BROWNIAN MOTORS DRIVEN BY POISSONIAN FLUCTUATIONS*

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(Received November 15, 2000)

Overdamped directed motion of Brownian motors in a spatially periodic system, induced by Poissonian fluctuations of various statistics and driven by thermal noise, is investigated. Two models of asymmetric as well as two models of symmetric Poissonian fluctuations are considered. Transport properties in dependence upon statistics of fluctuations imposed on the system are analyzed.

PACS numbers: 02.50.Ey, 05.10.Gg, 05.40.–a

1. Introduction

A subject which has lately been gaining interest is transport of Brownian particles (motors) moving in spatially periodic structures. In such systems directed motion of particles can be induced by zero-mean deterministic and/or random forces (the so-called ratchet effect) [1]. Why is this

* Presented at the XXIV International School of Theoretical Physics "Transport Phenomena from Quantum to Classical Regimes", Ustroń, Poland, September 25–October 1, 2000.

subject so fascinating? For at first glance it is counterintuitive. One expects that imposition of zero-mean forces on the system yields zero-mean reaction of the system. Now, the explanation of the transport phenomenon in such systems is rather obvious: the system is out of equilibrium and the detailed balance does not hold! The conditions for transport to occur in periodic systems are known. The symmetry should be broken. What kind of symmetry should be broken is a secondary question. For example, the reflection symmetry of the spatially periodic structure can be broken [1] or the statistical symmetry of fluctuations can be broken [2]. The symmetry can also be broken by correlation of various degree of freedom [3] and by correlation of various sources of noise [4]. Because real systems are rarely symmetric, the occurrence of transport should be rather a generic phenomenon than exceptional and specific one.

In this paper we study directed current of non-interacting particles in the system subjected to a spatially periodic potential. Particles are driven by both zero-mean thermal equilibrium fluctuations and zero-mean nonequilibrium fluctuations modeled by Poissonian white shot noise. The problem stated is not new since it has been previously studied [5, 6] and it has been shown that indeed the ratchet effect can occur and preferential direction of Brownian motion is induced by nonequilibrium fluctuations. However, our objective is to investigate the influence of various statistics of Poissonian fluctuations on properties of transport and eventually to find universal properties of transport in such systems.

The article is organized as follows. In Sec. 2 we define a mathematical model of Brownian motors (a ratchet system) with all characteristics of its ingredients. In Sec. 3 we present two models of asymmetric Poissonian fluctuations and equations which determine the stationary probability distribution $P(x)$ and the stationary probability current J of Brownian motors. Sec. 4 is devoted to symmetric Poissonian fluctuations. In Sec. 5 we analyze transport properties of Brownian motors and include main conclusions.

2. Model

We consider an ensemble of non-interacting Brownian particles moving in a one-dimensional spatially periodic potential $\hat{V}(\hat{x}) = \hat{V}(\hat{x} + L)$ of period L and of the barrier height $V_0 = \hat{V}_{\max} - \hat{V}_{\min}$, and driven by random forces. The dynamics of the system is modeled by an overdamped stochastic Langevin equation. The equation of motion in the dimensionless form is (the dimensionless variables are discussed in detail in [7])

$$\frac{dx}{dt} = f\,(x) + \varGamma\,(t) + \eta(t)\,, \tag{1}$$

where $x = \hat{x}/L$ is the dimensionless position, $t = \hat{t}/\tau$ the dimensionless time, $\tau = \gamma L^2/V_0$ the characteristic time, and γ is the friction coefficient. The deterministic rescaled force

$$f(x) = -\frac{dV(x)}{dx}\,, \tag{2}$$

and $V(x) = V(x+1)$ is a rescaled periodic potential of unit spatial period and of unit barrier height. The stochastic force $\Gamma(t)$ is Gaussian thermal equilibrium noise of the first two moments

$$\langle \Gamma(t) \rangle = 0\,, \qquad \langle \Gamma(t)\Gamma(s) \rangle = 2D_T \delta(t-s)\,, \tag{3}$$

where the rescaled noise strength $D_T = k_{\mathrm{B}}T/V_0$, k_{B} is the Boltzmann constant and T temperature of the system. Let us take a note that D_T is a relation between thermal energy of fluctuations and activation energy of the particle from the bottom to the top of the potential $\hat{V}(\hat{x})$.

The random force $\eta(t)$ models nonequilibrium fluctuations and is chosen to be Poissonian shot noise [8]

$$\eta(t) = \sum_{i=-\infty}^{\infty} z_i \delta(t - t_i) - \lambda \langle z_i \rangle\,, \tag{4}$$

where t_i are random instants of δ impulses, characterized by the Poissonian counting process $n(t)$ with the parameter λ. It means that the probability for appearing of k impulses in the time-interval $[0, t]$ is given by the Poisson distribution, namely,

$$P(n(t) = k) = \frac{(\lambda t)^k}{k!} \exp(-\lambda t)\,. \tag{5}$$

The parameter λ determines a mean number of the δ impulses per unit time (a mean frequency of impulses). The amplitudes $\{z_i\}$ of the δ impulses are mutually independent random variables and independent on the counting process $n(t)$. The amplitudes $\{z_i\}$ are distributed according to the common probability density $\rho(z)$. The process $\eta(t)$ is symmetric if the distribution $\rho(z)$ is symmetric, *i.e.* when $\rho(z) = \rho(-z)$. Otherwise, it is asymmetric noise. In the latter case

$$a = \lambda \langle z_i \rangle \neq 0 \tag{6}$$

corresponds in (4) to the negative value of the bias of Poissonian noise between δ spikes. The process $\eta(t)$ is a white noise (but nonequilibrated noise) with an average and correlation function given by

$$\langle \eta(t) \rangle = 0\,, \qquad \langle \eta(t)\, \eta(s) \rangle = 2D_S\, \delta(t-s)\,. \tag{7}$$

The noise intensity reads:

$$D_S = \frac{\lambda \left\langle z_i^2 \right\rangle}{2} \,. \tag{8}$$

The evolution equation for the probability distribution $P(x,t)$ of the process $x(t)$ has a form of the continuity equation [6],

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x} J(x,t) \,, \tag{9}$$

where the probability current

$$\begin{aligned} J(x,t) \;&=\; \left[f(x) - \lambda \left\langle z_i \right\rangle\right] P(x,t) - D_T \frac{\partial}{\partial x} P(x,t) \\ &\quad + \lambda \int\limits_{-\infty}^{\infty} \rho(z) \int\limits_{0}^{z} P(x-y,t)\, dy\, dz \,. \end{aligned} \tag{10}$$

The probability density $P(x,t)$ has to obey the following conditions

$$P\left(x,t\right) \geq 0 \,, \quad P\left(x,0\right) = P\left(x+1,0\right) \,, \quad \int\limits_{x_0}^{x_0+1} P\left(x,t\right)\, dx = 1 \,, \tag{11}$$

for arbitrary x_0. In the stationary state, when $t \to \infty$,

$$\begin{aligned} J \;&=\; \lim_{t\to\infty} J(x,t) = -D_T P^{'}(x) + [f(x) - \lambda \left\langle z_i \right\rangle] P(x) \\ &\quad + \lambda \int\limits_{-\infty}^{\infty} \rho(z) \int\limits_{0}^{z} P(x-y)\, dy\, dz \,, \end{aligned} \tag{12}$$

where $P(x)$ and J are steady-state probability density and probability current, respectively; the prime denotes a derivative with respect to x. The current J is the most important characteristics of transport of Brownian motors. The relation between the steady-state (dimensionless) averaged velocity $\langle v \rangle$ of Brownian motors and the probability current is simple [7, 9], namely,

$$\langle v \rangle = \langle \dot{x} \rangle = J \,. \tag{13}$$

The dimensional averaged velocity $\langle \hat{v} \rangle$ is given by the relation [7, 9]

$$\langle \hat{v} \rangle = v_0 \; J \,, \tag{14}$$

where the characteristic velocity $v_0 = L/\tau = V_0/\gamma L$ is determined by three quantities: the barrier height V_0 and period L of the potential $\hat{V}(\hat{x})$ and the friction coefficient γ (which, in turn, depends on viscosity of the system in which Brownian motors move as well as on linear sizes of the motors).

3. Asymmetric Poissonian fluctuations

In this section we specify asymmetric Poissonian fluctuations [5]. For asymmetric fluctuations, the probability density $\rho(z)$ of amplitudes $\{z_i\}$ of δ kicks is asymmetric. We consider two examples of such densities. Both are special cases of the Gamma distribution [8], namely,

$$\rho(z) = \rho_1(z) = A^{-1}\Theta(z)\exp(-z/A)\,, \qquad A > 0\,, \tag{15}$$

and

$$\rho(z) = \rho_2(z) = A^{-2}\Theta\,(z)\,z\exp\left(-z/A\right)\,, \qquad A > 0\,, \tag{16}$$

where $\Theta(z)$ is the Heaviside function. For the first distribution $\rho_1(z)$, the first two moments read

$$\langle z_i\rangle = A\,, \quad \left\langle z_i^2\right\rangle = 2A^2\,. \tag{17}$$

For the second distribution $\rho_2(z)$,

$$\langle z_i\rangle = 2A\,, \quad \left\langle z_i^2\right\rangle = 6A^2\,. \tag{18}$$

In these two cases, the amplitudes $\{z_i\}$ are positive: from time to time the δ impulse kicks the particle to the positive direction of x. On the other hand, the negative bias, $a = \lambda\langle z_i\rangle$, pushes the particle to the negative direction (let us remember that in summary $\langle\eta(t)\rangle = 0$). The main difference between these two distributions is (non)monotonicity and in consequence qualitatively different most probable values of amplitudes $\{z_i\}$ of δ impulses. In the first case, very small amplitudes (close to zero) are more probable. In the second case, the amplitudes $z_i = A$ are the most probable. Therefore, by manipulation of the parameter A, we have hoped for significant difference in transport properties for these two cases.

The equation determining the stationary probability distribution $P(x) = P_k(x)$, $k = 1, 2$, and stationary current $J = J_k$, $k = 1, 2$, is Eq. (12). It is an integro-differential equation and it is very difficult to handle it. This is why in each case we convert it to the form of differential equations: In the first case the result is

$$\begin{aligned} J_1 \;&=\; -D_T A P_1''(x) - \left[D_T + D_S - Af(x)\right] P_1'(x) \\ &\quad + \left[f(x) + Af'(x)\right] P_1(x)\,. \end{aligned} \tag{19}$$

In the second case,

$$\begin{aligned} J_2 \;&=\; -A^2 D_T P_2'''(x) + A\left[\left(Af(x) - \frac{2D_S}{3} - 2D_T\right) P_2(x)\right]'' \\ &\quad + \left[\left(2Af(x) - D_S - D_T\right) P_2(x)\right]' + f(x)\,P_2(x)\,. \end{aligned} \tag{20}$$

In order to obtain these equations, we have to change the integration order in the last term of Eq. (12). Next, observe that $\rho_k(z)$ $(k = 1, 2)$ obeys an ordinary differential equation of constant coefficients.

4. Symmetric Poissonian fluctuations

In this section we introduce symmetric Poissonian fluctuations [6]. For symmetric fluctuations, the probability density $\rho(z)$ of amplitudes $\{z_i\}$ of δ kicks is symmetric. We present two examples of such distributions: The exponential distribution,

$$\rho(z) = \rho_3(z) = 0.5\, A^{-1} \exp\left(-|z|/A\right), \qquad A > 0\,, \tag{21}$$

and a particular Gamma distribution, namely,

$$\rho\,(z) = \rho_4(z) = 0.5\, A^{-2}\, |z| \exp\left(-|z|/A\right), \quad A > 0\,. \tag{22}$$

The mean value of amplitudes $\langle z_i \rangle = 0$ and the second moments $\langle z_i^2 \rangle$ are the same as in the previous corresponding cases. The equations determining the stationary probability distribution $P(x) = P_k(x)$, $k = 3, 4$ and stationary current $J = J_k$, $k = 3, 4$ have the form

$$\begin{aligned} J_3 \;&=\; A^2 D_T P_3^{'''}(x) - A^2 \left[f(x) P_3(x)\right]^{''} \\ &\quad - \left(D_T + D_S\right) P_3^{'}(x) + f(x)\, P_3(x) \end{aligned} \tag{23}$$

in the case (21) and

$$\begin{aligned} J_4 \;&=\; -A^4 D_T P_4^{(5)}(x) + A^4 \left[f(x)\, P_4(x)\right]^{(4)} \\ &\quad + A^2 \left(2 D_T + \frac{D_S}{3}\right) P_4^{(3)}(x) - 2A^2 \left[f(x)\, P_4(x)\right]^{(2)} \\ &\quad - \left(D_T + D_S\right) P_4^{(1)}(x) + f(x)\, P_4(x) \end{aligned} \tag{24}$$

in the case (22). The superscripts (n), $n = 1, ..., 5$ denote the n-order derivative with respect to x. One can notice that order of the differential equation for $P(x)$ depends strongly on the form of the probability distribution of amplitudes $\{z_i\}$. *E.g.* for $\rho_4(z)$, the differential equation is of the 5-th order. It is interesting that from all these differential equations we can determine both $P_k(x)$ and J_k, $k = 1, ..., 4$, and the same conditions (11) are sufficient to solve them uniquely. In general, these equations cannot be solved by analytical means. Nevertheless, they can be solved numerically for an arbitrary form of the potential $V(x)$. However, exact analytical results can be obtained for a piecewise linear sawtooth-like potential. We choose the simplest form of such a potential, namely,

$$V(x) = \begin{cases} -2\frac{x-k}{1+2k}, & x \in [-1/2, k] \mod 1\,, \\ 2\frac{x-k}{1+2k}, & x \in [k, 1/2] \mod 1\,, \end{cases} \tag{25}$$

where $k \in (-1/2, 1/2)$ determines the asymmetry of the potential: For $k = 0$ it is reflection-symmetric, *i.e.*, $V(x) = V(-x)$; for $k \neq 0$ the reflection symmetry of $V(x)$ is broken. In Fig. 1, we present $V(x)$ in its graphical form.

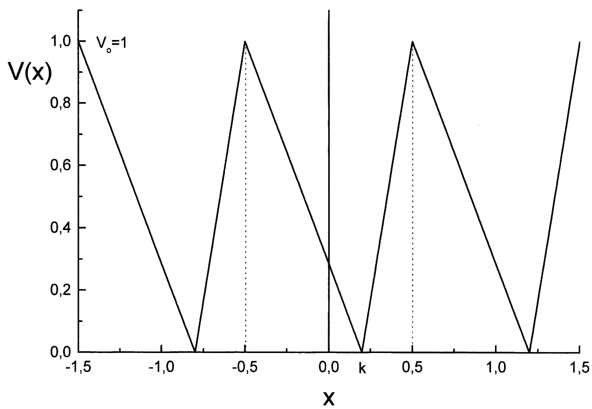

Fig. 1. The rescaled periodic potential $V(x) = V(x+1)$ of asymmetry determined by $k \in (-1/2, 1/2)$.

5. Analysis

We analyze the stationary averaged velocity of Brownian motors or, equivalently, the steady-state probability current, $J = \langle v \rangle$, which is determined by Eqs. (19), (20), (23) and (24) respectively. In order to solve these equations, we can proceed along the same way as in [5]. The method of solution of equations like considered here is also presented in [9, 10]. In this method, the probability density $P(x)$ and current J is determined by a nonhomogeneous system of linear algebraic equations and evaluation of the current is a matter of linear algebra. In each of four cases, J is a quotient of two determinants: the fifth degree in the case (19), the seventh degree in the cases (20) and (23), and the eleventh degree for (24). The explicit form of J is an extremely complex expression and therefore is not reproduced here. The detailed discussion of J in cases (15) and (21) is performed in [5] and [6], respectively. Here, we mention main and the most interesting findings which are visualized in Figs. 2 to 6.

(A) For the asymmetric distribution of amplitudes:

— Generally, the properties of transport are qualitatively the same for both distributions.

— There are two regimes of transport: diffusive and non-diffusive. In the diffusive regime, Poissonian fluctuations $\eta(t)$ induces both forward and backward transitions over the potential barrier. In the non-diffusive regime, Poissonian fluctuations $\eta(t)$ induces only forward transitions. This is the case when the bias $a = \lambda\langle z_i \rangle$ is smaller than the maximal value of the deterministic force $f(x)$. In Figs. 2 and 3 we display the stationary probability density $P(x)$ in these two

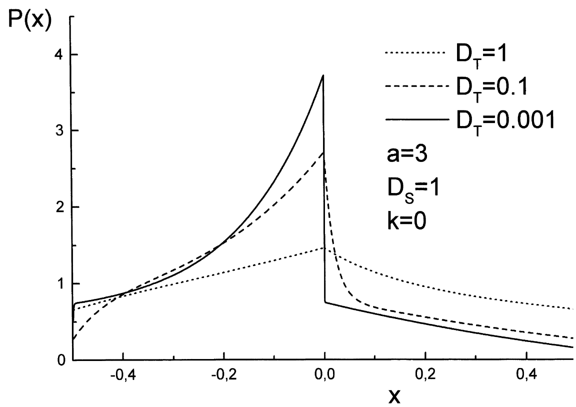

Fig. 2. The stationary probability density $P(x)$ in the diffusive regime for asymmetric Poissonian noise with exponentially distributed amplitudes (15) of δ impulses.

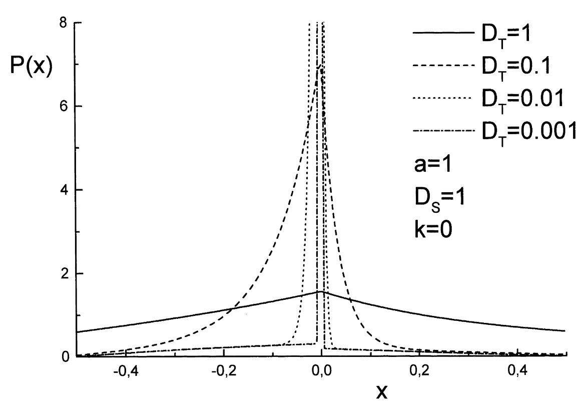

Fig. 3. The stationary probability density $P(x)$ in the non-diffusive regime for asymmetric Poissonian noise with exponentially distributed amplitudes (15) of δ impulses.

regimes. The main feature of $P(x)$ is its asymmetry even if the potential $V(x)$ is symmetric. The reason is asymmetry of the deterministic part $f(x) - \lambda\langle z_i \rangle$ of the force, *cf.* (1) and (4).

— $J > 0$ independently of asymmetry of the potential. This is because of positive amplitudes $\{z_i\}$ of δ kicks.

— There are optimal values of parameters which maximize the current. It is shown in Fig. 4 with a generic dependence of the stationary current on parameters of fluctuations.

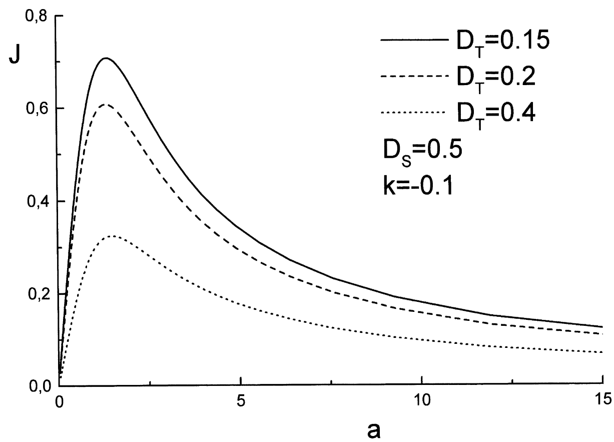

Fig. 4. Dimensionless probability current (averaged stationary velocity) J *vs* the bias of Poissonian noise in the case of the asymmetric Gamma distribution (16) of amplitudes of δ impulses for selected values of temperature.

(B) For the symmetric distribution of amplitudes:

— The current $J = 0$ if $V(x)$ is symmetric; $J \neq 0$ if $V(x)$ is asymmetric.

— Generally, the properties of transport are qualitatively the same for both distributions.

— However, for specific choice of parameter set, one can notice a remarkable difference: for the exponential distribution (21), the current J is monotonic function of D_S, while for the Gamma distribution (22) it is non-monotonic (see Figs. 5 and 6).

— The sign of J depends on asymmetry of the potential. For asymmetric potentials, the distance d_{mM} between a minimum and a neighboring maximum is different that the distance d_{Mm} between the maximum and the next neighboring minimum. Let us consider the case of the potential shown in Fig. 1 with asymmetry $k > 0$. Particles are symmetrically kicked by δ impulses into the left and into the right directions. However, the probability that the particle falls into the interval $(-1/2, k) \bmod 1$ is greater than it falls into the interval $(k, 1/2) \bmod 1$. As a consequence, motors move in the direction from the maximum to the minimum along both slighter and longer slope of the potential $V(x)$ (in the situation shown in Fig. 1, in the right direction).

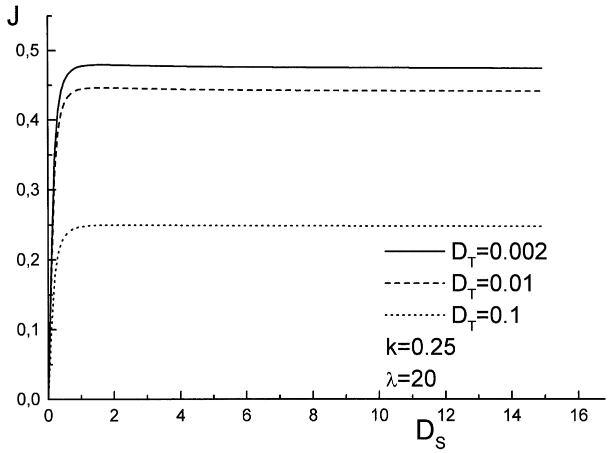

Fig. 5. Dimensionless probability current J *vs* the intensity D_S of Poissonian noise in the case of the symmetric exponentially distributed (21) amplitudes of δ impulses for selected values of temperature.

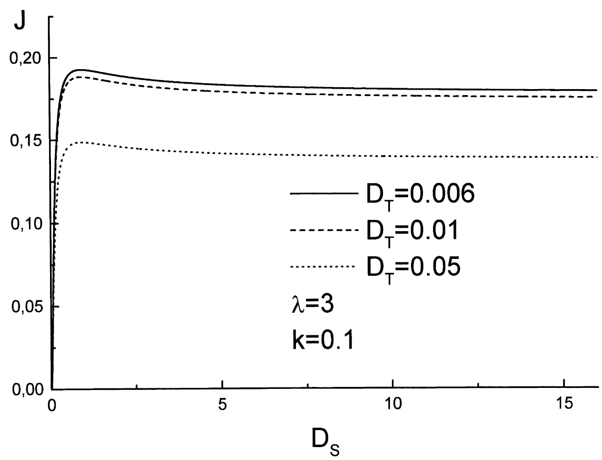

Fig. 6. Same as in Fig. 5 but for the symmetric Gamma distribution (22) of amplitudes of δ impulses.

The current is influenced by other parameters of the model such as the bias of Poissonian white fluctuations, temperature of the system and asymmetry of the potential. One can contemplate other statistics of Poissonian fluctuations to be implemented to the model considered. But the main conclusion is rather disappointing: differences caused by various sources of Poissonian fluctuations are mirror and they drive the Brownian motors qualitatively in the universal way. On the other hand, we analyzed only one transport coefficient, namely, the stationary mean velocity $\langle v\rangle$. Other important transport characteristics like efficiency of the ratchet system, the diffusion coefficient, $D = \lim_{t\to\infty}(\langle x^2(t) - \langle x(t)\rangle^2)/t$, or fluctuations of velocity, $\langle(\Delta v)^2\rangle = \langle v^2\rangle - \langle v\rangle^2$, have not been analyzed. As usual, more unsolved problems arise in the end of the paper than we have started with.

J. Ł. would like to thank DAAD for supporting his visit at the University of Augsburg, where part of the research was performed.

REFERENCES

[1] M. Bier, *Cont. Phys.* **38**, 371 (1997); F. Jülicher, A. Ajdari, J. Prost, *Rev. Mod. Phys.* **69**, 1269 (1997); P. Hänggi, R. Bartussek, *Nonlinear Physics of Complex Systems — Current Status and Future Trends*, Eds. J. Parisi, S.C. Müller, W. Zimmermann, Springer Series *Lecture Notes in Physics*, Vol. 476, Springer, Berlin 1996, p. 294.

[2] P. Hänggi, R. Bartussek, P. Talkner, J. Łuczka, *Europhys. Lett.* **35**, 315 (1996).

[3] M. Kostur, L. Schimansky-Geier, *Phys. Lett.* **A265**, 337 (2000).

[4] A. Fuliński, T. Telejko, *Phys. Lett.* **A152**, 11 (1990); J.H. Li, Z.Q. Huang, *Phys. Rev.* **E57**, 3917 (1998).

[5] T. Czernik, J. Kula, J. Łuczka, P. Hänggi, *Phys. Rev.* **E55**, 4057 (1997).

[6] J. Łuczka, T. Czernik, P. Hänggi, *Phys. Rev.* **E56**, 3968 (1997).

[7] J. Łuczka, *Physica* **A274**, 200 (1999).

[8] A. Papoulis, *Probability, Random Variables and Stochastic Processes*, Mc Graw Hill, New York 1984.

[9] J. Kula, M. Kostur, J. Łuczka, *Chem. Phys.* **235**, 27 (1998).

[10] T. Czernik, J. Łuczka, *Ann. Phys.* **9**, 721 (2000).