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COUPLED STATES OF TWO KINKS IN A SYSTEM WITH NONLINEAR DAMPING*

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The coupled states of two kinks and/or antikinks for scalar model with fourth-order potential in presence of linear and nonlinear frictional terms and absence of external force are constructed.

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1. Introduction and formulation of the problem

To study the transport phenomena in biological systems the model was suggested where the coherent excitation of polar modes that are stabilized by non-linear deformations of the system are used [1]. These deformations were described by a repulsive fourth-order term in the polarization field. A modification of this model that accounts for a linear frictional term was used in [2] for description of the non-linear dynamics of biological materials. In that paper the exact solution of equation of motion was constructed and the possibility of loss-free transport of energy in biological materials was pointed out. In [3] the restriction on the possibility to realize such a solution in physical systems or biological materials was indicated because it has an infinite energy. It was concluded on the results of calculation of the energy loss velocity. The velocity was fixed and determined by a constant damping coefficient only. Hence the conclusion may be derived that for stable propagation of solution in a frictional medium in absence of driving force the infinite energy is required. But another interpretation in such an active medium may be possible when the energy is undefined, namely, the system can be considered as unclosed.

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For further modification of the model of active medium the non-linear frictional effects should be accounted for. Such approach was used for description of the Josephson junctions, charge density waves and structural transitions [4, 5]. In these problems only one-soliton or one-kink solutions of nonlinear equations were studied. Here the solutions of the equation of motion that correspond to coupled states of two kinks and/or antikinks in scalar field theory with linear and nonlinear friction are constructed. Investigation of the new solutions allows to describe the additional peculiarities of physical phenomena under investigation. Besides, they may be useful for the problem of integrability [6].

To study two kink and/or antikink solutions in active medium let us consider a simple (1+1)-dimensional model of scalar field theory with fourth-order potential and take into account the linear and nonlinear frictional effects. If an external force is absent, the equation of motion has a form

$$\phi_{tt} - \phi_{xx} + \alpha\phi_t + \beta\phi_t^2 - \phi + \phi^3 = 0\,. \tag{1}$$

Here α and β are damping (frictional) coefficients, $\phi_{tt} = \partial^2\phi/\partial t^2$ and so on.

To solve this equation the method for solving nonlinear equation of mathematical physics extending the Hirota method to the case of degeneracy was used [7]. According to this method a function $\phi(x,t)$ should be replaced by new unknown function $F(x,t)$ by means of Cole–Hopf transformation with an arbitrary coefficient σ. Unlike as in the Hirota method, a concrete value of this coefficient is determined at the last stage of construction of a one-kink solution. Then $F(x,t)$ is represented as a formal series in powers of a parameter ε which does not have to be small

$$F(x,t) = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots\,, \tag{2}$$

By substituting this series for $F(x,t)$ into Eq. (1) and equating to zero coefficients for every degree of ε an infinite system of linear differential equations for new unknown functions $f_i(x,t), i = 1, 2, \dots$ may be obtained. To construct two-kink solutions the first three equations of this system are needed. They have the form

$$\begin{aligned}
f_{1,xtt} &- f_{1,xxx} + \alpha f_{1,xt} - f_{1,x} = 0\,,\\
f_{2,xtt} &- f_{2,xxx} + \alpha f_{2,xt} - f_{2,x} = 2f_{1,xt}f_{1,t} + f_{1,tt}f_{1,x}\\
&- 3f_{1,xx}f_{1,x} + \alpha f_{1,t}f_{1,x} - \beta\sigma f_{1,xt}^2\,,\\
f_{3,xtt} &- f_{3,xxx} + \alpha f_{3,xt} - f_{3,x} = 2f_{1,xt}f_{2,t} + 2f_{1,t}f_{2,xt} + f_{1,tt}f_{2,x}\\
&+ f_{1,x}f_{2,tt} - 2f_{1,x}f_{1,t}^2 - 3f_{1,xx}f_{2,x} - 3f_{1,x}f_{2,xx} - \left(\sigma^2 - 2\right) f_{1,x}^3\\
&+ \alpha f_{1,t}f_{2,x} + \alpha f_{1,x}f_{2,t} - 2\beta\sigma f_{1,xt}f_{2,xt} + 2\beta\sigma f_{1,x}f_{1,t}f_{1,xt}\\
&+ 2f_1f_{2,xtt} - 2f_1f_{2,xxx} + 2\alpha f_1f_{2,xt} - 2f_1f_{2,x}\,.
\end{aligned} \tag{3}$$

It is clear that for every i, function $f_i(x,t)$ is determined by previous functions only.

2. Two-kink solution

To construct the explicit expressions for solutions that describe the coupled states of two kinks/antikinks or kink and antikink, let us write down the functions f_1 and f_2 in the following form

$$\begin{aligned} f_1(x,t) &= \exp\left(k_1 x - \omega_1 t + \eta_1^{(0)}\right) + \exp\left(k_2 x - \omega_2 t + \eta_2^{(0)}\right) , \\ f_2(x,t) &= A\exp\left[2\left(k_1 x - \omega_1 t + \eta_1^{(0)}\right)\right] \\ &\quad + B\exp\left[(k_1+k_2)x - (\omega_1+\omega_2)t \right. \\ &\quad \left. + \eta_1^{(0)} + \eta_2^{(0)}\right] + C\exp\left[2\left(k_2 x - \omega_2 t + \eta_2^{(0)}\right)\right] . \end{aligned} \tag{4}$$

Here k_i, ω_i and $\eta_i, i = 1,2$ are parameters of solution which may be considered as analogues of wave number, multiplication of phase velocity on wave number and initial phase shift. Unlike in the Hirota method, there are no special requirements on the coefficients A, B and C and the condition $k_i \neq k_j$ does not apply.

An analysis of one kink (antikink) solution demonstrated that for one-particle function the parameter ω takes two values (negative and positive):

$$\omega = \frac{\alpha \pm \sqrt{\alpha^2 + 3\beta)}}{\beta} .$$

In appropriate way, the parameter k takes two values for every ω:

$$k^2 = \frac{2\alpha^2 - \alpha^2\beta - \beta^2 + \beta \pm \alpha(2-\beta)\sqrt{\alpha^2+3\beta}}{\beta} .$$

Here the minus sign corresponds to negative ω and the plus sign corresponds to positive ω. For fixed ω different signs of k correspond to kink and antikink.

Two-kink solutions may be constructed if the explicit expressions for coefficients A, B and C are obtained. It is possible to do by substituting Eqs. (4) into right hand side of the third of Eqs. (3) and equating to zero the coefficients for every exponential function. Additionally, it allows to truncate the series in Eq. (2) as in the Hirota method.

The results of calculation of the solutions that correspond to coupled states of two kinks and/or antikinks may be presented in the form (for simplicity, all initial phase shift are put equal to zero):

1) negative ω

a) two kinks; $k_1 = k_2 = k$ (two antikinks; $k_1 = k_2 = -k$)

$$\phi(x,t) = \frac{2\exp(\eta)\,[1+2A\exp(\eta)]}{1+2A\exp(\eta)\,[1+2A\exp(\eta)]}\,,$$

where $\eta = \pm kx - \omega t$ for kinks and antikinks, respectively, and

$$A = \frac{2\alpha\beta\sqrt{\alpha^2+3\beta} - 2\alpha^2\beta - 3\beta^2}{6\alpha^2\beta + 6\beta^2 - 6\alpha\beta\sqrt{\alpha^2+3\beta} - 4\alpha^2 - 6\beta + 4\alpha\sqrt{\alpha^2+3\beta}}\,;$$

b) kink and antikink; $k_1 = -k_2 = k$

$$\phi(x,t) = \frac{\exp(\eta_+)+\exp(\eta_-)+A\exp(2\eta_+)+B\exp(\eta_++\eta_-)+A\exp(2\eta_-)}{1+\exp(\eta_+)+\exp(\eta_-)+A\exp(2\eta_+)+B\exp(\eta_++\eta_-)+A\exp(2\eta_-)},$$

where $\eta_+ = kx - \omega t$, $\eta_- = -kx - \omega t$, $B = (K_2K_5 - K_3K_4)/K_1K_5$,

$$\begin{aligned}
K_1 &= 16\alpha^2 - 16\alpha\sqrt{\alpha^2+\beta} + 24\beta - 2\alpha^2\beta + 2\alpha\beta\sqrt{\alpha^2+3\beta}\,,\\
K_2 &= 16\alpha^2 - 16\alpha\sqrt{\alpha^2+3\beta} + 24\beta + 2\alpha^2\beta - 2\alpha\beta\sqrt{\alpha^2+3\beta} - 3\beta^2\,,\\
K_3 &= 2\alpha\beta\sqrt{\alpha^2+3\beta} - 2\alpha^2\beta - 3\beta^2\,,\\
K_4 &= 16\alpha^2 - 16\alpha\beta\sqrt{\alpha^2+3\beta} + 24\beta + 16\alpha^2\beta - 16\alpha\beta\sqrt{\alpha^2+3\beta} + 30\beta^2\,,\\
K_5 &= 6\alpha^2\beta + 6\beta^2 - 6\alpha\beta\sqrt{\alpha^2+3\beta} - 4\alpha^2 - 6\beta + 4\alpha\sqrt{\alpha^2+3\beta}\,.
\end{aligned}$$

2) positive ω

a) two kinks; $k_1 = k_2 = k$ (two antikinks; $k_1 = k_2 = -k$)

$$\phi(x,t) = \frac{2\exp(\eta)\,[1+2C\exp(\eta)]}{1+2\exp(\eta)\,[1+2C\exp(\eta)]}\,,$$

where $\eta = \pm kx - \omega t$ for kinks and antikinks, respectively,

$$C = -\frac{2\alpha^2\beta + 2\alpha\beta\sqrt{\alpha^2+3\beta} + 3\beta^2}{6\alpha^2\beta + 6\beta^2 + 6\alpha\beta\sqrt{\alpha^2+3\beta} - 4\alpha^2 - 6\beta - 4\alpha\sqrt{\alpha^2+3\beta}}\,;$$

b) kink and antikink; $k_1 = -k_2 = k$

$$\phi(x,t)=\frac{\exp(\eta_+)+\exp(\eta_-)+C\exp(2\eta_+)+D\exp(\eta_++\eta_-)+C\exp(2\eta_-)}{1+\exp(\eta_+)+\exp(\eta_-)+C\exp(2\eta_+)+D\exp(\eta_++\eta_-)+C\exp(2\eta_-)}\,,$$

where $\eta_+ = kx - \omega t$, $\eta_- = -kx - \omega t$, $D = (N_2N_5 + N_3N_4)/N_1N_5$,

$$\begin{aligned}
N_1 &= 16\alpha^2 + 16\alpha\sqrt{\alpha^2+3\beta} + 24\beta - 2\alpha^2\beta - 2\alpha\beta\sqrt{\alpha^2+3\beta}\,,\\
N_2 &= 16\alpha^2 + 16\alpha\sqrt{\alpha^2+3\beta} + 24\beta + 2\alpha^2\beta + 2\alpha\beta\sqrt{\alpha^2+3\beta} - 3\beta^2\,,\\
N_3 &= 2\alpha^2\beta + 2\alpha\beta\sqrt{\alpha^2+3\beta} + 3\beta^2\,,\\
N_4 &= 16\alpha^2 + 16\alpha\sqrt{\alpha^2+3\beta} + 24\beta + 16\alpha^2\beta + 16\alpha\beta\sqrt{\alpha^2+3\beta} + 30\beta^2\,,\\
N_5 &= 6\alpha^2\beta + 6\beta^2 + 6\alpha\beta\sqrt{\alpha^2+3\beta} - 4\alpha^2 - 6\beta - 4\alpha\sqrt{\alpha^2+3\beta}\,.
\end{aligned}$$

3. Conclusion

The two kink and/or antikink coupled states for the model with fourth-order potential in presence of linear and nonlinear frictional terms and absence of external force are constructed. Taking into account a linear friction only leads to one negative value of the parameter connected with kink velocity. If a nonlinear square friction is taken into account, the two values of this parameter with different signs appear. As a result the new states with positive velocity may be constructed in addition to the usual states with negative velocity. A special attention should be paid to the coupled states of kink and antikink because these solutions are localized but not as the waves of constant profile. The solutions that correspond to coupled states of topologically non-trivial configurations allow to describe the new peculiarities of transport phenomena in the active media. In particular, for a blood as a ferroelectric fluid it is possible to control its flow and domain structure not only by application of external force (usually, by application of electromagnetic field) but by using the medicines that can change the friction coefficients of a blood. In addition to one kink configurations of polarization of blood, the coupled states give the new possibilities to vary a velocity of blood flow and distribution of polarization in blood.

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