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# THE THERMODYNAMICS OF PORTFOLIOS*

E.W. Piotrowski

Institute of Theoretical Physics, University of Białystok
Lipowa 41, 15-424 Białystok, Poland
e-mail: ep@alpha.uwb.edu.pl

and J. Sładkowski

Institute of Physics, University of Silesia
Uniwersytecka 4, 40-007 Katowice, Poland
e-mail: sladk@us.edu.pl

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We propose a new method of valuation of portfolios and their respective investing strategies. To this end we define a canonical ensemble of portfolios that allows to use the formalism of thermodynamics.

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## 1. Introduction

Physicist, contrary to mathematicians, have only occasionally investigated economic systems. Recently, however, a growing number of papers of relevance to economics is being published in physics journals and conferences proceedings [1]. Moreover, H. Markowitz encouraged outsiders to engage in research on economics in his Nobel Prize Lecture [2]: *"I believe that microscopic market simulations have an important role to play in economics and finance. If it takes people from outside economics and finance – perhaps physicists – to demonstrate this role it won't be for the first time that outsiders have made substantial contribution to these fields"*. Physicists have good command of stochastic processes and statistical physics so it is hardly surprising that they successfully use their minds to analysing

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economic systems. Other physical concepts probably have analogue in economics, *e.g.* such notion as gauge symmetry can be identified in financial markets [3]. Another important and exciting branch of finances is the portfolio theory [4]. Here various physical concepts have direct analogies. Ultrametricity [5] commonly used in spin glasses can be used to describe distance between stocks [6]. Spin glasses seem to have lots in common with portfolio theory [7, 8]. We would like to propose a method of valuation of portfolios and investing strategies that stems from thermodynamics. Investors and portfolio managers set up their portfolio according to the market information available and their lore. They lose or gain. It is easy to gain when all prices are raising and lose when everything is falling down. But which part of the gain is due to skill and qualities of the investor? Which part is simple a result of the actual state of the market? Our model allows for a temperature like parameter that measures the quality and professionalism of the investor.

## 2. The portfolio description

A portfolio is a package of various assets (shares, bonds, derivative instruments, *etc.*) that can be exchanged on the market. If we denote by $a_i,\ i = 0, 1, \ldots N$ the unit of the $i$-th asset then the portfolio $P$ as

$$P = \sum_{i=0}^{N} \alpha_i a_i \,, \tag{1}$$

where $\alpha_i \in \mathbb{Z}$ is the number of units of the $i$-th asset in the portfolio $P$. The coefficients $\alpha_i$ can be negative because the stock exchange regulations allow for selling assets that the portfolio owner do not possesses (short selling). One usually supposes that one asset, say $\alpha_0$, in $P$ can be exchanged with any other asset at any time (money). An external observer describes the moves performed by the portfolio manager as a draw in the following lottery. Let $p_i, i = 1, ..., N$ be the probability of the purchase of the $w_i$ units of the $i$-th asset. The weights $w_i$ are given before the lottery come into operation. The value of the portfolio (return), $-c_{00\kappa}$, fulfils the following balance equation at an arbitrary moment after the draw:

$$c_{00\kappa}\,(c_0, \ldots, c_N) + \sum_{k=0}^{N} [\kappa = k]\, c_k w_k = 0\,, \tag{2}$$

where $\kappa$ denotes the random variable taking values from $0, \ldots, N$, the assets numbers. The expression [*sentence*] takes value 0 or 1 if the *sentence* is false or true, respectively (Iverson convention) [9]. The coefficient $c_i$ denotes the present relative price of a unit of the asset $a_i$, $c_i = \frac{u_i}{\overline{u}_i}$ where $u_i$ is the present price of the $i$-th asset and $\overline{u}_i$ its price at the moment of drawing. At the moment of drawing the balance equation takes the form:

$$c_{00\kappa}\,(1,\ldots,1) + \sum_{k=0}^{N} [\kappa = k]\, w_k = 0\,, \tag{3}$$

If $\alpha_0$ represents the basic currency (money) used to define the financial value of the portfolio then we have $c_0 = 1$ during considered period. One can show that the expectation value of $w_0\,[\kappa = 0]$ is the Legendre transform of the mean value of the portfolio.

## 3. The canonical portfolio

Let us consider a portfolio defined by the weights $w_i$ and the uniform distribution of probabilities $p_0 = p_1 = \ldots = p_N = \frac{1}{N+1}$. This portfolio has the same expectation value of the return $E(c_{00\kappa})$ as the uniform portfolio with weights $\frac{w_0}{N+1}, \ldots, \frac{w_N}{N+1}$ (every asset is included). It is interesting to find out how many of the portfolio owners (managers) are in this situation. To this end we consider two types of investors. The first group, called *zombies*, consists of investors whose moves on the market are fully deterministic. They react according to the market condition distributing their capital into various assets or not and act in the same way in the same situations. The second group, called *gamblers*, are indeterministic. Their moves can be different in analogous market situations. This does not mean that gamblers often change their strategies. We simply cannot predict their moves because part of their knowledge and past experiences are not available for us as external observers. Is there a qualitative way of measuring information on investor's behaviour? We do not consider here mechanisms leading to disclosure of information and its consequences. It seems reasonable that the measure $S(p_0, \ldots, p_N)$ we would like to use for measuring the investor behaviour (or more precisely the lottery defined by the probabilities $p_i$) be additive in the following sense. It should not matter if the portfolio is constructed in one draw with the probabilities $p_0, p_1, p_2, \ldots, p_N$ or with two subsequent draws with probabilities $\frac{p_0}{p_0+p_1}, \frac{p_1}{p_0+p_1}$ and $p_0 + p_1, p_2, \ldots, p_N$. This leads to the equation:

$$\begin{aligned} S\,(p_0, p_1, \ldots, p_N) \;&=\; S\,(p_0 + p_1, p_2, \ldots, p_N) \\ &\quad + (p_0 + p_1)\, S\left(\frac{p_0}{p_0 + p_1}, \frac{p_1}{p_0 + p_1}\right)\,, \end{aligned} \tag{4}$$

where the arguments describe probabilities of drawing (that is buying) of the assets. We will look for a solution fulfilling

$$S\left(p_0, p_1, \ldots, p_N\right) = S\left(p_0\right) + S\left(p_1, p_2, \ldots, p_N\right) . \tag{5}$$

Equations (4) and (5) after some standard algebraic manipulations lead to the following solution for the function $S$, see [10]

$$S\left(p_0, p_1, \ldots, p_N\right) = -\sum_{k=0}^{N} p_k \ln p_k . \tag{6}$$

For obvious reasons, we will call this function entropy. The number $K := \mathrm{e}^S$ gives the effective number of assets in the portfolio: $K$ is equal $M$ if the weights are distributed uniformly between $M$ assets, and is 1 if the portfolio contain a single asset.

Let us now consider the "Cartesian product" of two statistically independent portfolios consisting of $N+1$ and $M+1$ assets, respectively. Then

$$\begin{aligned} S_{(M+1)\times(N+1)} &= -\sum_{k=0}^{M}\sum_{l=0}^{N} p_k p'_l \ln\left(p_k p'_l\right) \\ &= -\sum_{k=0}^{N} p_k \ln\left(p_k\right) - \sum_{l=0}^{N} p'_l \ln\left(p'_l\right) = S_{M+1} + S_{N+1} \end{aligned} \tag{7}$$

so the entropy is additive. We would like to compare returns ("achievements") of portfolio managers (owners). Therefore we classify the investors according to the value of their portfolio. This would allows us to divide the appropriate returns into two parts corresponding to manager's lore and tide of the market. We will choose the portfolios that maximize the entropy to represent the classes of investors. An "external" observer would know only the expectation value of returns. (We suppose that draws are not correlated and the market is efficient.) Perhaps, it would be better to assume that the values of the random variable $\kappa$ correspond to whole sectors of the markets (*e.g.* oil companies) rather then to separate assets [1, 11]. This means that we are looking for a conditional extrema of the function $S(p_0, \ldots, p_N)$. The conditions are:

$$\sum_{k=0}^{N} p_k = 1 \tag{8a}$$

and

$$-c_{00}\left(c_0, \ldots, c_N\right) = E\left(\sum_{k=0}^{N}\left[\kappa = k\right] c_k w_k\right) . \tag{8b}$$

We have weakened the balance condition (2) to be fulfilled only on the expectation values level. The Lagrange method of finding conditional extrema requires the following differential form to vanish

$$dS\left(p_0,\ldots,p_N\right)+\beta dE\left(\sum_{k=0}^{N}\left[\kappa=k\right]c_kw_k\right)+\gamma d\sum_{k=0}^{N}p_k=0\,, \tag{9}$$

where $\beta$ and $\gamma$ are Lagrange multipliers. The substitution of equation (6) leads to the condition

$$-\ln p_k-1+\beta c_kw_k+\gamma=0 \tag{10}$$

which gives explicit dependence of the probabilities $p_k$ characterizing the maximal entropy portfolio on the prices $c_k$

$$p_k=\exp\left(\beta c_kw_k+\gamma+1\right)\,. \tag{11}$$

The multiplier $\gamma$ can be eliminated by explicit normalization and the elimination leads to Gibbs-like probability distribution

$$p_k\left(c_0,\ldots,c_N\right)=\frac{\exp\left(\beta c_kw_k\right)}{\sum_{k=0}^{N}\exp\left(\beta c_kw_k\right)}\,. \tag{12}$$

Now we are in a position to define *the canonical statistical ensemble* of investors which consists of all investors that have got the same return. The canonical ensemble describes all strategies (zombies and gamblers) leading to the same return. It can be represented by the portfolio maximizing the entropy (*canonical portfolio*). Choosing any other representative would mean lower entropy and as a result would give a bias to one strategy (knowledge). If one recalls that $K=\exp S$ gives the effective number of assets in the portfolio then one immediately gets that the entropy $S$ measures also the diversification of the portfolios. The canonical portfolio is the safest one in the class of the same return. We will also define the temperature $T$ of the canonical ensemble as $T:=\frac{1}{\beta}$ and the statistical sum $Z$ as $Z\left(c_0,\ldots,c_N\right):=\sum_{k=0}^{N}\exp(\beta c_kw_k)$. If we keep the weights $w_k$ constant then the changes of the prices $c_k$ imply appropriate changes of the value of the portfolio $-c_{00}$ (return). The expected infinitesimal change of $c_{00}$ is given by

$$-dc_{00}\left(c_0,\ldots,c_N\right)=dE\left(c_\kappa w_\kappa\right)\,. \tag{13}$$

Having in mind that $S=-\sum_{k=0}^{N}p_k\ln p_k=\ln Z-\beta E\left(c_\kappa w_\kappa\right)$ we can write

$$\begin{aligned} dS &= \frac{\beta\sum_{k=0}^{N}w_k\exp\left(\beta w_kc_k\right)dc_k}{Z}-\beta dE\left(c_kw_k\right) \\ &= \beta\left(\sum_{k=0}^{N}E\left(\left[\kappa=k\right]w_k\right)dc_k-dE\left(c_\kappa w_\kappa\right)\right)\,. \end{aligned} \tag{14}$$

This implies that if we treat $S$ as an independent variable

$$dc_{00}\left(c_0,\ldots,c_N,S\right)+\sum_{k=0}^{N}\overline{w}_k dc_k = TdS\,, \tag{15}$$

where $\overline{w}_k$ denotes the mean content of the $k$-th asset in the portfolio. Now, the temperature of the portfolio $T$ is equal to

$$T=\frac{\partial c_{00}}{\partial S}\,. \tag{16}$$

This temperature measures the change of the portfolio value caused by its entropy change. As in classical thermodynamics, we can formulate two principles.

**The I principle of the canonical ensemble**: *The change of value of a canonical portfolio $-c_{00}$ consists of two parts. The first one is equal to the change the investors knowledge $\delta Q$ and the second is equal to change of the values of content of the portfolio:*

$$dc_{00}+\delta Q+\sum_{k=0}^{N}\overline{w}_k dc_k = 0\,. \tag{17}$$

**The II principle of the canonical ensemble**: *The value of lost investors knowledge $-\delta Q$ is proportional to the increase in the entropy of its canonical ensemble:*

$$\delta Q + TdS = 0\,. \tag{18}$$

Note that we have written $\delta Q$ instead of the obvious $dQ$ because, in general, such a function $Q$ does not exist. Following the development of thermodynamics we will define the *free value* $-F$ of the canonical portfolio as:

$$F\left(c_0,\ldots,c_N,T\right):=c_{00}\left(c_0,\ldots,c_N,S\right)-TS\,. \tag{19}$$

This allows for the formulation of the two principles of the canonical ensemble in a balance-like equation:

$$dF\left(c_0,\ldots,c_N,T\right)+SdT+\sum_{k=0}^{N}\overline{w}_k dc_k = 0 \tag{20}$$

which combined with the definition of the entropy $S$ gives

$$-F = T \ln Z \tag{21a}$$

and

$$-c_{00} = \frac{\partial \ln Z}{\partial \beta} \,. \tag{21b}$$

The name free value can be justified as follows. Let us suppose that during the market evolution (changes of the prices $c_k$) the class of the investor does not change ($T$ = const). Then the changes of values of the assets in the portfolio, $\sum_{k=0}^{N} \overline{w}_k dc_k$ are measured by changes of the potential $F$. Such a process can be referred to as isothermal. $-T$ can be interpreted as price of a unit of the entropy $S$. The entropy of a canonical portfolio increases when the absolute value of the parameter $T$ increases. This means that an amateur pays more than a professional investor for errors of the same order. But if the temperature is negative an amateur gets more from erroneous decisions. A portfolio with $T < 0$ and small entropy can only be constructed by a specialist who uses his knowledge in reverse. After changing signs of the weights $w_k$ (short position) such a portfolio gives excellent returns. Note that due to the additivity of entropy the temperature of the portfolio constructed by merging two portfolios with temperature $T$ but built in different market sectors equals $T$.

It would be helpful to give some flesh to the above consideration by working out a numerical example. Let us consider two investors Alice and Bob. Both have the same initial capital, say \$1 and, besides money, there are only two assets available on the market, $a_1$ and $a_2$. Bob (a zombie) divided his capital into two equal parts and spent them on both assets. Alice (a zombie, a pro?) spent a quarter of her capital on $a_1$ and kept the rest. Suppose now that after some time the price of the asset $a_1$ went down by 20% and the price of the asset $a_2$ increased by 30% that is the relative prices are now $c_1 = 0.8$ and $c_2 = 1.3$. This means that Alice's and Bob's portfolios gave the returns $c_{00}^{\mathrm{A}} = 0.95$ and $c_{00}^{\mathrm{B}} = 1.05$, respectively (we have neglected the interest rates). The temperatures are $T^{\mathrm{A}} = -0.46$ and $T^{\mathrm{B}} = 2.55$, respectively (only two decimal positions are kept). Bob has got better return then Alice even though $T^{\mathrm{B}} > T^{\mathrm{A}}$. This is because Bob alone makes profit from his knowledge ($T^{\mathrm{B}}$ is positive). But the authors would prefer Alice to Bob as an investment adviser because when one listen to her advises and acts contradictory to them one gets better return then Bob's. Alice's knowledge is greater than Bob's because if $\mid T^{\mathrm{B}} \mid > \mid T^{\mathrm{A}} \mid$ then $S^{\mathrm{B}} > S^{\mathrm{A}}$.

## 4. Conclusions

We have proposed a method that allows numerically measure investors qualities. Inspired by thermodynamics, we were able to define canonical ensembles of portfolios, the temperature of portfolios and, possibly, various thermodynamics-like potentials. We have used the relative prices $c_k$ of the assets that have direct interpretation. The theory of financial market would prefer a covariant description of the portfolio. This can be easily achieved by replacing the parameters $c_{00}, c_1, \ldots, c_N$ by their natural logarithms (that is integrals of the instantaneous interest rates)

$$\ln \mid c_{00} \mid \ \rightarrow \ c_{00} \,, \tag{22a}$$

$$\ln \mid c_k \mid \ \rightarrow \ c_k \,, \tag{22b}$$

$$\frac{\mid c_k \mid}{\mid c_{00} \mid} w_k \ \rightarrow \ w_k \,. \tag{22c}$$

We imagine that analogous methods can be developed for valuation of credit repayment scenarios or bonds under the stochastic behaviour of interest rates.

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