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NOTE ON THE ALGEBRAIC STRUCTURE OF SUPERINTEGRABLE SYSTEMS*

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The algebraic structure underlying Winternitz system is found to be universal for maximally superintegrable systems.

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1. Introduction

According to the well known definition [1] a dynamical system of N degrees of freedom is integrable if there exist N globally defined mutually Poisson-commuting isolating integrals of motion $F_k, k = 1, ..., N$. Under some mild global assumptions (for example, compactness, *i.e.* a confining character of interparticle forces) the hypersurfaces $F_k =$ const. are N-dimensional tori invariant under the dynamical map determined by the Hamiltonian. The new, action-angle variables are defined as follows:

$$J_k = \frac{1}{2\pi}\sum_i \int\limits_{\gamma_k} p_i dq^i\,, \quad k = 1, ..., n\,, \tag{1}$$

where γ_k are generators of the homotopy group of invariant tori and the orientation is chosen such that $J_k > 0$; the canonically conjugated angle variables φ^k parametrize the circles S^1 once the invariant torus is identified with $(S^1)^N$. In terms of new variables the equations of motion read

$$\begin{aligned} \dot{J}_k &= 0\,, \\ \dot{\varphi}^k &= \omega^k(J) = \frac{\partial H(J)}{\partial J_k}\,. \end{aligned} \tag{2}$$

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In the generic case there are no other global isolating integrals of motion. If there are such integrals, the system is called superintegrable. The maximal number of additional integrals is $N-1$; the corresponding system is called maximally superintegrable.

Maximal superintegrability implies that all trajectories are closed or, equivalently, all ratios $\omega^k(J)/\omega^l(J)$ are rational numbers. The additional integrals can be chosen as follows: if $n^k\omega^k(J) = n^l\omega^l(J)$ with (no summation), n^k, n^l integers, then $\cos(n^k\varphi^k - n^l\varphi^l)$ is globally defined constant of motion [2].

It is easy to find the general form of maximally superintegrable system. Typically, $\omega^k(J)$ are continuous functions. Therefore

$$\omega^k(J) = m^k\omega(J)\,, \tag{3}$$

where $m^k \in \boldsymbol{Z}$ and $\omega(J)$ is some function of action variables. Due to Eq. (2)

$$\omega(J)m^k = \frac{\partial H}{\partial J_k}\ \ ,\quad k = 1,...,N \tag{4}$$

Eq. (4) implies

$$H = H\left(\sum_{k=1}^{N} m^k J_k\right)\,. \tag{5}$$

Important examples of superintegrable systems are isotropic harmonic oscillator, Kepler problem, Calogero–Moser model [3, 4] and Winternitz system [5].

To a superintegrable system one can associate a nontrivial symmetry algebra. To see this let us note that the Poisson bracket of two integrals of motion (which, for superintegrable systems, is generically nonvanishing) is again a constant of motion so it must be a function of $2N-1$ independent integrals. Generically, the resulting algebra is nonlinear. For some cases it linearlizes on some submanifold (as for the Kepler problem on hypersurface of constant energy) or can be linearized by judicious choice of basic integrals. The latter was shown by Evans [6] to be the case for Winternitz system. In the present note we show that the Evans result can be immediately extended to any superintegrable system.

2. The dynamical algebra of superintegrable systems

Let us assume that we have a superintegrable system of N degrees of freedom. In the "confining" part of phase space one introduces the action-angle variables $J_k, \varphi^k, k = 1,...,N$. As explained in Sec. 1, superintegrability implies $\omega^k(J) = m^k\omega(J)$. Let m be the least common multiple of $\mid m^k \mid$ and let $l_k \equiv m/m^k$. Define new complex valued variables

$$a_k \equiv \sqrt{\frac{J_k}{l_k}} \mathrm{e}^{-il_k\varphi^k}, \qquad a_k^* \equiv \sqrt{\frac{J_k}{l_k}} \mathrm{e}^{il_k\varphi^k}; \tag{6}$$

it can happen that $J_k/l_k < 0$ but the phases of square roots can be chosen arbitrarily. The variables a_k are defined globally. Therefore, they are expressible in terms of original canonical variables q^i, p_i. The basic Poisson bracket read

$$\{a_k, a_l\} = 0, \qquad \{a_k^*, a_l^*\} = 0, \qquad \{a_k, a_l^*\} = -i\delta_{kl} \tag{7}$$

i.e. we get Weyl algebra W_N. It follows also immediately from Eqs. (2),(6)

$$\dot{a}_k = im\omega(J)a_k, \qquad \dot{a}_k^* = -im\omega(J)a_k^*. \tag{8}$$

Therefore

$$T_{ik} \equiv a_i a_k^* \tag{9}$$

are integrals of motion and it follows easily from the discussion of Sec. 1 that all $2N-1$ independent integrals are expressible in terms of T_{ik}. For example,

$$H = H\left(m\sum_{i=1}^{N} T_{ii}\right). \tag{10}$$

The functions T_{ik} obey $u(N)$ algebra

$$\{T_{ij}, T_{kl}\} = i\delta_{il}T_{kj} - i\delta_{jk}T_{il}. \tag{11}$$

Therefore, a proper choice of finite set of linearly independent (but functionally dependent) integrals of motion linearizes the algebra.

Following Evans [6] we introduce additional functions

$$P_{ij} = a_i a_j, \qquad Q_{ij} = a_i^* a_j^* \tag{12}$$

which close together with T_{ij} to $\mathrm{sp}(2N, \boldsymbol{R})$ algebra

$$\begin{aligned}
\{T_{ij}, P_{kl}\} &= -i\delta_{ik}P_{il} - i\delta_{jl}P_{ik}, \\
\{T_{ij}, Q_{kl}\} &= i\delta_{ik}Q_{jl} + i\delta_{il}Q_{jk}, \\
\{P_{ij}, Q_{kl}\} &= i\delta_{ik}T_{jl} + i\delta_{il}T_{jk} + i\delta_{jk}T_{il} + i\delta_{jl}T_{ik}.
\end{aligned} \tag{13}$$

The conjugacy relations are

$$Q_{ij}^* = P_{ij}, \qquad T_{ij}^* = T_{ji}. \tag{14}$$

3. Conclusions

We have shown that linear structure of the algebra of integrals of motion, revealed by Evans [6] in the case of Winternitz system, is the general property of maximally superintegrable systems. The integrals linearising the algebra are, in general quite complicated, single-valued functions on phase space. Their complexity is the price one has to pay for having a linear structure. One can also ask what are the consequences for quantum theory. It is well known that the quantum theory in the quasiclassical regime $\hbar \to 0$, $n \to \infty$ (n stands collectively for quantum numbers), $n\hbar =$ const. is achieved by imposing $J_k = n_k \hbar$; then

$$E_{\{n\}} = H\left(\hbar \sum_{k=1}^{N} m^k n_k\right) \tag{15}$$

showing large degeneracy of the spectrum of superintegrable systems.

This result can be achieved as follows: define

$$\tilde{a}_k = \sqrt{J_k}\mathrm{e}^{-i\varphi^k}, \qquad \tilde{a}_k^* = \sqrt{J_k}\mathrm{e}^{+i\varphi^k}\,. \tag{16}$$

These variables are defined globally and $\{\tilde{a}_k, \tilde{a}_l^*\} = -i\delta_{kl}$. One can try to quantize them; due to in general very complicated character of the transformation $(q,p) \to (\varphi, J)$ an attempt to construct the quantum counterparts of $\tilde{a}_k, \tilde{a}_k^*$ is plagued by ordering problems. In quasiclassical regime reorderings produce, however, higher order terms in $\hbar$ so they are irrelevant. Consequently, in this regime one car define $\hat{\tilde{a}}_k, \hat{\tilde{a}}_k^{+}$ obeying

$$[\hat{\tilde{a}}_k, \hat{\tilde{a}}_l^{+}] = \hbar\delta_{kl} \tag{17}$$

and identify $J_k \sim \hat{\tilde{a}}_k^{+}\hat{\tilde{a}}_k \equiv \hat{\tilde{N}}_k$.

If one attempts to make the some reasoning with $\tilde{a}_k$ replaced a_k, Eq. (6), it appears immediately that some energy levels are missing if some $l_k \neq 1$. This can be understood if we note that $\hat{a}_k \sim f(\hat{\tilde{N}}_k)\hat{\tilde{a}}_k^{l_k}$ and $f(X)$ is singular at $X = 0$. One has then to be very careful in bookkeeping of powers of $\hbar$.

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REFERENCES

[1] V. Arnold, *Mathematical Methods of Classical Mechanics*, Springer 1978.

[2] L. Landau, *Mechanics*, PWN, Warsaw 1966, in Polish.

[3] F. Calogero, *J. Math. Phys.* **12**, 419 (1971).

[4] J. Moser, *Adv. Math.* **16**, 1 (1975).

[5] P. Winternitz *et al.*, *Sov. J. Nucl. Phys.* **4**, 444 (1966); J. Fris *et al.*, *Phys. Lett.* **16**, 354 (1967); A. Makarov *et al.*, *Nuovo Cim.* **52**, 1061 (1967).

[6] N.W. Evans, *Phys. Lett.* **A147**, 483 (1990).