CURRENT REVERSALS IN CHAOTIC RATCHETS*

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The problem of the classical deterministic dynamics of a particle in a periodic asymmetric potential of the ratchet type is addressed. When the inertial term is taken into account, the dynamics can be chaotic and modify the transport properties. By a comparison between the bifurcation diagram and the current, we identify the origin of the current reversal as a bifurcation from a chaotic to a periodic regime. Close to this bifurcation, we observed trajectories revealing intermittent chaos and anomalous deterministic diffusion. We extend our previous analysis of this problem to include multiple current reversal and the orbits in phase space.

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1. Introduction

There is an increasing interest in recent years in the study of the transport properties of nonlinear systems that can extract usable work from unbiased nonequilibrium fluctuations. These, so called ratchet systems, can be modeled, for instance, by considering a Brownian particle in a periodic asymmetric potential and acted upon by an external time-dependent force of zero average. For recent reviews see [1–4]. This recent burst of work is motivated in part by the challenge to explain the unidirectional transport of molecular motors in the biological realm [4]. Another source of motivation arises from the potential for new methods of separation of particles, polyelectrolytes and macromolecules [5–10], and more recently in the recognition of the "ratchet effect" in the quantum domain [11–16].

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In order to understand the generation of unidirectional motion from nonequilibrium fluctuations, several models have been used. In Ref. [1], there is a classification of different types of ratchet systems; among them we can mention the "Rocking Ratchets", in which the particles move in an asymmetric periodic potential subject to spatially uniform, time-periodic deterministic forces of zero average. Most of the models, so far, deal with the over-damped case in which the inertial term due to the finite mass of the particle is neglected. However, more recently, this oversimplification was overcome by treating properly the effect of finite mass [17–22].

In particular, in two recent papers [17,18], the authors studied the effect of finite inertia in a deterministically rocked, periodic ratchet potential. They consider the deterministic case in which noise is absent. The inertial term allows the possibility of having both regular and chaotic dynamics, and this deterministically induced chaos can mimic the role of noise. They showed that the system can exhibit a current flow in either direction, presenting multiple current reversals as the amplitude of the external force is varied.

In Ref. [18] the role of the chaotic dynamics in the current was analyzed in detail, establishing for the first time a close connection between the current and the bifurcation diagram when a control parameter of the model is varied. In this paper we elaborate on this idea by studying the multiple current reversals and the orbits in phase space.

The outline of the paper is as follows: in the next section we introduce the equations of motion that define the model, and in the next section we present the numerical results. We end with some concluding remarks in the last section.

2. The ratchet potential model

Let us consider the one-dimensional problem of a particle driven by a periodic time-dependent external force, under the influence of an asymmetric periodic potential of the ratchet type. The time average of the external force is zero. Here, we do not take into account any kind of noise, and thus the dynamics is deterministic. The equation of motion is given by

$$m\ddot{x} + \gamma\dot{x} + \frac{dV(x)}{dx} = F_0 \cos(\omega_D t)\,, \tag{1}$$

where m is the mass of the particle, γ is the friction coefficient, $V(x)$ is the external asymmetric periodic potential, F_0 is the amplitude of the external force and ω_D is the frequency of the external driving force. The ratchet potential is given by

$$V(x) = V_1 - V_0 \sin\frac{2\pi(x-x_0)}{L} - \frac{V_0}{4}\sin\frac{4\pi(x-x_0)}{L}\,, \tag{2}$$

where L is the periodicity of the potential, V_0 is the amplitude, and V_1 is an arbitrary constant. The potential is shifted by an amount x_0 in order that the minimum of the potential is located at the origin.

Let us define the following dimensionless units: $x' = x/L$, $x'_0 = x_0/L$, $t' = \omega_0 t$, $w = \omega_D/\omega_0$, $b = \gamma/m\omega_0$ and $a = F_0/mL\omega_0^2$. Here, the frequency ω_0 is given by $\omega_0^2 = 4\pi^2 V_0 \delta/mL^2$ and δ is defined by

$$\delta = \sin(2\pi|x'_0|) + \sin(4\pi|x'_0|)\,.$$

The frequency ω_0 is the frequency of the linearized motion around the minima of the potential, thus we are scaling the time with the natural period of motion $\tau_0 = 2\pi/\omega_0$. The dimensionless equation of motion, after renaming the variables again without the primes, becomes

$$\ddot{x} + b\dot{x} + \frac{dV(x)}{dx} = a\cos(wt)\,, \tag{3}$$

where the dimensionless potential can be written as

$$V(x) = C - \frac{1}{4\pi^2\delta}\left[\sin 2\pi(x - x_0) + \frac{1}{4}\sin 4\pi(x - x_0)\right]\,, \tag{4}$$

and is depicted in Fig. 1. The constant C is such that $V(0) = 0$, and is given by $C = -(\sin 2\pi x_0 + 0.25\sin 4\pi x_0)/4\pi^2\delta$. In this case, $x_0 \simeq -0.19$, $\delta \simeq 1.6$ and $C \simeq 0.0173$.

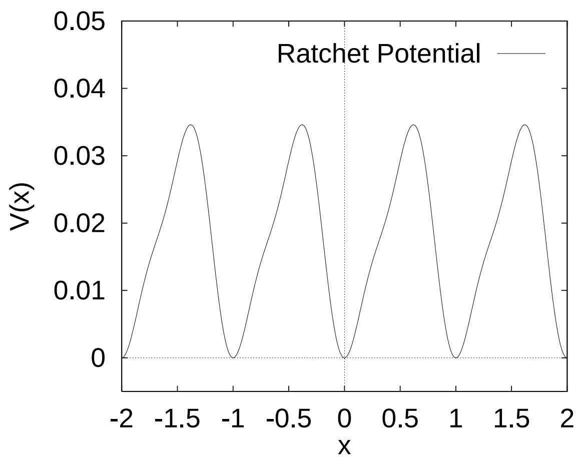

Fig. 1. The dimensionless ratchet periodic potential $V(x)$.

In the equation of motion Eq. (3) there are three dimensionless parameters: a, b and w, defined above in terms of physical quantities. We will vary the parameters in order to understand the role of each in the dynamics. The parameter $a = F_0/mL\omega_0^2$ is the ratio of the amplitude of the external force and the force due to the potential $V(x)$. This can be seen more clearly using the expression for ω_0^2 in terms of the parameters of the potential. In this case, the ratio becomes

$$a = \frac{1}{4\pi^2\delta}\frac{F_0}{(V_0/L)}\,,$$

that is, except for a constant factor, a is the ratio of F_0 and the force V_0/L, where V_0 is the amplitude and L the periodicity of the potential (see Eq. (2)).

The parameter b is simply the dimensionless friction coefficient, and w is the ratio of the driving frequency of the external force and ω_0. We will discuss in more detail these parameters in the next section.

The extended phase space in which the dynamics is taking place is three-dimensional, since we are dealing with an inhomogeneous differential equation with an explicit time dependence. This equation can be written as a three-dimensional dynamical system, that we solve numerically, using the fourth-order Runge–Kutta algorithm. The equation of motion Eq. (3) is non-linear and thus allows the possibility of periodic and chaotic orbits. If the inertial term associated with the second derivative $\ddot{x}$ were absent, then the dynamical system could not be chaotic.

The main motivation behind this work is to study in detail the origin of the current reversal in a chaotically deterministic rocked ratchet as found in [18]. In order to do so, we have to study first the current J itself, that we define as the time average of the average velocity over an ensemble of initial conditions. Therefore, the current involves two different averages: the first average is over M initial conditions, that we take equally distributed in space, centered around the origin and with an initial velocity equal to zero. For a fixed time, say t_j, we obtain an average velocity, that we denoted as v_j, and is given by

$$v_j = \frac{1}{M}\sum_{i=1}^{M}\dot{x}_i(t_j)\,. \tag{5}$$

The second average is a time average; since we take a discrete time for the numerical solution of the equation of motion, we have a discrete finite set of N different times t_j; then the current can be defined as

$$J = \frac{1}{N}\sum_{j=1}^{N} v_j\,. \tag{6}$$

This quantity is a single number for a fixed set of parameters a, b, w.

Besides the orbits in the extended phase space, we can obtain the Poincaré section, using as a stroboscopic time the period of oscillation of the external force. With the aid of Poincaré sections we can distinguish between periodic and chaotic orbits, and we can obtain a bifurcation diagram as a function of the parameter a. As was shown in [18], there is a connection between the bifurcation diagram and the current.

3. Numerical results

Using the definition of the current J given in the previous section, we calculate J fixing the parameters $b = 0.1$ and $w = 0.67$ and varying the parameter a. The current shows, as stressed before [17,18], multiple current reversals and a complex variation with a, as shown in Fig. 2(b). We can observe strong fluctuations as well as portions where the current is approximately constant. The challenge here is to explain this high variability in the current with the aid of what we know from the nonlinear chaotic dynamics of the system.

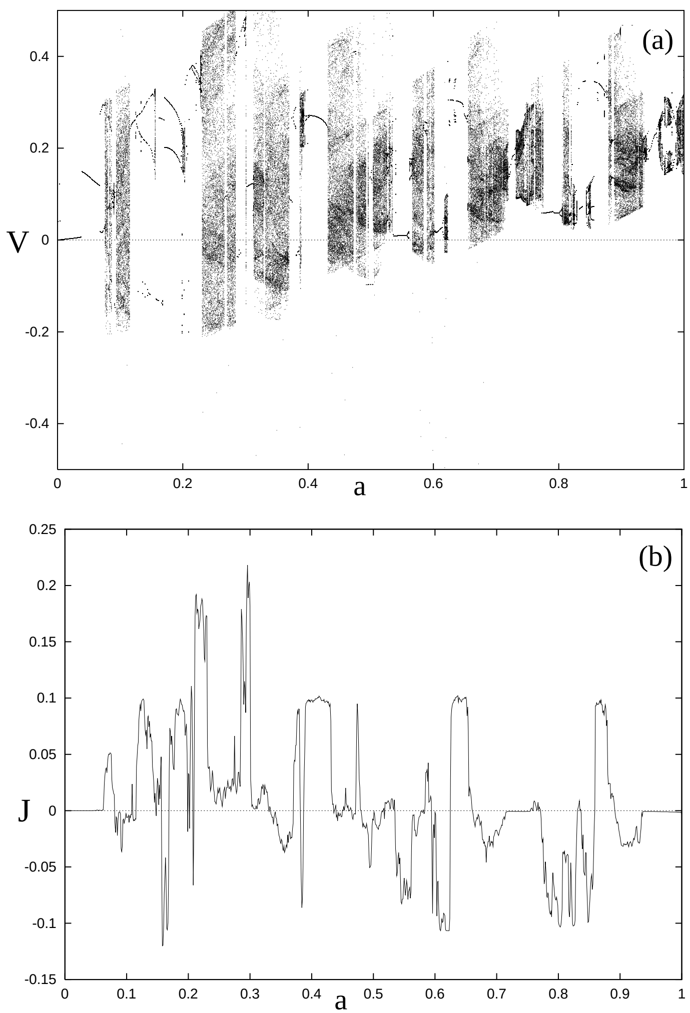

Fig. 2. For $b = 0.1$ and $w = 0.67$ we show: (a) the bifurcation diagram as a function of a, (b) the current J as a function of a. We can see multiple current reversals.

Associated with this current, there is a correspondent bifurcation diagram as a function of a, as depicted in Fig. 2(a). The complexity in this diagram is a consequence of the richness in the dynamics of the particle in the non-linear ratchet potential. We notice that the bifurcation diagram for the ratchet is qualitatively similar to the bifurcation diagram of a harmonically forced pendulum with friction [23]. We can imagine the problem of the pendulum as a particle in a *symmetric* periodic potential that varies in time. In this sense, our ratchet problem is an asymmetric generalization of the pendulum where a spatial symmetry breaking occurs. There is a recent work [21] that studied the pendulum and the ratchet in the context of symmetry breaking.

In order to understand the first part of the current, let us analyze the case of small values of a, where we chose $b = 0.1$ and $w = 0.67$. In Fig. 3(a) we show the current as a function of a, and in Fig. 3(b) we depict the bifurcation diagram in the same range of a. Let us imagine that an ensemble of particles are initially located at the minimum of the ratchet potential around the origin, and that all these particles have an initial velocity equal to zero. For $a = 0$, we have no external force and thus, all these particles remain in the minimum around the origin and therefore the current is zero. For very small values of a, we still have a zero current, since the particles have friction and tend to oscillate in this minimum. However, there is a critical value of a for which the particles start to overcome the potential barriers around the minimum and transport along the ratchet potential in a periodic or chaotic way. This critical value can be calculated as follows: remember that a is a dimensionless quantity defined as

$$a = \frac{1}{4\pi^2\delta}\frac{F_0}{(V_0/L)} \,.$$

Here, F_0 is the amplitude of the external force and V_0/L is the order of magnitude of the force exerted by the potential. Thus, we expect the current J be different from zero when F_0 is on the order of V_0/L, that is, $F_0 \sim V_0/L$. In this case, the critical value of a is $a_c \sim 1/4\pi^2\delta$, since $F_0/(V_0/L) \sim 1$. Using the value of $\delta \simeq 1.6$ we obtain $a_c \sim 0.1$, which is on the order of magnitude of the values that we obtain numerically. Above this value, the current starts to grow since more and more particles contribute to the current.

At the beginning, the current is dominated by transport due to periodic orbits, but for larger values of a, some of the orbits in the ensemble become chaotic and the transport is not as efficient as before, resulting in a current that starts to oscillate erratically. In fact, in this region, there exist the possibility of coexistence of multiple attractors in the phase space. For example, in Fig. 3(a), we have two coexistent attractors: a periodic and

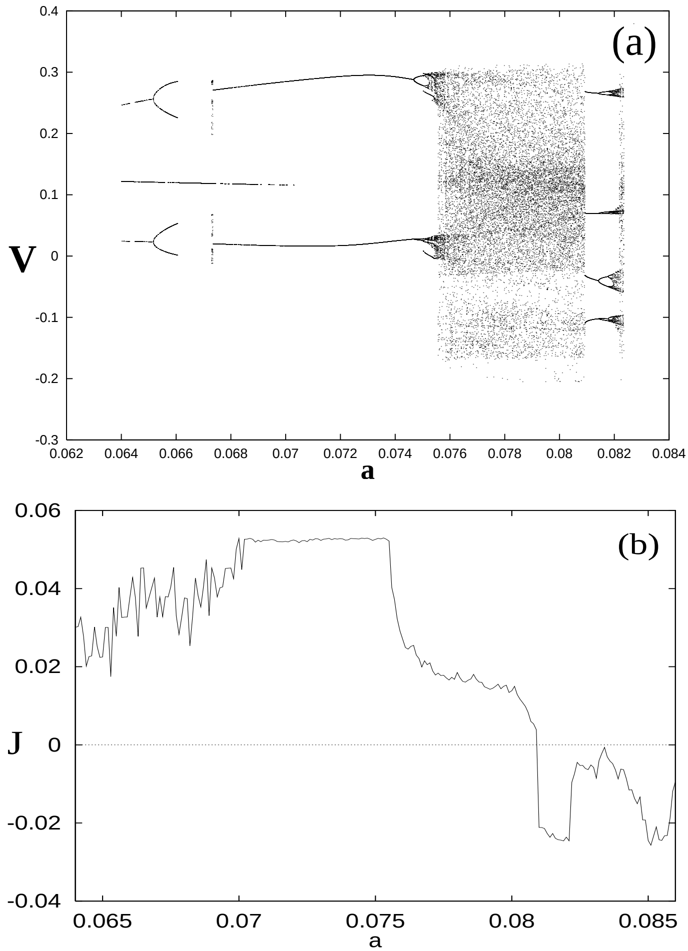

Fig. 3. For $b = 0.1$ and $w = 0.67$ we show: (a) the bifurcation diagram as a function of a, (b) the current J as a function of a. The range in the parameter a corresponds to the first current reversal.

a chaotic one, around $a = 0.067$. In this case, depending on the initial conditions, some orbits in the ensemble can end up in the periodic attractor, and the rest in the chaotic attractor.

For values of a even larger, all the orbits enter a chaotic region through a period-doubling bifurcation, and the current starts to decrease inside this chaotic band. Finally, exactly at the bifurcation point where a periodic window opens, the current drops to zero and becomes negative in a very abrupt way [18].

Let us focus first on the range of the control parameter where the first current reversal takes place. This occurs around $a \simeq 0.08$ as shown in Fig. 3. We can observe a period-doubling route to chaos and after a chaotic region, there is a saddle-node bifurcation taking place at the critical value $a_c \simeq 0.08092844$. It is precisely at this bifurcation point that the current reversal occurs. After this bifurcation, a periodic window emerges, with an orbit of period four. In Figs. 3(a), 3(b) we are analyzing only a short range of values of a, where the first current reversal takes place. If we vary a further, we can obtain multiple current reversals, as shown in Fig. 2(b).

In order to understand in more detail the nature of the current reversal, let us look at the orbits just before and after the transition. The reversal occurs at the critical value $a_c \simeq 0.08092844$. If a is below this critical value a_c, say $a = 0.07$, then the orbit is periodic, with period two. For this case we depict, in Fig. 4(a), the position of the particle as a function of time. We notice a period-two orbit, as can be distinguish in the bifurcation diagram for $a = 0.07$. This orbit transport particles to the positive direction and the corresponding velocity is a periodic function of time of period two, as shown in Fig. 4(b). The phase space for this orbit is depicted in Fig. 4(c). We notice that the particle oscillates for a while around the minima of the ratchet potential, before moving to the next one. The spatial asymmetry of the potential is apparent in this orbit in phase space.

In Fig. 5(a) we show again the position as a function of time for $a = 0.081$, which is just above the critical value a_c. In this case, we observe a period-four orbit, that corresponds to the periodic window in the bifurcation diagram in Fig. 3(a). This orbit is such that the particle is "climbing" in the negative direction, that is, in the direction in which the slope of the potential is higher. We notice that there is a qualitative difference between the periodic orbit that transport particles to the positive direction and the periodic orbit that transport particles to the negative direction: in the latter case, the particle requires twice the time than in the former case, to advances one well in the ratchet potential. A closer look at the trajectory in Fig. 5(a) reveals the "trick" that the particle uses to navigate in the negative direction: in order to advance one step to the left, it moves first one step to the right and then two steps to the left. The net result is a negative current.

The period-four orbit is apparent in Fig. 5(b), where we show the velocity as a function of time. In Fig. 5(c) we depict the corresponding phase space for this case. The transporting orbit is more elaborate because it involves motion to the positive and negative directions, as well as oscillations around the minima.

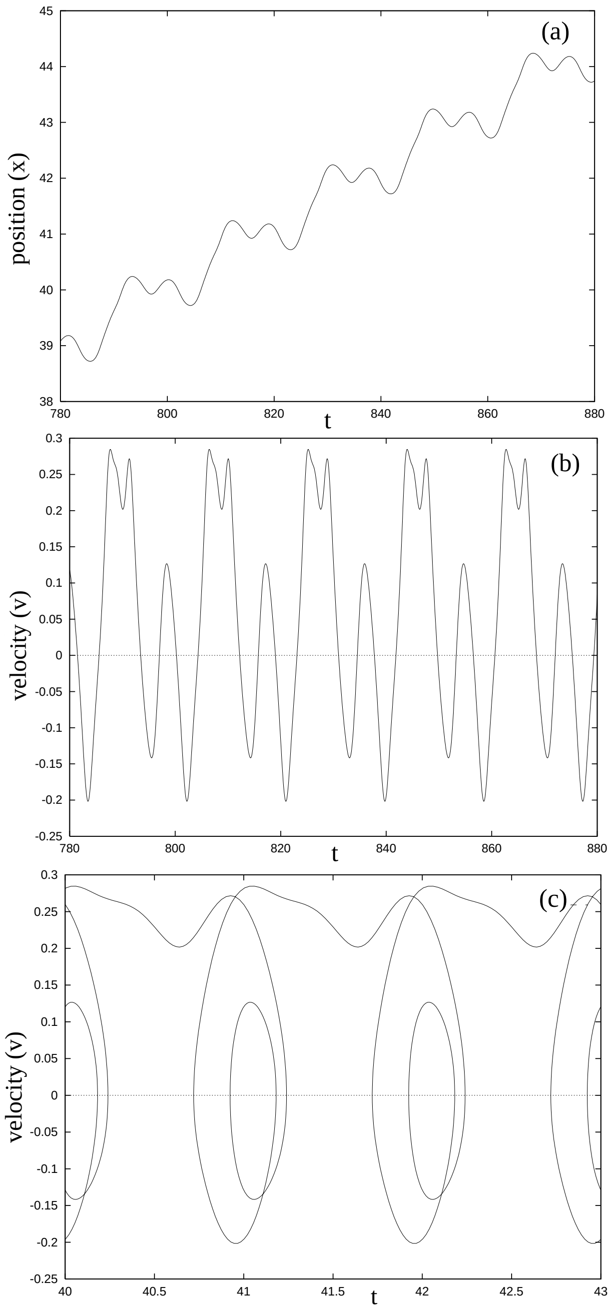

Fig. 4. For $b = 0.1$ and $w = 0.67$ and $a = 0.07$ we show: (a) the trajectory of the particle as a function of time, (b) the velocity as a function of time and (c) the phase space. This case corresponds to positive current.

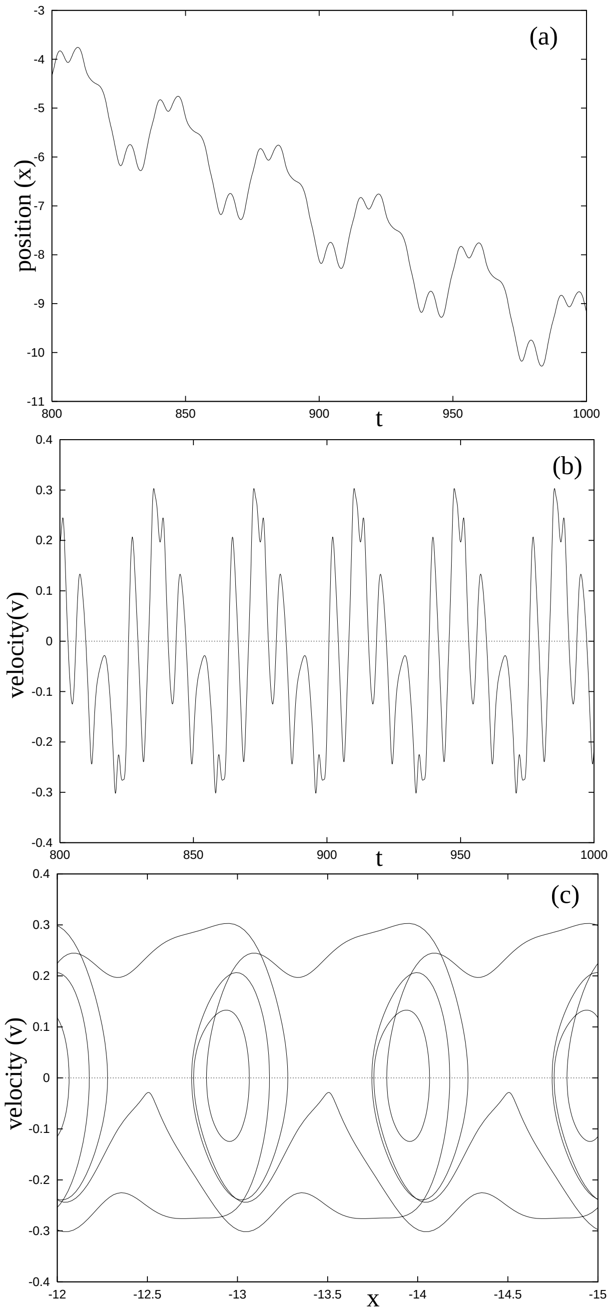

Fig. 5. For $b = 0.1$ and $w = 0.67$ and $a = 0.081$ we show: (a) the trajectory of the particle as a function of time, (b) the velocity as a function of time, (c) the phase space. This case corresponds to negative current.

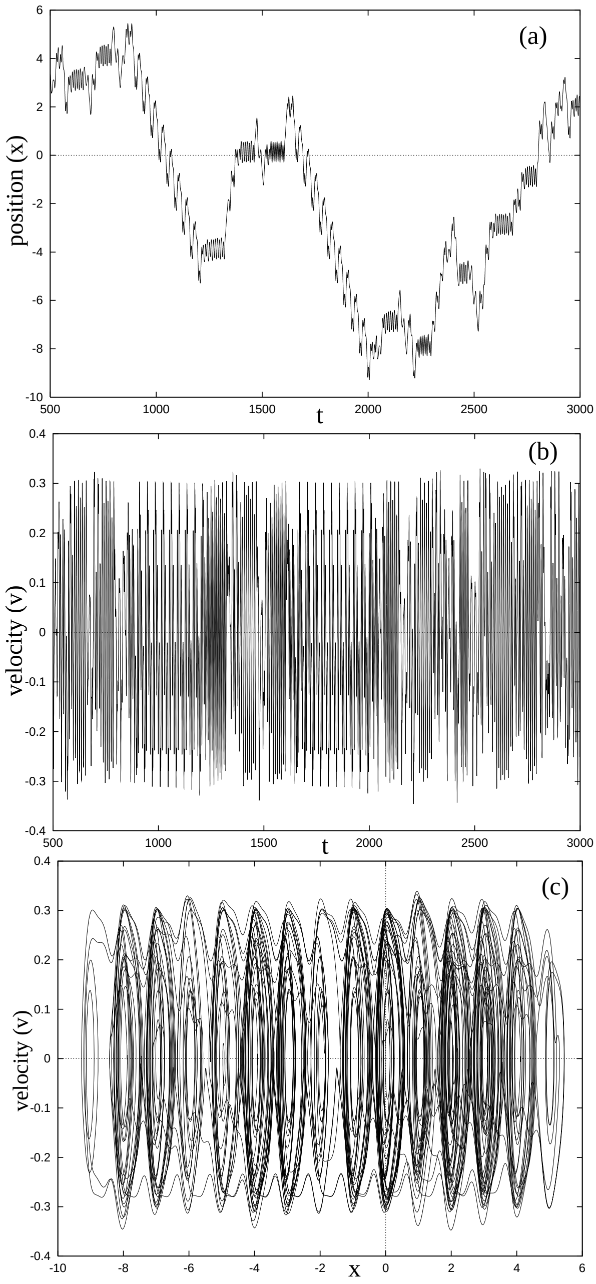

Fig. 6. For $b = 0.1$ and $w = 0.67$ and $a = 0.08092844$ we show: (a) the trajectory of the particle as a function of time, (b) the velocity as a function of time, (c) the phase space. This case corresponds to a near the bifurcation, where the dynamics becomes intermittent and there is anomalous diffusion.

In Fig. 6(a), we show a typical trajectory for a just below a_c. The trajectory is chaotic and the corresponding chaotic attractor is depicted in Fig. 7. In this case, the particle starts at the origin with no velocity; it jumps from one well in the ratchet potential to another well to the right or to the left in a chaotic way. The particle gets trapped oscillating for a while in a minimum (sticking mode), as is indicated by the integer values of x in the ordinate, and suddenly starts a running mode with average constant velocity in the negative direction. In terms of the velocity, these running modes, as the one depicted in Fig. 5(a), correspond to periodic motion. This can be seen more clearly in Fig. 6(b), where we plot the velocity as a function of time in the same range of values as the orbit in Fig. 6(a). In Fig. 6(c) we show the corresponding phase space.

The phenomenology can be described as follows. For values of a above a_c, as in Fig. 5(a), the attractor is a periodic orbit. For a slightly less than a_c there are long stretches of time (running or laminar modes) during which the orbit appears to be periodic and closely resembles the orbit for $a > a_c$, but this regular (approximately periodic) behavior is intermittently interrupted by finite duration "bursts" in which the orbit behaves in a chaotic manner. The net result in the velocity is a set of periodic stretches of time interrupted by burst of chaotic motion, signaling precisely the phenomenon of intermittency [24]. As a approach a_c from below, the duration of the running modes in the negative direction increases, until the duration diverges at $a = a_c$, where the trajectory becomes truly periodic.

To complete this picture, in Fig. 7, we show two attractors: (1) the chaotic attractor for $a = 0.08092$, just below a_c, corresponding to the trajectory in Fig. 6(a), and (2) the period-4 attractor for $a = 0.08093$, corresponding to the trajectory in Fig. 5(a). This periodic attractor consist of four points in phase space, which are located at the center of the open circles. We obtain these attractors confining the dynamics in x between -0.5 and 0.5, that is, we used the periodicity of the potential $V(x+1) = V(x)$, to map the points in the x axis modulo 1. Thus, even though the trajectory transport particles to infinity, when we confine the dynamics, the chaotic structure of the attractor is apparent. As a approaches a_c from below, the dynamics in the attractor becomes intermittent, spending most of the time in the vicinity of the period-four attractor, and suddenly "jumping" in a chaotic way for some time, and then returning close to the period-four attractor again, and so on. In terms of the velocity, the result is an intermittent time series as the one depicted in Fig. 6(b).

In order to characterize the deterministic diffusion in this regime, we calculate the mean square displacement $\langle (x - \langle x \rangle)^2 \rangle$ as a function of time. We obtain numerically that $\langle (x - \langle x \rangle)^2 \rangle \sim t^{\alpha}$, where the exponent $\alpha \simeq 3/2$. This is a signature of anomalous deterministic diffusion, in which

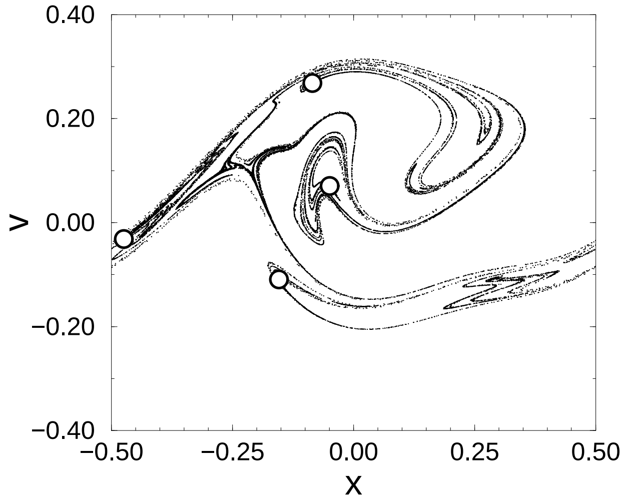

Fig. 7. For $b = 0.1$ and $w = 0.67$ we show two attractors: a chaotic attractor for $a = 0.08092$, just below a_c, and a period-four attractor, for $a = 0.08093$, consisting of four points located at the center of the open circles. See Ref. [18].

$\langle (x - \langle x \rangle)^2 \rangle$ grows faster than linear, that is, $\alpha > 1$ (super diffusion). Normal deterministic diffusion corresponds to $\alpha = 1$. In contrast, the trajectories in Figs. 4(a) and 5(a) transport particles in a ballistic way, with $\alpha = 2$. The relationship between anomalous deterministic diffusion and intermittent chaos has been explored recently, together with the connection with Lévy flights [25]. The character of the trajectories, as the one in Fig. 6(a), remains to be analyzed more carefully in order to determine if they correspond to Lévy flights.

4. Concluding remarks

In summary, we have studied the chaotic dynamics of a particle in a ratchet potential under the influence of an external periodic force. We establish a connection between the bifurcation diagram and the current and identify the mechanism by which the current reversal in deterministic ratchets arises: it corresponds to a bifurcation from a chaotic to a periodic regime. Near this bifurcation, the chaotic trajectories exhibit intermittent chaos and the transport arises through deterministic anomalous diffusion with an exponent greater than one. The richness and the complexity of the bifurcation diagram and the associated current, urge us to study their connection in more detail in the near future.

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