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STAR PRODUCTS AND TOPOLOGICAL QUANTUM GROUPS

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A well-behaved topological quantum algebra structure on a quantized enveloping topological algebra is given by a star product on the corresponding exact compact connected Poisson–Lie group of its triangular Lie bialgebra.

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1. Introduction

Deformations of different groups and algebras [1, 2] has attracted great attention during the last few years. These mathematical objects called quantum groups or quantum algebras originate in quantum inverse method [3] and have found many interesting important physical applications. The R-matrix formulation of the quantum group theory [4], based on the fundamental relation of QISM (the FRT relation) has given an additional impulse for the investigation of these deformed algebras.

As it is well known, quantum groups can be seen as non commutative generalization of the topological space which have a group structure. Such a structure induces an abelian Hopf algebra structure [5] on the algebra of smooth functions on the group. Quantum groups are defined then as a non abelian Hopf algebras [6]. A way to generate them consists of deforming the abelian product of the Hopf algebra of functions into a non abelian one (*-product), using the so called quantization by deformation or star-quantization [7–9].

The existence of a star product has been studied by Vey [10], Neroslavsky and Vlassov [11], who proved the existence of a star product on a symplectic manifold with a vanishing third De Rham cohomology group and by De Wilde and Lecopmte [12] in the case of an arbitrary symplectic manifold. From a geometrical point of view, Omori and al [13] and Fedesov [14] also constructed star products for arbitrary symplectic manifold. The relation between the De Wilde and Lecompte approach and the Fedesov one is established by Deligne [15]. Recently Kontsevich [16], by using different methods has also construct and classify differential star products on an arbitrary Poisson manifold.

This quantization technique gives a deformed product once it is assigned a Poisson bracket on the algebra of smooth functions. In order to obtain that the deformed algebra is a Hopf algebra, namely a quantum group, the starting group G has to be endowed with a Poisson–Lie structure. Finally, using the duality procedure, this quantization leads to the structure of the quantum algebra on the quantized enveloping algebra of the Lie algebra corresponding to the above Lie group G .

Quantization deformation of Poisson–Lie group has been studied by Drinfeld [17] and Moreno and Valero [18] who proved that every exact Poisson–Lie group can be quantized. And by Etingof and Kazhdan [19, 20] who shows that any Poisson–Lie group admits a local quantization and also by Bonneau and al [21] and Pincezon and Bidegain [22] who proved respectively that a reductive(general) Poisson–Lie group admits a local formal quantization such that the comultiplication is the same as in the classical case. But the problem to find a concrete star product (twist) on a Poisson–Lie group is not yet solved in general. The only case when it has been performed [23] concerns the q-deformed Heisenberg algebra, and in [24] a quantization of Lie–Poisson SL(2) is given up the second order in the deformation parameter. The star-product approach is used also to give a quantum Lie algebra in [25], to realise both $\mathrm{SU}_q(n)$ and virasoro algebra in [25], to study quasi-quantum group structure in [26] and deformed yangians in [27]. The purpose of the present paper is to show explicitly how the star product on a compact exact Poisson–Lie group leads to the structure of well-behaved topological quantum algebra on the quantized enveloping algebra of the Lie algebra of the Lie group, a quantum matrix group structure on the quantized algebra of smooth functions over the Lie group and that equivalent star-products generate isomorphic topological quantum algebras.
This paper is organized as follows, the second Section is devoted to a review of basic definitions of quantum topological algebras, the third Section shows explicitly the main result which states that a star product on a compact connected Poisson–Lie group leads to the structure of a well behaved topological quantum algebra on the quantized enveloping algebra of the Lie bialgebra corresponding to the above Poisson–Lie group, in Section 4 we review the relation between stars products and quantum groups (quantum matrix groups) and the last Section shows that two equivalent star products generate two isomorphic well behaved topological quantum algebras.

2. Quantum topological algebras

Given a topological locally convex Hausdorff vector spaces V_1 and V_2, we denote $V_1\hat{\otimes}V_2$ the complete topological vector space projective tensor product of V_1 and V_2.

Definition 1 *A topological Hopf algebra is a Frechet or dual Frechet and nuclear topological vector space A with the following continuous and linears maps:*

- *Product:* $m : A\hat{\otimes}A \longrightarrow A$
- *Coproduct:* $\Delta : A \longrightarrow A\hat{\otimes}A$
- *Antipode:* $S\colon A \longrightarrow A$
- *Unity:* $\mathbf{1} : \boldsymbol{k} \longrightarrow A$
- *Counity:* $\varepsilon\colon A \longrightarrow \boldsymbol{k}$

Satisfying the following relations:

$$\begin{aligned}(\Delta\hat{\otimes}id)\Delta &= (id\hat{\otimes}\Delta)\Delta\,,\\ m(S\hat{\otimes}id)\Delta &= m(id\hat{\otimes}S)\Delta = \mathbf{1}\circ\varepsilon\,,\\ m(id\hat{\otimes}\mathbf{1}) &= m(\mathbf{1}\hat{\otimes}id) = id\,,\\ (id\hat{\otimes}\varepsilon)\Delta &= (\varepsilon\hat{\otimes}id)\Delta = id\,. \end{aligned} \tag{1}$$

Definition 2 *A pair (A, R) consisting of a topological Hopf algebra A and an invertible element $R \in A\hat{\otimes}A$ will be called a quasi-triangular topological Hopf algebra if*

$$\Delta^{op} = R\Delta R^{-1}\,, \tag{2}$$

$$(\Delta\hat{\otimes}id)R = R^{13}R^{23}\,, \tag{3}$$

$$(id\hat{\otimes}\Delta)R = R^{13}R^{12}\,. \tag{4}$$

Here $\Delta^{op} = P\circ\Delta$, P is the permutation operator.

A quasitriangular topological Hopf algebra is called triangular if $R_{21} = R_{12}^{-1}$.

Coboundary topological Hopf algebras are defined in the same way, but with (3) and (4) changed by

$$R_{12}.(\Delta\hat{\otimes}id)R = R_{23}.(id\hat{\otimes}\Delta)R \tag{5}$$

together with the relation $R_{21} = R_{12}^{-1}$ and $(\varepsilon\hat{\otimes}\varepsilon)R = 1$.

The symbols $R^{13}, R^{12}, R^{23}, R^{21}$ have the following meaning: if $R = \sum_i a_i \hat{\otimes} b_i$ then

$$R^{13} = \sum_i a_i \hat{\otimes} 1 \hat{\otimes} b_i\,, \quad R^{23} = \sum_i 1 \hat{\otimes} a_i \hat{\otimes} b_i\,,$$
$$R^{12} = \sum_i a_i \hat{\otimes} b_i \hat{\otimes} 1\,, \quad R^{21} = \sum_i b_i \hat{\otimes} a_i\,.$$

For a quasitriangular topological Hopf algebra, we deduce from (2), (3) that R satisfies the quantum Yang Baxter equation

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}\,. \tag{6}$$

3. Star product and quantum topological algebras

Let G be a compact connected Lie group, and $F(G)$ be the topological algebra of smooth functions on G. Moreover, setting the following continuous applications:

$$\Delta(f)(x,y) = f(xy), \;\; f \in F(G), x, y \in G\,,$$
$$S(f)(x) = f(x^{-1})\,,$$
$$\delta_e(f)(x) = f(e)\,,$$

where δ_e is the Dirac distribution we get a well-behaved topological Hopf algebra structure on F [29].

Now let $D^{'}(G) = (F(G))^*$ be the space of compactly supported distributions on G, with strong dual topology; following [29], the transposition defines a well-behaved topological Hopf algebra structure on $D^{'}(G)$ where the product is the convolution one, the unit is the Dirac distribution δ_e and the counit is the evaluation on 1. In order to check the coproduct, we introduce the map: $\delta : G \longrightarrow D^{'}(G)$; defined by

$$\langle \delta_x, f \rangle = f(x), \quad f \in F(G), x \in G$$

it is easily seen that δ actually defines a topological inclusion of G as a subset of $D^{'}(G)$, so in the sequel we identify G and $\delta(G)$. This being done, on has $G^{\perp} = 0$, so $\overline{\mathrm{vect}(G)} = D^{'}(G)$ and the coproduct:

$$\Delta_0 : D^{'}(G) \longrightarrow D^{'}(G) \hat{\otimes} D^{'}(G) = D^{'}(G \times G)$$

is given on G by

$$\Delta_0(x) = x \hat{\otimes} x, \;\; x \in G\,. \tag{7}$$

The choice of spaces $F(G)$ and $D^{'}(G)$ is not any: one can notice that some classics sub-spaces of $F(G)$ are not topological Hopf algebras. For example, the smooth function with compact support space is not a sub-Hopf algebra of $F(G)$ because the coproduct of a function indefinitely differential with compact support is not necessarily with compact support (it is possible to translate this in $D^{'}(G)$ by the existence of distribution couples of which the product of convolution is not defined). Let g_0 be the algebra of G, g its complexification, and $U(g)$ the corresponding enveloping algebra. We identify, as usual, $U(g)$ and the algebra of left invariant differential operators of finite order on G; if we define a linear map:

$$i : U(g) \longrightarrow D^{'}(G)$$

$$i(A)(f) = \delta(A(f)), \ \ A \in U(g), f \in F(G)\,. \tag{8}$$

It is easy to check that i is a morphism, so we identify U and $i(U)$ and consider in the sequel that $U(g) \subset D^{'}(G)$; and the topological Hopf algebra structure of $D^{'}(G)$ is exactly the extension of the usual topological Hopf algebra structure of $U(g)$.

The left and right regular representations of G on $F(G)$ are representations of $D^{'}(G)$; defined by the formula:

$$\langle R(T)f, J\rangle = \langle f, S_0(T).J\rangle\,,$$

$$\langle L(T)f, J\rangle = \langle f, J.T\rangle \qquad \forall f \in F(G), TJ \in D^{'}(G)\,. \tag{9}$$

Given, $X \in U(G) \subset D^{'}(G), X^l(X^r)$ is exactly the left(right) invariant differential operator corresponding to X, then:

$$\begin{aligned} \langle X^l(f), Y\rangle &= \langle f, Y.X\rangle\,, \\ \langle X^r(f), Y\rangle &= \langle f, S_0(X).Y\rangle \end{aligned} \tag{10}$$

with S_0 is the classical antipode of $U(g)$. We dote the compact connected Lie group G with Poisson structure, such that for the usual coproduct on the Hopf Topological algebara $F(G)$, the Poisson bracket $\{,\}$ satisfies:

$$\Delta\{f, \mathrm{g}\} = \{\Delta(f), \Delta(\mathrm{g})\}, \qquad\qquad f, \mathrm{g} \in F(G)\,. \tag{11}$$

Equivalently we can consider the Lie algebra g; its dual g^* has a bracket :

$$\phi^* : g^* \wedge g^* \to g^* \tag{12}$$

such that its dual ϕ is a 1-cocycle for the adjoint action. If ϕ is the coboundary of some $r \in g \wedge g$ (solution of the classical Yang–Baxter equation)

$$[\![r, r]\!] = 0\,, \tag{13}$$

where the Schouten bracket is defined as follows

$$[\![r,r]\!] = [r_{12},r_{13}] + [r_{12},r_{23}] + [r_{13},r_{23}]\,,$$

then the poisson Lie structure is said triangular. In this case the Poisson–Lie structure on G is given explicitly by:

$$\{\phi,\psi\} = \sum_{i,j} r^{ij}(X_i^r(\phi)X_j^r(\psi) - X_i^l(\phi)X_j^l(\psi))\,, \tag{14}$$

where $X_i^r = (R_{\rm g})_* X_i$ and $X_i^l = (L_{\rm g})_* X_i$ are the right and left vectors fields on the group G, (X_i) is a basis of g , $(R_{\rm g})_*, (L_{\rm g})_*$ are the derivatives mapping corresponding to the right and left translation .

Now we give the following definition [29, 30, 32]

Definition 3 *We denote $F(G)[[h]]$ the space of formal power series in the parameter h with coefficients in $F(G)$. A star product on the Poisson–Lie group is defined as a bilinear map*

$$F(G) \times F(G) \longrightarrow F(G)[[h]]$$

$$(\phi,\psi) \longmapsto \phi * \psi = \sum_j h^j C_j(\phi,\psi) \tag{15}$$

such that

(i) when the above map is extended to $F(G)[[h]]$, it is formally associative

$$(\phi * \psi) * \chi = \phi * (\psi * \chi)\,; \tag{16}$$

(ii) $C_0(\phi,\psi) = \phi.\psi = \psi.\phi\,;$

(iii) $C_1(\phi,\psi) = \{\phi,\psi\}\,;$

(iv) the two cochains $C_k(\phi,\psi)$ are bidifferential operators on $F(G)$.

(i) and condition *(ii)* express that we have an associative algebra *(i)* and condition *(ii)* express that we have an associative algebra deformation in the sense of Gerstenhaber [31], while condition *(iii)* ensures that the corresponding commutator

$$[f,{\rm g}]_* = \tfrac{1}{2}(f * {\rm g} - {\rm g} * f)$$

is a deformation in the sense of Gerstenhaber of the Lie algebra $(F(G),\{,\})$. Equivalence of two Gerstenhaber deformations is the associative case (two star products $*$ and $*'$) is defined by the existence of formal series of (differentials) operators $T = \sum_r h^r T_r$ with $T_0 = id$ such that

$$Tf * T\mathrm{g} = T(f * \mathrm{g})\,.$$

By equivalence one may consider only star-products vanishing on constants $C_r(f,c) = C_r(c,f) = 0,\ \ r \geq 1, c \in R, f \in F(G)$ and assume that $C_1 = \{,\}$.

In this definition the Hopf algebra $F(G)[[h]]$, with a new product $*$ and unchanged coproduct, is considered to be a topological Hopf algebra. we recall that deformations with unchanged coproduct are called preferred deformations [29]. This condition is imposed to quantization because of the invariance property of the Poisson–Lie group bracket

$$\Delta(\{\phi,\psi\}) = \{\Delta(\phi), \Delta(\psi)\}\,.$$

It is therefore natural to impose the same compatibility condition of the star-product with respect to the coproduct of $F(G)$, i.e.:

$$\Delta(\phi * \psi) = (\Delta(\phi) * \Delta(\psi)) \tag{17}$$

is satisfied. The star-product on the right side is canonically defined on $F(G)\hat{\otimes}F(G)$ by

$$(\phi\hat{\otimes}\psi) * (\phi^{'}\hat{\otimes}\psi^{'}) = (\phi * \phi^{'})\hat{\otimes}(\psi * \psi^{'})\,. \tag{18}$$

Remark: If all C_k are a left (right)-invariant bidifferential operators then the corresponding star product is called a left (right)-invariant one.

Now, using the fact that the enveloping algebra $U(g)$ is isomorphic to the algebra of left (right) invariant differential operators on G, we deduce that if C_i is a left-invariant two -cochain then there is an element $F_i \in U(g)\hat{\otimes}U(g) \subset\ \ D^{'}(G)\hat{\otimes}D^{'}(G)$ such that:

$$C_i^l(\phi,\psi) = F_i^l(\phi\hat{\otimes}\psi) \tag{19}$$

and similarly for the right invariant two cochain there exist an element $H_i \in U(g)\hat{\otimes}U(g) \subset\ \ D^{'}(G)\hat{\otimes}D^{'}(G)$ such that:

$$C_j^r(\phi,\psi) = H_j^r(\phi\hat{\otimes}\psi)\,. \tag{20}$$

If we introduce the two elements of $U(g)\hat{\otimes}U(g)[[h]] \subset\ \ D^{'}(G)\hat{\otimes}D^{'}(G)[[h]]$

$$F = 1 + \sum_{i\geq 1} F_i h^i\,,$$

$$H = 1 + \sum_{j\geq 1} H_j h^j\,.$$

Then, we obtain the following:

Proposition 1 *The associativity of the left-invariant star-product implies*

$$(\Delta_0\hat{\otimes}id)F.(F\hat{\otimes}1) = (id\hat{\otimes}\Delta_0)F.(1\hat{\otimes}F) \tag{21}$$

and the associativity of the right-invariant star-product leads to the following equality

$$(S_0^{\hat{\otimes}2}(H)\hat{\otimes}1).(\Delta_0\hat{\otimes}id)S_0^{\hat{\otimes}2}(H) = (1\hat{\otimes}S_0^{\hat{\otimes}2}(H)).(id\hat{\otimes}\Delta_0)S_0^{\hat{\otimes}2}(H)\,. \tag{22}$$

Proof: Writing the left-invariant star product by the following expression

$$(\phi *^l \psi) = \mu(F^l(\phi\hat{\otimes}\psi))\,, \tag{23}$$

where μ is the usual multiplication on the algebra of smooth functions over the group and $F = 1 + \frac{\hbar}{2}r + \sum_{i\geq 2} F_i h^i$, then for any element X in the enveloping algebra, we have

$$\begin{aligned}
&\langle X, \phi *^l (\psi *^l \chi)\rangle \\
&= \langle X, \mu(id\hat{\otimes}\mu)((id\hat{\otimes}\Delta_0)F^l.F^l_{23}(\phi\hat{\otimes}\psi\hat{\otimes}\chi))\rangle\,, \\
&= \langle (id\hat{\otimes}\Delta_0)\Delta_0(X), (id\hat{\otimes}\Delta_0)F^l.F^l_{23}(\phi\hat{\otimes}\psi\hat{\otimes}\chi)\rangle\,, \\
&= \langle (id\hat{\otimes}\Delta_0)(F)(1\hat{\otimes}F)(id\hat{\otimes}\Delta_0)\Delta_0(X), (\phi\hat{\otimes}\psi\hat{\otimes}\chi)\rangle\,.
\end{aligned} \tag{24}$$

Similarly we found that

$$\langle X, (\phi *^l \psi) *^l \chi\rangle = \langle (\Delta_0\hat{\otimes}id)(F)(F\hat{\otimes}1)(\Delta_0\hat{\otimes}id)\Delta_0(X), (\phi\hat{\otimes}\psi\hat{\otimes}\chi)\rangle\,, \tag{25}$$

so, comparing (24)and (25) we obtain the result (21); the same proof is valid for the right-invariant one.

Proposition 2 *Assume that F is a left-invariant star product on the group G, then $S_0^{\hat{\otimes}2}(F)$ is a right-invariant star product on the group G.*

Proof: by applying the operator $(S_0\hat{\otimes}S_0\hat{\otimes}S_0)$ to the equation (21) and using the fact that $(S_0\hat{\otimes}S_0)\circ\Delta_0 = \Delta_0^{op}\circ S_0$ we found obviously the equation (22). We define in the following The star product on the compact Poisson–Lie group by the expression

$$\phi * \psi = \mu((S_0^{\hat{\otimes}2})^{-1}(F^{-1})^r.F^l(\phi\hat{\otimes}\psi))\,. \tag{26}$$

In fact, the product defined in this way is associative; $\forall X \in U(g)$, we have:

$$\begin{aligned}
&\langle(\phi * \psi) * \chi, X\rangle \\
&= \langle\mu(S_0^{\hat{\otimes}2})^{-1}((F^{-1}))^r.F^l(\mu((F^{-1})^r.F^l(\phi\hat{\otimes}\psi))\hat{\otimes}\chi)), X\rangle\,, \\
&= \langle\mu(\mu\hat{\otimes}id)((\Delta_0\hat{\otimes}1)((S_0^{\hat{\otimes}2})^{-1}((F^{-1}))^r).(\Delta_0\hat{\otimes}1)F^l.((S_0^{\hat{\otimes}2})^{-1}((F^{-1}))^r\hat{\otimes}1) \\
&.(F^l\hat{\otimes}1)(\phi\hat{\otimes}\psi\hat{\otimes}\chi)), X\rangle\,, \\
&= \langle\mu(\mu\hat{\otimes}id)((\Delta_0\hat{\otimes}id)((S_0^{\hat{\otimes}2})^{-1}((F^{-1}))^r).((S_0^{\hat{\otimes}2})^{-1}((F^{-1}))^r\hat{\otimes}1).(\Delta_0\hat{\otimes}id)F^l \\
&.(F^l\hat{\otimes}1)(\phi\hat{\otimes}\psi\hat{\otimes}\chi)), X\rangle\,, \\
&= \langle(\Delta_0\hat{\otimes}id)((S_0^{\hat{\otimes}2})^{-1}((F^{-1}))^r).((S_0^{\hat{\otimes}2})^{-1}((F^{-1}))^r\hat{\otimes}1).(\Delta_0\hat{\otimes}id)F^l \\
&.(F^l\hat{\otimes}1)(\phi\hat{\otimes}\psi\hat{\otimes}\chi)), (\Delta_0\hat{\otimes}id)\Delta_0(X)\rangle\,, \\
&= \langle(\phi\hat{\otimes}\psi\hat{\otimes}\chi), ((F^{-1})\hat{\otimes}1)(\Delta_0\hat{\otimes}id)((F^{-1}))(\Delta_0\hat{\otimes}id)\Delta_0(X)(\Delta_0\hat{\otimes}id)F.(F\hat{\otimes}1))\rangle\,, \\
&= \langle(\phi\hat{\otimes}\psi\hat{\otimes}\chi), (1\hat{\otimes}(F^{-1}))(id\hat{\otimes}\Delta_0)((F^{-1}))(id\hat{\otimes}\Delta_0)\Delta_0(X)(id\hat{\otimes}\Delta_0)F.(1\hat{\otimes}F))\rangle\,, \\
&= \langle(id\hat{\otimes}\Delta_0)((S_0^{\hat{\otimes}2})^{-1}((F^{-1}))^r)(1\hat{\otimes}(S_0^{\hat{\otimes}2})^{-1}((F^{-1}))^r)(id\hat{\otimes}\Delta_0)F^l \\
&.(1\hat{\otimes}F^l))(\phi\hat{\otimes}\psi\hat{\otimes}\chi), (id\hat{\otimes}\Delta_0)\Delta_0(X)\rangle\,, \\
&= \langle\phi * (\psi * \chi), X\rangle\,. \qquad (27)
\end{aligned}$$

The star-product define a deformation of a quotient algebra $\boldsymbol{F}_e(G)$ defined as the set of element of $\boldsymbol{F}(G)$ in a neighbour containing the identity of G modulo the following relation of equivalence

$$\phi \sim \psi \quad \text{if} \;\; \langle X, \phi - \psi\rangle = 0 \quad \text{for any} \;\; X \in \;\; U(g) \subset D^{'}(G)\,,$$

where $\langle , \rangle$ is the pairing between $\boldsymbol{F}_e(G)$ and $U(g) \subset D^{'}(G)$.
The dual $\boldsymbol{F}_e^*(G)$ of $\boldsymbol{F}_e(G)$ is nothing but the set of distributions on G with support at the unit element of G. So, if using the L. Schwartz theorem which states That the set of distributions on G with support at the identity element of G is the enveloping algebra of the Lie algebra of the Lie group, we deduce that a star product provide a deformation of the enveloping algebra.

The quantized enveloping algebra $U(g)[[h]]$ is endowed with a structure of Hopf algebra where the multiplication algebra is the ordinary convolution on $\boldsymbol{F}_e^*(G)$ and the coproduct Δ_F is given by

$$\langle\Delta_F(X), \phi\hat{\otimes}\psi\rangle = \langle X, \phi * \psi\rangle \qquad (28)$$

for all $\phi, \psi \;\in\; \boldsymbol{F}(G)$, and $X \in U(g)$ in fact:

$$\langle\Delta_F(X), \phi\hat{\otimes}\psi\rangle = \langle X, \phi * \psi\rangle$$

for $X \in U(g)$, $\phi, \psi \in F(G)$ Explicitly:

$$\begin{aligned}
\langle X, \phi * \psi\rangle &= \langle X, \mu((S_0^{\hat{\otimes}2})^{-1}(F^{-1}))^r F^l(\phi\hat{\otimes}\psi)\rangle \\
&= \langle\Delta_0(X), (S_0^{\hat{\otimes}2})^{-1}(F^{-1})^r F^l(\phi\hat{\otimes}\psi)\rangle
\end{aligned}$$

using the fact that:

$$X^{l(r)}_{/e}(\phi) = X_{/e}(\phi) = X(\phi)(e)\,,$$

we obtain:

$$\langle X, \phi * \psi\rangle = \mu((\Delta_0(X))^l((S_0^{\hat{\otimes}2})^{-1}(F^{-1}))^r F^l(\phi\hat{\otimes}\psi))(e,e)\,.$$

Then we must calculate first the quantity:

$$I = ((\Delta_0 X)^l (S_0^{\hat{\otimes}2})^{-1}(F^{-1})^r F^l(\phi\hat{\otimes}\psi))(\mathrm{g},\mathrm{g})\,.$$

For this, we use the fact that for $X \in U(g) \subset D^{'}(G)$ we have

$$\begin{aligned} X^l_{/\mathrm{g}}(\phi) &= X^l(\phi)(\mathrm{g}) = \langle \delta_\mathrm{g} *_c X, \phi\rangle\,, \\ X^r_{/\mathrm{g}}(\phi) &= X^r(\phi)(\mathrm{g}) = \langle S_0(X) *_c \delta_\mathrm{g}, \phi\rangle\,, \end{aligned}$$

where $*_c$ is the convolution product on $U(g) \equiv D^{'}(e) \subset D^{'}(G)$ and δ_g is the Dirac distribution at $\mathrm{g} \in G$ so:

$$\begin{aligned} I &= \langle (\delta_\mathrm{g}\hat{\otimes}\delta_\mathrm{g}) *_c \Delta_0(X), (S_0^{\hat{\otimes}2})^{-1}(F^{-1})^r F^l(\phi\hat{\otimes}\psi)\rangle\,, \\ &= \langle (\delta_\mathrm{g}\hat{\otimes}\delta_\mathrm{g}) *_c \Delta_0(X), \langle F^{-1} *_c (\delta_\mathrm{g}\hat{\otimes}\delta_\mathrm{g}), \langle (\delta_\mathrm{g}\hat{\otimes}\delta_\mathrm{g}) *_c F, \phi\hat{\otimes}\psi\rangle\rangle\rangle\,. \end{aligned}$$

Next we use the following notation; for $X \in D^{'}(G)$, its dual (denoted $\tilde{X}$) $\in F(G)$, then:

$$\begin{aligned} I &= \langle F^{-1} *_c (\delta_\mathrm{g}\hat{\otimes}\delta_\mathrm{g}), \langle (\delta_\mathrm{g}\hat{\otimes}\delta_\mathrm{g}) *_c F, \phi\hat{\otimes}\psi\rangle\rangle.(\Delta_0(X))\tilde{}(\mathrm{g},\mathrm{g})\,, \\ &= (F^{-1})\tilde{}.(\phi\hat{\otimes}\psi).(F)\tilde{}.(\Delta_0(X))\tilde{}(\mathrm{g},\mathrm{g}) \end{aligned}$$

and if we use the following property of the convolution product:

$$(Y *_c X)\tilde{} = (X)\tilde{}.(Y)\tilde{}\,,$$

we have:

$$\begin{aligned} I &= ((F^{-1})\tilde{}.(\phi\hat{\otimes}\psi).(\Delta_0(X) *_c F)\tilde{})(\mathrm{g},\mathrm{g})\,, \\ &= \langle (\delta_\mathrm{g}\hat{\otimes}\delta_\mathrm{g}) *_c (\Delta_0(X) *_c F), F^{-1})\tilde{}.(\phi\hat{\otimes}\psi)\rangle\,, \\ &= \langle (F^{-1}) *_c (\delta_\mathrm{g}\hat{\otimes}\delta_\mathrm{g}) *_c (\Delta_0(X) *_c F), (\phi\hat{\otimes}\psi)\rangle\,. \end{aligned}$$

Then we have:

$$\begin{aligned} \langle X, \phi * \psi\rangle &= \mu((\Delta_0 X)^l (S_0^{\hat{\otimes}2})^{-1}(F^{-1})^r F^l(\phi\hat{\otimes}\psi))(e,e)\,, \\ &= \langle (F^{-1}) *_c (\delta_\mathrm{g}\hat{\otimes}\delta_\mathrm{g}) *_c (\Delta_0(X) *_c F), (\phi\hat{\otimes}\psi)\rangle\,, \\ &= \langle (F^{-1}) *_c \Delta_0(X) *_c F, (\phi\hat{\otimes}\psi)\rangle\,, \end{aligned}$$

which implies that:

$$\Delta_F(X) = F^{-1} *_c \Delta_0(X) *_c F\,.$$

Thus

$$\Delta_F(X) = F^{-1}.\Delta_0(X).F\,. \tag{29}$$

We can easily show that the twisted coproduct Δ_F is coassociative, in fact, by using the main equation (21) and the coassociativity of the classical coproduct we obtain

$$(\Delta_F \hat{\otimes} id)\Delta_F(X) = (id \hat{\otimes} \Delta_F)\Delta_F(X)\,.$$

For the antipode of the quantized enveloping algebra, we recall first that the antipode S_0 of $U(g)$ satisfies the following equation

$$m(S_0 \hat{\otimes} id)\Delta_0(X) = m(id \hat{\otimes} S_0)\Delta_0(X) = \varepsilon(X)1\,, \tag{30}$$

where m is the usual multiplication on the enveloping algebra $U(g)$. F and F^{-1} can be respectively split as

$$F = \sum_k a_k \hat{\otimes} b_k, \qquad F^{-1} = \sum_k c_k \hat{\otimes} d_k$$

and set $u = m(id \hat{\otimes} S_0)(F^{-1})$ is an invertible element of $U(g)[\![h]\!] \subset D^{'}(G)[\![h]\!]$, then we can easily show that the antipode of the quantized enveloping algebra $U(g)[\![h]\!]$ is given by:

$$S_F(X) = u.S_0(X).u^{-1}\,, \tag{31}$$

where $u^{-1} = m(S_0 \hat{\otimes} id)F$.

Similarly, we can prove that

$$m(id \hat{\otimes} S_F)\Delta_F(X) = \varepsilon(X)1\,. \tag{32}$$

In other words, by using (9), (10) we obtain that the antipode of the deformed algebra of functions on the group is given by

$$S_h(f) = S((S_0(u^{-1}))^r u^l f)\,.$$

Now if we introduce the following element defined by Drinfeld [17]

$$R_F = F_{21}^{-1}.F\,, \tag{33}$$

then we can easily show that R_F define a quasitriangular structure on the quantized enveloping algebra $U(g)[\![h]\!]$. In fact if using polynomial notation [18, 29], we obtain

$$\begin{aligned}(\Delta_F\hat{\otimes}id)R_F &= (F^{-1}\hat{\otimes}1)(\Delta\hat{\otimes}id)R_F(F\hat{\otimes}1)\,, \\ &= R_F(x,z)R_F(y,z)\,. \end{aligned} \tag{34}$$

So,

$$(\Delta_F\hat{\otimes}id)R_F = (R_F)_{13}.(R_F)_{23}\,, \tag{35}$$

where we have used the definition(33) in the first, sixth and seventh equalities and the relation (21) writing in polynomial notation for the remaining ones. Similarly, we obtain

$$(id\hat{\otimes}\Delta_F)R_F = (R_F)_{13}.(R_F)_{12}\,. \tag{36}$$

From the fact that $\phi * 1 = 1 * \phi = \phi$ for all $\phi\ \in \boldsymbol{F}_e(G)$ we deduce that

$$(id\hat{\otimes}\varepsilon)F = (\varepsilon\hat{\otimes}id)F = 1 \tag{37}$$

consequently,

$$(\varepsilon\hat{\otimes}id)(R_F) = (id\hat{\otimes}\varepsilon)(R_F) = 1 \tag{38}$$

and from the definition(33)we deduce that

$$(R_F)_{21}.R_F = 1\,. \tag{39}$$

Now using again the expression (33)we obtain

$$\begin{aligned}(\Delta_F)^{op} &= P(\Delta_F)\,, \\ &= P(F^{-1}).\Delta_0.P(F) = P(F^{-1}).F.\Delta_F.F^{-1}.P(F) \end{aligned} \tag{40}$$

then

$$(\Delta_F)^{op} = R_F.\Delta_F.(R_F)^{-1}\,. \tag{41}$$

From (34) and (41) we show that R_F satisfies the quantum Yang–Baxter equation

$$(R_F)_{12}.(R_F)_{13}.(R_F)_{23} = (R_F)_{23}.(R_F)_{13}.(R_F)_{12}\,. \tag{42}$$

Now using the equations (35), (36) and (42) it is easily seen that the R-matrix R_F satisfy

$$(R_F)_{23}(id\hat{\otimes}\Delta_F)R_F = (R_F)_{12}.(\Delta\hat{\otimes}id)R_F\,,$$

which together with (38)and (41) implies that $(U(g)[\![h]\!], \Delta_F, R_F, S_F)$ is also a coboundary Hopf algebra, which is an obvious result since a triangular Hopf algebra is a coboundary one.

Then we have established that a star-product on a compact Poisson–Lie group (G, r) leads to a well-behaved topological quantum algebra $(U(g)[\![h]\!], \Delta_F, R_F, S_F)$, where $F = 1 + \frac{\mathrm{h}}{2}r + \sum_{i\geq 2} F_i h^i$ and $R_F = 1 + hr + \ldots$.

Theorem 1 *Let $(U(g)[\![h]\!], \Delta, R, S)$ be a well-behaved triangular topological Hopf algebra then it can be obtained by a star-product on the connected and simply connected compact Poisson–Lie group (G, r) corresponding to the Lie algebra g, where $R = 1 + hr + \ldots$.*

4. Star products and quantum groups

The relevance of the previous procedure is that we can get many concrete solutions of the QYBE by taking different representations of the universal object R. Thus, if we take a finite dimensional representation ρ of g in the algebra of $n \times n$ complex matrices $M(n, C)$, we have

$$S = (\rho \hat{\otimes} \rho)(R)\,, \tag{43}$$

which satisfies the QYBE

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}\,, \tag{44}$$

and the unitary condition

$$SS_{21} = 1\,, \tag{45}$$

where $S_{21} = \sigma(S)$. We see that this quantization procedure leads in a natural way to QYBE.

Finally, let us consider again the matrix representation $\rho : g \to M(n, C)$ consequently the group G is realized as subgroup of $GL(n, C)$. Let $T = (t_{ij})_{i,j=1}^{n}$ be the matrix of coordinate functions on G

$$t_{ij}(\mathrm{g}) = \mathrm{g}_{ij}\,. \tag{46}$$

Left and right actions of G on matrix coordinates on G given by

$$\begin{aligned} (X^l t_{ij})(\mathrm{g}) &= (\mathrm{g}X)_{ij} = \sum_k g_{ik}X_{kj}\,, \\ (X^r t_{ij})(\mathrm{g}) &= (X\mathrm{g})_{ij} = \sum_k X_{ik}g_{kj}\,. \end{aligned} \tag{47}$$

for $X \in g$.

On the other hand, let $\hat{F} = (\rho \hat{\otimes} \rho)(F)$ and defining $T_1 = T \hat{\otimes} 1$ and $T_2 = 1 \hat{\otimes} T$, the $*$-product between matrix coordinates of G elements can be expressed in a elegant manner by

$$T_1 *_h T_2 = \hat{F}^{-1} T \hat{\otimes} T \hat{F} \tag{48}$$

applying the flip operator to both sides of this expression we get

$$T_2 *_h T_1 = \sigma(\hat{F}^{-1}) T \hat{\otimes} T \sigma(\hat{F}) \,, \tag{49}$$

and combining the two above equations we obtain the relation

$$S T_1 *_h T_2 = T_2 *_h T_1 S \,, \tag{50}$$

which is the well-known formula that gives the commutation relations between the matrix coordinate functions of G defining the quantum group $F_h(G)$

5. Equivalent star product on a Poisson–Lie group

First, we recall [34,35], that two star products $*_1$ and $*_2$ on a Poisson–Lie group are said to be equivalent if there is a series

$$T = id + \sum_{r=1}^{\infty} h^r T_r \,,$$

where the T_r are linear operators on $C^{\infty}(G)$ such that

$$T(f *_1 \mathrm{g}) = Tf *_2 T\mathrm{g} \,.$$

Let F and $\bar{F}$ be the two corresponding 2-cocycles *i.e.*, two invertible element of the Hopf algebra $U(g)[\![h]\!] \subset D^{'}(G)[\![h]\!]$ such that

$$(\Delta \hat{\otimes} id) F.F_{12} = (id \hat{\otimes} \Delta) F.F_{23} \,,$$

$$(\Delta \hat{\otimes} id) \bar{F}.\bar{F}_{12} = (id \hat{\otimes} \Delta) \bar{F}.\bar{F}_{23} \,,$$

and let $A = (U((g)[[h]], \Delta_F, R_F, S_F)$ and $\bar{A} = (U(g)[[h]], \Delta_{\bar{F}}, R_{\bar{F}}, S_{\bar{F}})$ be the resulting well-behaved topological quantum algebras, where

$$\Delta_F = F.\Delta_0.F^{-1}, \quad R_F = F_{21}^{-1}.F \,, \tag{51}$$

$$\Delta_{\bar{F}} = \bar{F}.\Delta_0.\bar{F}^{-1}, \quad R_{\bar{F}} = \bar{F}_{21}^{-1}.\bar{F} \,, \tag{52}$$

then it is easily seen that $\bar{A}$ can be obtained from A by applying the twist $\hat{F} = F^{-1}.\bar{F}$. In fact

$$\Delta_{\bar{F}} = \hat{F}.\Delta_F.\hat{F}^{-1} \tag{53}$$

and

$$R_{\bar{F}} = \hat{F}_{21}^{-1}.R_F.\hat{F} \,. \tag{54}$$

If the two star products are equivalents *i.e.* the corresponding elements F and $\bar{F}$are related by the following expression

$$\bar{F} = \Delta_0(E^{-1}).F.(E\hat{\otimes}E) \tag{55}$$

for some invertible element E of $U(g)[[h]] \subset D^{'}(G)[[h]]$, then the coproduct $\Delta_{\bar{F}}$ can be rewritten as

$$\Delta_{\bar{F}}(X) = (E^{-1}\hat{\otimes}E^{-1})\Delta_F(E.X.E^{-1}).(E\hat{\otimes}E)\,. \tag{56}$$

The two twisted antipodes are related by the following expression

$$S_{\bar{F}} = E^{-1}S_0(E^{-1}).S_F.S_0(E).E\,. \tag{57}$$

Similarly, the triangular structures are related by

$$R_{\bar{F}} = (E^{-1}\hat{\otimes}E^{-1}).R_F.(E\hat{\otimes}E)\,. \tag{58}$$

So, the induced isomorphism maps the triangular structures as well. This say that the processes of quantization-deformation can only give a genuinely new triangular topological quantum algebra is the two cocycle F corresponding to the star-product is cohomologically(relatively to the Hopf algebra cohomology, (see [37]) non trivial, for example, if the second group of cohomology for the Hopf topological algebra $U(g)[\![h]\!]$ vanish then all star-products on the connected and simply connected compact Poisson–Lie group corresponding to the Lie bialgebra g are equivalents.

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