient to know p_j and the difference vector

$$(x_{j+1} - x_j) = \varepsilon_j \hat{p}_j\,. \tag{50}$$

It is straightforward to solve for $\hat{p}_j$ in terms of p_j and x_j

$$\hat{p}_j = \frac{(x_j p_j)p_j - p_j^2 x_j}{\sqrt{(p_j x_j)^2 - p_j^2 x_j^2}}\,, \tag{51}$$

and the sign ε_j should be positive or negative depending upon whether m^2/z_j is larger or smaller than $z_j x_j^2$ (it is useful to note that $2(p_j x_j) = (m^2/z_j + z_j x_j^2)$). Therefore, if we prescribe the first vertex vector x_1 (this is always chosen in our process as the original q energy momentum vector) then the directrix vectors as well as the vertices are determined recursively up to a sign:

$$\begin{aligned} k_j &= p_j + \varepsilon_j \hat{p}_j \,, \\ x_{j+1} &= x_j + \varepsilon_j \hat{p}_j \,. \end{aligned} \tag{52}$$

It is evident that the other sign will determine the corresponding ℓ_j.

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