SEMICLASSICAL STABILITY ANALYSIS OF A TWO-PHOTON LASER***

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We explain in this paper in simple terms the behavior of two-photon lasers and describe recent results that have led to the realization of the first continuous-wave two-photon optical laser. We stress the differences between one- and two-photon lasers to develop an appreciation of their dynamics and the difficulties associated with achieving two-photon lasing. We find similarities and significant differences between the one- and two-photon polarizations of the medium, population inversion and mode-pulling formula. The theory is treated semiclassically by using Maxwell–Bloch equations. We study the linear stability analysis of the steady state of the system which is taken to be contained in a ring cavity. The results are illustrated with an application to a specific atomic system in a long sample of sodium vapor as an amplifying medium, in which the possibility of short pulse train generation is exhibited.

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1. Overview

The phenomenon of the laser action involving one-photon emission per atomic transition has long been successfully explained [1]. Consider a single photon that is incident on an excited atom and resonant with an internal atomic transition leading to a lower energy state. This photon may stimulate the atom to jump to its lower energy state and emit an energy conserving photon having essentially the same frequency as the stimulating photon. Lasers come in many different designs but a simple and often used scheme

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consists of a cavity which is two highly reflecting mirrors between which the field can build up. If that was all, of course nothing would happen, since the mirrors are lossy. To compensate for this loss, we have to pump the cavity, which is done by a beam of excited atoms. When an atom in the beam enters the cavity, it feels the field existing there, and if the photon in the cavity is in resonance with a dipole transition in the atom, the atom can jump to this lower state by stimulated emission of a photon to the cavity mode. If the atomic pump rate *i.e.* in this scheme, the number of atoms flying through the cavity per unit time is high enough, the gain of photons in the cavity (due to stimulated emission) will exceed the loss (through the mirrors) and hence the field can build up between the mirrors and we have lasing. The possibility of achieving laser action involving two-photon per atomic transition was first suggested in [2,3]. It has received much attention in recent years, not only because this novel type of laser may be potential as a high power optical amplifier and as a tunable source, but also because the self-organization features of a system with nonlinear interaction between the field and matter are interesting. Unfortunately, the practical benefits of the laser have not been realized nor have the theories been tested because it has been difficult to achieve two-photon lasing [4] due to the lack of suitable gain media. An obvious question that might arise here is: What is a two-photon laser? The answer to this question is quite simple if we compare and contrast the origin of gain in one- and two-photon lasers.

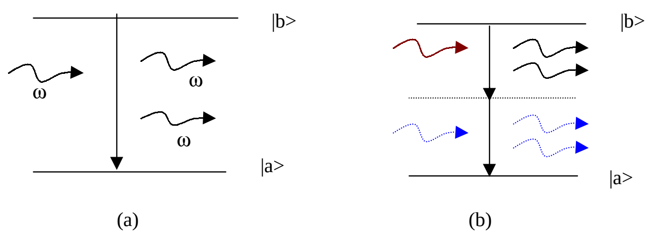

Fig. 1. (a) The one-photon stimulated emission process. The states have opposite parity. (b) The two-photon stimulated emission process. The states have the same parity.

In this process, an incident photon can stimulate an atom from the upper state $| b'\rangle$ to the lower state $| a'\rangle$ and two-photon are scattered by the atom. The scattered photons have the same frequency, phase and direction of the incident photon which give the laser its unique coherence properties. The stimulated emission rate is proportional to the incident photon flux and proceeds most efficient when the frequency of the incident photon ω is equal to the transition frequency $\omega_{b'a'}$ and when the states have opposite parity (connected by an allowed electric dipole transition matrix element). In contrast, the gain in two-photon lasers is due to the two-photon stimulated emission process. In this process, two incident photons can stimulate an atom from the upper state $\mid b\rangle$ to the lower state $\mid a\rangle$ and four photons can take on any value so long as $2\omega' \simeq \omega_{ba}$, where ω_{ba} is the two-photon transition frequency. The process is the most efficient when the states have the same parity (not connected by an allowed electric dipole transition matrix element). The scattered photons have the same frequency, phase and direction of the incident photons give the laser coherence properties different than these of normal one-photon laser [5-13].

The problem for an experimental realization of a two-photon laser is that two-photon coupling in general is very small, since it is hard to find an atom where two-levels of the same parity have an almost resonant intermediate level with opposite parity and separated by photons of optical frequencies. The existence of the almost intermediate level is demanded for reaching a sufficiently large two-photon coupling. This problem has, however, been circumvented in two ways; *(i)* between high-lying Rydberg states in alkali atoms, quite large two-photon matrix elements can be found, due to existence of an almost resonant intermediate level of the opposite parity. The photons involved in such a transition are, however, quite small, actually in the microwave region. The corresponding laser pumped by this transition is called a micromaser and has been the subject of intense theoretical studies [14–16], and was the first two-photon micromaser realized experimentally [17]. The two-photon micromaser has many interesting properties, which distinguishes it from two-photon laser. This difference mainly arises because the spontaneous life-time of Rydberg atoms is much longer than for lower lying states and the atoms will decay spontaneously in the cavity, whereas this does not happen in the micromaser. Any decay is quantum noise, and the effects characterizing the two-photon micromaser are therefore in some sense washed out because of this noise and therefore not seen in the two-photon laser; and *(ii)* in the optical regime, a clever scheme has been adopted by Mossberg and collaborators in which they use dressed atoms (The energy levels of an atom in a strong field are eigenstates of the Hamiltonian describing field+atom. These energy levels form a ladder and differ substantially from those of an atom in zero field. An atom exposed to a strong field is called a dressed atom.) as pump media, and the two-photon transition is then between the dressed-atom energy levels. We shall not enter a discussion of the properties of two-photon dressed-atom laser, but refer instead to the literature [18–23].

Although a definitive experimental implementation of a Two-Photon Laser (TPL) has not been possible until very recently [23], TPL's have been the subject of continued theoretical attention since the early days of the laser era [13,14]. The theoretical interest of the TPL lies in the intrinsic nonlinear nature of the two-photon interaction. This fact makes this system a potential source for non-classical light and thus the major part of the literature has been devoted to the quantum description of such a laser [24–29]. Contrarily there is not much work on semi-classical modeling of TPL and, in particular, there still lacks a complete understanding of its stability and dynamical properties [30]. On the other hand the two-photon resonant interaction has been investigated in detail when the temporal width τ_0 of the pulse is much less than both of population and polarization lifetimes *i.e.* $\tau_\circ \ll \tau_1, \tau_2$ holds [31,32]. (Here τ_1 and τ_2 are population and polarization lifetimes respectively). Recently nonlinear propagation of picosecond pulses interacting with a three-level system in the intermediate region $\tau_1 > \tau_0 > \tau_2$ has been carried out [33]. TPL's are a recurrent theme in the literature and have attracted considerable theoretical semiclassically [34–36] as well as quantum mechanically [24–29]. On the experimental side, a TPL has been realized and studied extensively in the microwave region of the spectrum [24] and in the optical regime by Mossberg and collaborators [28–30] in a clever, slightly different scheme, where the atomic pump transition in between levels of dressed atoms. In order to obtain the pure two-photon laser, in which the laser works without one-photon processes contributions, some conditions must be fulfilled. These conditions imply that, *(i)* the detuning of the fields with respect to the one-photon transitions are much larger than the relaxation rates and the two-photon cavity detuning in order to ensure that the cavity is tuned to the two-photon transition at the time that it is highly detuned from the one photon transition; *(ii)* the deviations of the field frequencies are negligible in order to ensure far off-resonant one-photon processes; and *(iii)* the ac-Stark shifts do not modify the above far off-resonance conditions [37–42]. If the above conditions are verified we can apply the adiabatic elimination of the dipoles and of the intermediate level population.

Recent work [43] in view of continuing technological improvements in micro-cavities even at optical frequencies has motivated the examination of certain aspects of the two-photon laser theory that are fundamental to the process. These aspects have their counterpart in the usual single-photon laser but rather different behavior is to be expected in the two-photon case, owing to the essential nonlinearity of the process even at weak signal. We have here in mind a degenerate two-photon laser with the atom pumped to the upper state connected to the lower one of the lasing transition by a two-photon process. Although not realized as yet in this pure form, it probably is a matter of short time before that occurs [26,27]. The situation here is somewhat different from the dressed states scheme that has already been demonstrated experimentally some time ago by Mossberg and collaborators [18–20].

The issue we have in mind has to do with the steady-state behavior of the system, taking into account the spatio-temporal dependence of the relevant magnitudes such as the field strength and the inversion. This is most conveniently accomplished in a semi-classical formalism in terms of the Maxwell–Bloch equations. Related treatments based on either single rate equations [34], discussing threshold conditions, or the Maxwell–Bloch equations without the spatial dependence, have been presented in the literature [45–47]. What we have studied and presented below is essentially the generalization of the complete Maxwell–Bloch equations, usually employed in the single-photon laser theory, to the two-photon case. We have found it most convenient to use a formulation presented some time ago by Narducci in the semi-classical theory of the single-photon laser [48].

2. Derivation of equations

We consider the coupled set of Maxwell–Bloch equations, in the usual rotating wave approximation, which govern our two-level atom when the dipole forbidden transition is replaced by a non-degenerate two-photon one, in which pairs of photons with the different frequency are created or absorbed, and we analyze the stability of the steady state. We adopt a semi-classical laser model based on a microscopic two-level Hamiltonian. We assume a collection of identical homogeneously broadened two-level atoms, with energies ω_1 and ω_2 such that $(\omega_2 > \omega_1)$ with $\omega_2 - \omega_1 = \hbar\omega_{21}$, ω_{21} the atomic transition frequency and a generated unidirectional single-mode classical electric field

$$E(z,t) = \tfrac{1}{2}\{E_j \mathrm{e}^{i(k_j z - \omega_j t)} + \mathrm{c.c.}\}, \quad j = a, b \tag{1}$$

inside a ring cavity. Here E_j is the real field amplitude, k_j the wave-number, z the cavity axial direction and ω_j represents the unloaded cavity frequency $(j = a, b)$. The atoms interact with the field in the dipole approximation via a two-photon transition, where these states are assumed to have the same parity, and thus are not connected by a one-photon transition. By using the rotating wave approximation one obtains the following equations for the probability amplitudes of the form

$$\begin{aligned} ih\frac{\partial a_1}{\partial t} &= \omega_1 a_1 - \frac{1}{4}\sum_m r_{11}^{(m)} \mid E_m \mid^2 a_1 \\ &\quad - \frac{1}{4}\mu^{(2)} E_a E_b a_2 \mathrm{e}^{-i((k_a+k_b)z-(\omega_a+\omega_b)t)}, \end{aligned} \tag{2}$$

$$
\begin{aligned}
ih\frac{\partial a_2}{\partial t} &= \omega_2 a_2 - \frac{1}{4}\sum_m r_{22}^{(m)} \mid E_m \mid^2 a_2 \\
&\quad -\frac{1}{4}\mu^{(2)} E_a^* E_b^* a_1 \mathrm{e}^{i((k_a+k_b)z-(\omega_a+\omega_b)t)}, \qquad (3)
\end{aligned}
$$

where $\mu^{(2)}$ the effective dipole matrix element for the two-photon transition, r_{jj} is the ac-Stark shift and are given by

$$
\begin{aligned}
\mu^{(2)} &= \frac{1}{h}\sum_{i>2}\mu_{1i}\mu_{i2}\left(\frac{1}{\omega_{i2}+\omega_a}+\frac{1}{\omega_{i2}+\omega_b}\right), \\
r_{jj}^{(m)} &= \frac{2}{h}\sum_{i>2}\frac{\mu_{ji}\mu_{ij}\omega_{ji}}{\omega_{ij}^2-\omega_m^2}.
\end{aligned}
$$

Adopting the plane-wave approximation and by using equations (2) and (3), we reduce the Maxwell–Bloch equations to

$$
\begin{aligned}
\frac{\partial \bar{E}_a}{\partial z}+\frac{1}{c}\frac{\partial \bar{E}_a}{\partial t} &= -i(\beta_a\bar{D}+\xi_a)\bar{E}_a-\frac{i\alpha_a}{2}\bar{P}\bar{E}_b^*, \qquad (4)\\
\frac{\partial \bar{E}_b}{\partial z}+\frac{1}{c}\frac{\partial \bar{E}_b}{\partial t} &= -i(\beta_b\bar{D}+\xi_b)\bar{E}_b-\frac{i\alpha_b}{2}\bar{P}\bar{E}_a^*, \qquad (5)\\
\frac{\partial \bar{P}}{\partial t} &= -(\gamma_1-i(\Delta_{21}-\Delta_s))\bar{P}-\frac{i\mu^{(2)}}{2h}\bar{E}_a\bar{E}_b\bar{D}, \qquad (6)\\
\frac{\partial \bar{D}}{\partial t} &= \frac{i\mu^{(2)}}{4h}\{\bar{P}\bar{E}_a^*\bar{E}_b^*+\bar{P}^*\bar{E}_a\bar{E}_b\}-\gamma_2(\bar{D}-1), \qquad (7)
\end{aligned}
$$

where $\bar{E}_j, \bar{P}$ and $\bar{D}$ are the normalized output field, two-photon polarization and population difference, respectively, γ_1 and γ_2 are the decay rates of two-photon polarization and population difference, respectively. α_j denotes the unsaturated gain constant per unit length of the active medium ($\alpha_j = 2\pi N\omega_j\mu^{(2)}/c\varepsilon_0$), where N is the number of atoms per unit volume, ε_0 the vacuum electric permeability and c the speed of light. $\Delta_s = \frac{1}{4h}\sum_{m=a,b}(r_{22}^{(m)}-r_{11}^{(m)}) \mid E_m \mid^2$. We denote by $\Delta_{21}=\omega_a+\omega_b-\omega_{21}$, the detuning of the cavity mode from two-photon off resonance. $\beta_j = \frac{2\pi\omega_j N}{2c}(2r_{22}^{(m)}-r_{11}^{(m)})$ and $\xi_j = \frac{2\pi\omega_j N}{2c}(r_{22}^{(m)}+r_{11}^{(m)})$. It is to be noted that when we put $E_a = E_b$, we get the results of [18]. The term proportional to $(\gamma_1+i(\Delta_{21}-\Delta_s))$ is similar to that of the one-photon case. Equations (4)–(7) here are non-linear in $\bar{E}_j$, as is the case for the one-photon two-level system [48]. The major difference between the two cases is that the equations governing this system involve non-linearity of higher order.

In the case of a degenerate two-photon model we assume the frequencies of the two fields to be equal and we deal with the only field $E(z,t) =$

$\frac{1}{2}\{E_0 e^{i(kz-\omega t)} + \text{c.c.}\}$ that interacts with both the dipole-allowed atomic transitions. The degenerate two-photon laser model constitutes a limiting case of the non-degenerate one. Thus, the laser behavior predicted from a non-degenerate two-photon laser model with equal cavity losses for both fields coincides with that predicted from a degenerate two-photon laser model (when analyzing the steady solution of the two-photon laser [49]). The fact that there is a difference in a factor 2 in the gain parameter that makes the pump parameter different in both cases was commented. In particular this implies that the minimum population inversion required for laser action in the non-degenerate case must be twice that in the degenerate case, and was interpreted by [34]. What occurs is that the validity conditions of both models do not coincide: in the degenerate case the field interacts with both atomic transitions, whereas in the non-degenerate case each field interacts with only one transition. This explains, roughly, the factor 2 in the gain parameter.

Maxwell–Bloch equations for a degenerate two-photon laser with the atom pumped to the upper state connected to the lower one of the lasing transition by a two-photon process and its steady state have been derived and discussed previously in [49]. For simplicity, we neglect the Stark shift terms. The Maxwell–Bloch equations (4)–(7) reduced to

$$\frac{\partial \bar{F}}{\partial z} + \frac{1}{c}\frac{\partial \bar{F}}{\partial t} = -\alpha \bar{P}\bar{F}^*, \tag{8}$$

$$\frac{\partial \bar{P}}{\partial t} = -(\gamma_1 + i\delta_{Ac})\bar{P} - \gamma_1 \bar{F}^2 \bar{D}, \tag{9}$$

$$\frac{\partial \bar{D}}{\partial t} = \gamma_2 \left\{ \tfrac{1}{2}(\bar{P}\bar{F}^{*2} + \bar{P}^*\bar{F}^2) - \bar{D} + 1 \right\}, \tag{10}$$

where $\bar{F}, \bar{P}$ and $\bar{D}$ are the normalized output field, two-photon polarization and population difference, respectively, ($\bar{F} = \sqrt{\mu^{(2)}/\hbar\gamma_1\gamma_2}\bar{E}_0$). In this case, the unsaturated gain constant per unit length of the active medium is given by $\alpha = 2N\omega_c(\mu^{(2)})^{2/3}/2c\hbar\varepsilon_0\gamma_1$. We denote by $\delta_{Ac} = \omega_A - 2\omega_c$ the detuning of the cavity mode from two-photon resonance. ω_A is the atomic transition frequency.

The model is completed by appropriate boundary conditions which, in the case of a traveling wave ring-cavity resonator, take the form

$$\bar{F}(0,t) = R\bar{F}\left(L, t - \frac{\Lambda - L}{c}\right), \tag{11}$$

where L is the length of the active medium; while the full length of the ring resonator is Λ, and R is the amplitude reflectivity of two of the mirrors. For simplicity, the remaining optical surfaces that are needed to complete the ring are assumed to be ideal reflectors.

3. Steady state

In order to gain some physical understanding of the process and discuss some aspects of the threshold conditions, we analyze first the steady-state behavior of the system. To study the steady state, we consider the equations in the long-time limit by setting the time derivatives equal to zero, for an active medium detuned by an arbitrary amount δ_{Ac} from the resonant cavity mode. Under these conditions, the output field is expected to oscillate with a carrier frequency ω_L which is neither equal to ω_c nor $\omega_A/2$, but to some intermediate value determined by the cavity and atomic parameters. For this reason, we look for steady-state solutions of the type

$$\bar{F}(z,t) = \bar{F}_{\rm st}(z)\mathrm{e}^{-i\delta\omega t}, \tag{12}$$

$$\bar{P}(z,t) = \bar{P}_{\rm st}(z)\mathrm{e}^{-i2\delta\omega t}, \tag{13}$$

$$\bar{D}(z,t) = \bar{D}_{\rm st}(z), \tag{14}$$

where $\delta\omega$ is the frequency offset of the operating laser line from the resonant mode (*i.e.* $\delta\omega = \omega_L - \omega_c$). The atomic variables can be determined at once as functions of the stationary field profile

$$\bar{P}_{\rm st}(z) = -\bar{F}^2_{\rm st}(z)\frac{1-i\varDelta}{1+\varDelta^2+\mid \bar{F}_{\rm st}(z)\mid^4}, \tag{15}$$

$$\bar{D}_{\rm st}(z) = \frac{1+\varDelta^2}{1+\varDelta^2+\mid \bar{F}_{\rm st}(z)\mid^4}, \tag{16}$$

where the detuning parameter $\varDelta$ is defined as $\varDelta = (\delta_{Ac} - 2\delta\omega)/\gamma_1$. The steady state polarization and the field envelope are generally out of phase from one another by an amount that depends on the detuning δ_{Ac} and the position of the operating laser line. On resonance, however, $\bar{P}_{\rm st}$ and $\bar{F}_{\rm st}$ have the same phase. The steady state population difference (inversion) saturates at high intensity levels in the sense that $\bar{D}_{\rm st} \to 0$ as $\mid \bar{F}_{\rm st} \mid \to \infty$. To determine the value of the output field and the form of its longitudinal profile in steady state, it is convenient to represent the field amplitude in terms of its modulus only, because here we simply assume no phase change during the steady state evolution

$$\frac{d\bar{F}_{\rm st}(z)}{dz} = \frac{\alpha\bar{F}^3_{\rm st}(z)}{1+\varDelta^2+\bar{F}^4_{\rm st}(z)}\,. \tag{17}$$

The boundary condition, expressed in terms of the field modulus is given by $F_{\rm st}(0) = RF_{\rm st}(L)$. The output laser intensity can be calculated as [13],

$$F^4_{\rm st}(L) = \frac{2\alpha L}{1-R^2}F^2_{\rm st}(L) - \frac{1+\varDelta^2_j}{R^2}, \tag{18}$$

where $\Delta_j = (\delta_{Ac} - 2\delta\omega_j)/\gamma_1$, $\delta\omega_j$ is the operating laser frequency. Equation (18) has two roots and at laser threshold the intensity is not vanishing. There is coexistence of three solutions (above threshold): the trivial and two other solutions with intensity different from zero. One solution grows with the pumping parameter up to an asymptotic value for pumping going to infinite. The other solution decreases towards the zero solution as the pumping grows to infinity. This means that the threshold is not a second order phase transition as in the case of single photon lasers.

The quantity $c \mid \ln R \mid /\gamma_1 \Lambda$ represents the decay rate of the cavity field, and $2\pi c/\Lambda$ is the spacing between adjacent cavity resonances. After introducing the abbreviations $K = c \mid \ln R \mid /\Lambda, \alpha_1 = 2\pi c/\Lambda$, we obtain

$$\delta\omega_j = \omega_L - \omega_c = \frac{K\delta_{Ac} + \alpha_1\gamma_1 j}{\gamma_1 + 2K}, \tag{19}$$

where the sub-index j reminds us of the possible existence of multiple solutions. This is the well known mode-pulling formula. It shows that the laser operating frequency is a weighted average of the atomic resonant frequency and the frequency of one of the cavity modes.

4. Linear stability analysis

The general stability analysis of the Maxwell–Bloch equations (8)–(10) is a rather difficult problem. The main source of complication originates from the spatial dependence of the field and of the atomic variables. In an attempt to get around this problem, mostly linear stability analysis have been carried out within the uniform field limit. While this may not appear to be a very realistic approach, there are good reasons, in fact, why useful information can be extracted even from this limiting case: *(i)* we can reformulate the Maxwell–Bloch problem in terms of a new set of atomic and field variables that are not very sensitive to limited departures from the ideal limit. For this reason it is not necessary to operate with unrealistically low values of the gain or the mirror transmittivity; *(ii)* the mean field limit is a good indicator of instabilities and functions as a rough diagnostic tool. This is fortunate because the numerical solutions of the time-dependent Maxwell–Bloch equations require considerable efforts and some guidance can produce significant saving of time. The resonant case, is not very complicated and can be studied exactly with limited effort. For this reason, in this section we limit ourselves to the exact analysis of the resonant laser problem, without any restrictions on the gain of the active medium or the reflectivity of the mirrors. Our starting point is the full set of Maxwell–Bloch equations (8)–(10) with $\delta_{Ac} = 0$. Because the phase of the stationary field is undetermined, it is possible to select $F_{\rm st}(z)$ as a real quantity. In principle, a random fluctuation of the cavity field could force the growth of the imaginary part through a process called phase instability. In this section we simply assume that no phase instability can develop, so that both the field and polarization variables remain real during the linearized evolution. The steady state of this system of equations is given in equations (15), (16) and (17). To study the stability of this steady state, we set

$$F(z,t) = F_{\rm st}(z) + {\rm e}^{\lambda t}\delta f(z),$$

$$P(z,t) = P_{\rm st}(z) + {\rm e}^{\lambda t}\delta p(z),$$

$$D(z,t) = D_{\rm st}(z) + {\rm e}^{\lambda t}\delta d(z) \tag{20}$$

into equations (8)–(10), upon neglecting fluctuation terms of order higher than one. The linearized equation of the field fluctuation takes the form

$$\frac{d}{dz}\delta f(z) = M(z)\delta f(z), \tag{21}$$

where

$$\begin{aligned} M(z) \;=\; & -\frac{\lambda}{c} + \frac{\lambda+3\gamma_1}{\lambda+\gamma_1}\frac{\alpha F_{\rm st}^2}{1+F_{\rm st}^4} - \frac{\alpha F_{\rm st}^6}{1+F_{\rm st}^4}\frac{2\gamma_1+\lambda}{\lambda+\gamma_1} \\ & \times\frac{2\gamma_1\gamma_2}{(\lambda+\gamma_1)(\lambda+\gamma_2)+\gamma_1\gamma_2F_{\rm st}^4}. \end{aligned} \tag{22}$$

The formal solution of equation (14) is

$$\delta f(z) = \delta f(0)\exp\left(\int_0^z dz' M(z')\right) = \delta f(0){\rm e}^{\Psi(z)}. \tag{23}$$

The problem is that $\bar{F}_{\rm st}$ is not known in closed analytic form. We can get around this difficulty with a change of independent variable from z to $\bar{F}_{\rm st}$, if we take advantage of the fact that $dz = d\bar{F}_{\rm st}/\ (d\bar{F}_{\rm st}/dz)$ and that $d\bar{F}_{\rm st}/dz$ is known explicitly from equation (10) and the field fluctuation takes the form

$$\delta f(z,t) = {\rm e}^{\lambda t}f(z) = {\rm e}^{\lambda t}\delta f(0){\rm e}^{\Psi(z)}. \tag{24}$$

Next, imposing the boundary condition

$$\delta f(0,t) = R\delta f\left(L, t-\frac{\Lambda - L}{c}\right) \tag{25}$$

we obtain the characteristic equation

$$\lambda_n = -i\alpha_n - \frac{c}{2\Lambda}\frac{(\lambda_n + 3\gamma_1) \mid \ln R \mid}{\lambda_n + \gamma_1} - \frac{c}{4\Lambda}\frac{\lambda_n + 2\gamma_1}{\lambda_n + \gamma_1} \times \ln\left(\frac{(\lambda_n + \gamma_1)(\lambda_n + \gamma_2) + \gamma_1\gamma_2 F_{\mathrm{st}}^4(L)}{(\lambda_n + \gamma_1)(\lambda_n + \gamma_2) + \gamma_1\gamma_2 R^2 F_{\mathrm{st}}^4(L)}\right), \tag{26}$$

where $\alpha_n = 2\pi nc/\Lambda$. The characteristic equation (26) depends on the cavity linewidth K ($c/\Lambda\gamma_1 = K/ \mid \ln R \mid$) of the population difference, and the gain of the active medium through the output field intensity $\bar{F}_{\mathrm{st}}^2$. The characteristic equation (26) is similar to that for the one-photon two-level system with the following substitutions $\bar{F}_{\mathrm{st}} \to \bar{F}_{\mathrm{st}}^2$, and in a factor 2 in the denominator of the second and third terms of the equation which govern the two-photon laser case equation (26). The origin of the term $-i\alpha_n$ here can be traced back to the equality $\exp(0) = \exp(2\pi ni)$ for $n = 0, \pm 1, \pm 2\dots$. Note that setting $\exp(0) = 1$ would be a mistake because it would eliminate practically the entire spectrum of eigenvalues. At this point, we have reduced the linearized problem (14) to the solution of an infinite number of characteristic equations, one for each value of α_n. The existence of an infinite number of eigenvalues is not surprising in view of the space–time dependent nature of the field and atomic variables and of the boundary conditions of the laser resonator. One is reminded of the ordinary vibration problems, linear string, two-dimensional membrane, *etc.*, except that here we are dealing simultaneously with three fluctuation variables equation (20), and thus on physical grounds, one expects three characteristic roots $\lambda_n^{(1)}, \lambda_n^{(2)}, \lambda_n^{(3)}$ for each value of n. Because α_n represents the frequency separation between the n^{th} empty cavity resonance and the selected reference mode, it is easy to interpret the set of roots $\lambda_n^{(i)}$, $i = 1, 2, 3$, as descriptive of the growth or decay of an initial fluctuation that develops in correspondence to the n^{th} mode of the cavity. This interpretation forms the basis for a classification of the possible unstable behaviors of the system. If $\mathrm{Re}\,\lambda_0$ is positive and $\mathrm{Re}\,\lambda_n (n \neq 0)$ are all negative, an initial fluctuation of the resonant mode will grow exponentially and evolve with the same carrier frequency as the stationary state. Thus, the linearized dynamics of the laser can be described only in terms of the behavior of the resonant mode fluctuation (all the other fluctuations are damped because $\mathrm{Re}\,\lambda_n < 0, n \neq 0$) and the instability will be of the single-mode type. If, on the other hand, $\mathrm{Re}\,\lambda_n < 0$ and, for some value of n, $\mathrm{Re}\,\lambda_n > 0$, the n^{th} cavity mode will support the growth of a fluctuation whose carrier frequency is different from that of the stationary state. Here, we have suggested the existence of a one-to-one correspondence between the index n, that appears in equation (26), and the longitudinal cavity modes. Our informal suggestion is founded on physical grounds. The main conceptual difficulty with this interpretation is that the notion of "mode" is not well defined when the resonator mirrors have a finite reflectivity, and the elementary cavity excitations have a finite lifetime. In fact, in solving the linearized problem, we have not even introduced resonator eigenfunctions, as one normally would in a standard boundary value problem. For this reason, we continue to refer to $\lambda_n^{(i)}$ as the set of linearized eigenvalues of the n^{th} cavity resonator.

A complete analysis of equation (26), particularly with regard to role played by the basic laser parameters, gain, internode spacing, reflectivity and the atomic decay rates, has not been carried out. Equation (26) predicts that both single and multimode unstable behavior can be established with confidence. We begin our analysis by scaling all the relevant rates of the problem to the linewidth γ_1 of the active medium. In this way, equation (26) takes the form

$$\begin{aligned}\bar{\lambda}_n \;=\; & -i\bar{\alpha}_n - \frac{c}{2\gamma_1 \Lambda}\frac{(\bar{\lambda}_n+3)\mid \ln R\mid}{\bar{\lambda}_n+1} - \frac{c}{4\gamma_1 \Lambda}\frac{\bar{\lambda}_n+2}{\bar{\lambda}_n+1} \\ & \times \ln\left(\frac{(\bar{\lambda}_n+1)(\bar{\lambda}_n+\bar{\gamma})+\bar{\gamma}F_{\mathrm{st}}^4(L)}{(\bar{\lambda}_n+1)(\bar{\lambda}_n+\bar{\gamma})+\bar{\gamma}R^2F_{\mathrm{st}}^4(L)}\right), \end{aligned} \tag{27}$$

where $\bar{\lambda}_n = \lambda_n/\gamma_1$, $\bar{\alpha}_n = \alpha_n/\gamma_1$, and $\bar{\gamma} = \gamma_2/\gamma_1$. A numerical study of this problem shows that single-mode instabilities $\bar{\alpha}_n = 0$ tend to be favored in the presence of high gain and laser cavity losses $\bar{K} > 1$. These conditions are difficult to realize in a practical system. In general, it appears from equation (27) that single-mode instabilities require a scaled cavity linewidth which is sufficiently larger than unity. In order to keep the calculations presented in this paper as realistic as possible, we have chosen to apply our model for a real atomic system, (for the transition $4p_{3/2}$–$6p_{3/2}$ in Potassium). The reason for choosing this transition is the result of a compromise. On one hand, one wants the energy of the photons involved to be as large as possible, and preferably in the optical regime. On the other hand, it is hard to find a two-photon transition in the optical regime with a large coupling, since a large two-photon coupling demands the existence of an almost resonant intermediate level with opposite parity. The transition mentioned above involves photons with an energy of $\simeq 7980$ cm^{-1} *i.e.* near-infrared, and has a two-photon coupling that is orders of magnitude larger than the other candidates we looked at, due to the almost resonant $5s$ state. Besides the atom, we should also choose a cavity. In the model presented in this paper, we are assuming that only one mode of the cavity field is excited. For this to be true, the cavity should be rather small, since it then supports fewer modes, and these will be better separated in energy. Another advantage of having a small cavity is that the two-photon coupling $\mu^{(2)}$ will be larger, since it is proportional to V^{-1} (following the notation of Loudon) [50], V being the cavity volume. We have chosen the cavity volume $V = 10^{-15}$ m^3. In figure 2 the largest real parts of the linearized eigenvalues are plotted

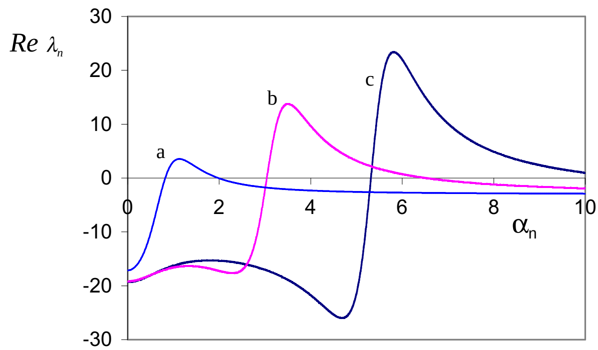

Fig. 2. The largest real parts of the linearized eigenvalues are plotted as functions of $\bar{\alpha}_n$ viewed as a continuous variable. For all the curves displayed in the figure we have selected $R = 0.8, \bar{\gamma} = 0.1$, and $\bar{k} = 3.55$, where (a) — $\alpha L = 1$, (b) — $\alpha L = 3$ and (c) — $\alpha L = 5$.

as functions of $\bar{\alpha}_n$ viewed as a continues variable. For all the curves displayed in this figure we have selected $R = 0.8$, $\alpha_1 = 100$, $\bar{\gamma} = 0.1$ and for different values of αL. We show that unstable situation for several values of the relevent parameters (the only physical meaningful values of $\bar{\alpha}_n$ are all the positive and negative multiples of the intermode spacing $\bar{\alpha}_1$). Multi-mode instabilities are not bounded by the high-loss requirement, but they still require large values of the gain to reach their threshold. In figure 3 the largest real parts of the linearized eigenvalues are plotted as functions of $\bar{\alpha}_n$ viewed as a continues variable. With the same parameters as in figure 2 but for different values of $\bar{\gamma}$ and $\alpha L = 6$. This figure gives an example of some typical real parts of the linearized eigenvalues for parameter values that lead to multimode instability. As shown in this figure, the beat frequency due to the superposition of the stationary solution and of the unstable sidebands is sensitive to the value of $\bar{\gamma}$. The important feature is the monotonic shift of the positive real parts of the eigenvalues towards higher and higher values of $\bar{\alpha}_n$ for increasing values of the gain. In this case the role of γ_1 is played by γ_2, so that, for example, the quantities α_n and K must be normalized to γ_2. As a consequence, our analysis holds not only when the sample contained in the cavity is a two-level system, but also when it is, for example, a Kerr medium.

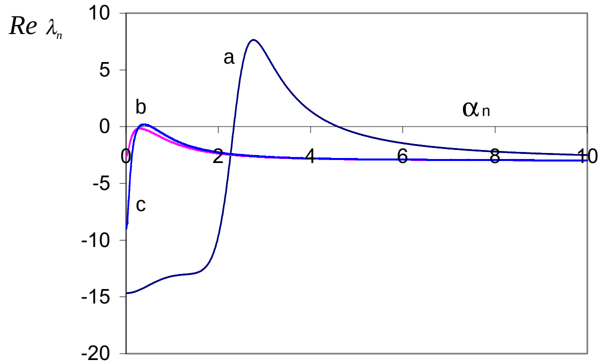

Fig. 3. The largest real parts of the linearized eigenvalues are plotted as functions of $\bar{\alpha}_n$ viewed as a continuous variable. For all the curves displayed in the figure we have selected $R = 0.8$, $\bar{k} = 3.55$ and $\alpha L = 6$, where (a) — $\bar{\gamma} = 0.1$, (b) — $\bar{\gamma} = 0.5$ and (c) — $\bar{\gamma} = 0.7$.

The most salient distinctive features of the two-photon lasers are: the laser-off solution is always stable (thus implying the necessity of triggering for laser action) and the laser-on solution is stable for pump values above (and not below) the laser second (or instability) threshold. Moreover, self-pulsing emission is still possible in autonomous class-B two-photon lasers [52] (lasers for which the polarization decay rate largely exceeds the population and photon decay rates and on which no external modulation is exerted), a behavior that is in contrast with most laser models. The electric and magnetic dipole interaction of a system of two-level atoms with an electromagnetic field is considered in the nonlinear regime in [53] through response theory. They have pointed at the order of the optical Bloch equations with respect to the nonlinear response theory. Also, they showed that in the rotating wave and near-resonance approximations and up to order e^2 (e is electron charge) the results of the nonlinear response theory reduce to those of the Bloch equations.

In conclusion, we have derived the general Maxwell–Bloch equations for the system consisting of the two-level atoms with dipole forbidden transition, placed in a two-photon one. The treatment has been carried out in the framework of the semiclassical laser theory. We have calculated the spatial behavior of the field strength and have shown the effect of the additional non-linearity due to the two-photon coupling. We have generalized the stability analysis of the steady-state solution of the complete Maxwell–Bloch equations, usually employed in the homogeneously broadened single-photon laser theory [48,51], to the two-photon case. Although the model is rather idealized, its general features should be relevant to a real single-mode system.

The analysis presented in this paper has been inspired by the comparison between the linear stability analysis technique and the so-called weak side-band approach [54]. In our case the linear stability analysis not only agrees with the results of the weak sideband approach, but extends its range of applicability, particularly in the case when the cavity detuning must be taken into account.

The problem we have formulated and solved in this paper has an interesting counterpart in the microwave regime where one can tailor at will, in combination with the choice of the principal quantum number of the pumped Rydberg state. The experimental realization of such a scenario should be relatively easy with present day technology. In our treatment we have focused on the degenerate two-photon laser. It would thus be interesting to study the non-degenerate case. We could imagine having a transition in which one photon is visible, and the other is, say, infrared. The frequencies of these two photons could be chosen in such a way that we would obtain a large two-photon coupling and hence this laser type would be easier to realize. In this laser type, we would also expect Stark shift to play a dominant role. We hope to report on such issues in a forthcoming paper. The single-mode and multimode instabilities in one-photon lasers and related optical systems have been discussed in [51]. Our present discussion generalizes these early studies.

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