LOCAL CHARACTERISTICS OF RANDOM MOTION*

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(Received October 30, 2000)

Markovian diffusion processes yield a system of conservation laws which couple various conditional expectation values (local moments). Solutions of that closed system of deterministic partial differential equations stand for a regular alternative to erratic (irregular) sample paths that are associated with weak solutions of the original stochastic differential equations. We investigate an issue of local characteristics of motion in the non-Gaussian context, when moments of the probability measure may not exist. A particular emphasis is put on jump-type stochastic processes with the Ornstein–Uhlenbeck–Cauchy process as a fully computable exemplary case.

PACS numbers: 02.50.Ga

1. Local characteristics of the Brownian motion

Lets us consider a Markov process X_t in R^1. We can fully characterize an associated random dynamics by means of a transition density $p(y,s,x,t)$ with $0 \leq s < t$ and an initial density $\rho_0(x) = \rho(x,t_0), 0 \leq t_0$. For the stochastic process to be properly defined we impose the so-called stochastic continuity condition:

$$\lim_{t\to s^+} \frac{1}{t-s} \int\limits_{|y-x|>\varepsilon} p(y,s,x,t)dx = 0 \tag{1}$$

to be valid for arbitrary (every) $\varepsilon > 0$ and for almost every $y \in R^1$. That is known to imply that along a sample path ω there holds

$$\lim_{t\to s^+} P[\omega : |X_t(\omega) - X_s(\omega)| \geq \varepsilon] = 0.$$

* Presented at the XIII Marian Smoluchowski Symposium on Statistical Physics, Zakopane, Poland, September 10–17, 2000.

This condition needs to be respected by both diffusion and jump-type propagation scenarios and is known to be necessary but still insufficient for the Markov process to have continuous trajectories.

Various characteristics of random motion can be directly associated with properties of sample trajectories (consider for concreteness the iterated logarithm laws in the Brownian motion). Those characteristics of random dynamics that are free of irregularities of sample paths, need to involve some forms of averaging. That refers either to global averaging present while evaluating moments of a probability measure (provided they exist) or to the evaluation of local moments of that measure.

Given a transition probability of a Markov process, we can attempt to derive a number of local expectation values, like *e.g.* the forward drift of the process:

$$b(x,s) = \lim_{t\to s^+} \frac{1}{t-s} \int\limits_{|y-x|\leq\delta} (y-x)p(x,s,y,t)dy \tag{2}$$

and the diffusion (coefficient if a constant) function

$$D(x,s) = \lim_{t\to s^+} \frac{1}{t-s} \int\limits_{|y-x|\leq\delta} (y-x)^2 p(x,s,y,t)dy\,. \tag{3}$$

where the δ cutoff is needed to guarantee a convergence of the integral. In principle, the drift and diffusion functions should take values independent of the δ-cutoff and/or exist when the cutoff is removed (once we let δ go to ∞). However, for the jump-type processes, local moments (*e.g.* drifts and diffusion functions) in the absence of a cutoff are generically nonexistent, while in a cutoff version they display an explicit and nontrivial δ-dependence.

As yet, our discussion extends to both continuous and discontinuous processes. Let us explicitly reveal when a diffusion process enters the game. By assuming that holds for

$$\lim_{t\to s^+} \frac{1}{t-s} \int\limits_{R} |y-x|^{2+\gamma} p(x,s,y,t)dy = 0 \tag{4}$$

for any $\gamma > 0$, we set a sufficient condition for the continuity of sample paths (to hold true almost surely), that also allows to remove the previous δ-cutoff from local moment formulas. The resulting process is a Markovian diffusion process with the forward drift $b(x,s)$ and diffusion function $D(x,s)$. Our further discussion will be carried under a simplifying assumption that a diffusion function $D(x,s)$ actually is a constant which we denote by D.

Our previous considerations, when specialized to Markovian diffusion-type processes, can be cast in another form. Namely, we can depart from a

formal infinitesimal version of a stochastic differential equation for a random variable $X_t = X(t)$ taking values in R^1:

$$dX_t = b(X(t), t)dt + \sqrt{2D}dW_t\,, \tag{5}$$

where $W_t = W(t)$ is a Wiener process and $b(x, t)$ stands for the forward drift of the diffusion-type process X_t.

If we assign a probability density $\rho_0(x)$ with which the initial data $x_0 = X(0)$ for the stochastic differential equation are distributed (weak solutions enter the scene), then the emergent Fick law would reveal a statistical tendency of particles to flow away from higher probability residence areas. This feature is encoded in the corresponding Fokker–Planck equation (equivalently, a continuity equation):

$$\partial_t \rho = -\nabla \cdot (v\rho)\,, \tag{6}$$

where a diffusion current velocity is

$$v(x, t) = b(x, t) - D\frac{\nabla\rho(x, t)}{\rho(x, t)}\,. \tag{7}$$

Clearly, the local diffusion current (a local flow that might be experimentally observed for a cloud of suspended particles in a liquid) $j = v\rho$ is nonzero in the nonequilibrium situation and quantifies a non-negligible matter transport which occurs as a consequence of the Brownian motion, on the ensemble average.

It is interesting to notice that the local velocity field $v(x, t)$ obeys the natural (local) conservation law, which we quite intentionally pattern after the moment identities (hierarchy of conservation laws) valid for the Boltzmann and Kramers equations. The pertinent momentum conservation law originates directly from the rules of the Itô calculus for Markovian diffusion processes, and from the first moment equation in the diffusion approximation of the Kramers theory:

$$\partial_t v + (v \cdot \nabla)v = \nabla(\Omega - Q)\,. \tag{8}$$

The general form of an auxiliary potential $\Omega(x, t)$ reads:

$$\Omega(x, t) = 2D\left[\partial_t\phi + \frac{1}{2}\left(\frac{b^2}{2D} + \nabla \cdot b\right)\right] \tag{9}$$

and can be interpreted as a constraint equation for the admissible forward drift $b(x, t) = 2D\nabla\phi(x, t)$. It is useful to mention that the above momentum

conservation law may be regarded to account for effects of external (conservative) volume forces $-\nabla(-\Omega)$ which are determined by *a priori* arbitrary, but bounded from below, continuous function $\Omega(x,t)$. Then, the allowed forward drift must be disentangled from the Ricatti-type equation (9) as a function of Ω.

In the above there appears a contribution from a probability density ρ-dependent potential $Q(x,t)$. It is given in terms of the so-called osmotic velocity field $u(x,t)$:

$$\begin{aligned} Q(x,t) &= \tfrac{1}{2}u^2 + D\nabla\cdot u\,, \\ u(x,t) &= D\nabla\ln\rho(x,t)\,, \end{aligned} \tag{10}$$

and is generic to a local momentum conservation law respected by isothermal Markovian diffusion processes. Notice that in case of the free Brownian motion (admitted, if we set $\Omega=0$), we would have $v(x,t)=-u(x,t)$ for all times. An equivalent form of the pressure-type potential Q is $Q=2D^2\frac{\triangle\rho^{1/2}}{\rho^{1/2}}$.

It is interesting to observe that a gradient field *ansatz* for the diffusion current velocity ($v=\nabla S$):

$$\partial_t\rho = -\nabla\cdot[(\nabla S)\rho] \tag{11}$$

allows to transform the momentum conservation law of a Markovian diffusion process to the universal Hamilton–Jacobi form:

$$\Omega = \partial_t S + \tfrac{1}{2}|\nabla S|^2 + Q\,, \tag{12}$$

where $Q(x,t)$ was defined before. By applying the gradient operation we immediately recover the previous local momentum conservation law.

Notice that the above Hamilton–Jacobi-type equation is sensitive to any additive (constant or time-dependent) modification of the potential Ω. In the above, the contribution due to Q is a direct consequence of an initial probability measure choice for the diffusion process, while Ω alone does account for an appropriate forward drift of the process.

The simplest realisation of the outlined theoretical framework can be provided by invoking a standard Ornstein–Uhlenbeck process. Namely, let us consider an Itô equation (in its symbolic differential version) for infinitesimal increments of the velocity random variable, exhibiting the systematic frictional resistance:

$$dV(t) = -\beta V(t)dt + \beta\sqrt{2D}dW(t)\,, \tag{13}$$

where $W(t)$ is the normalized Wiener process. One can easily infer the corresponding second Kolmogorov (Fokker–Planck) equation

$$\partial_t p(v_0,v,t) = \beta D\triangle_v + \beta\nabla_v\cdot[vp(v_0,v,t)] \tag{14}$$

for the transition probability density of the time homogeneous process in the velocity space alone. The pertinent transition probability density reads:

$$p(v_0, v, t) = \left(\frac{m}{2\pi kT(1 - \mathrm{e}^{-2\beta t})}\right)^{1/2} \exp\left\{\frac{m}{2kT}\frac{(v - v_0\mathrm{e}^{-\beta t})^2}{1 - \mathrm{e}^{-\beta t}}\right\}. \tag{15}$$

Let us consider an instantaneous velocity $V_t = v$, that has been achieved in the course of the Ornstein–Uhlenbeck random evolution beginning from a certain $V_0 = v_0$. We can evaluate a conditional expectation value (local mean with respect to the law of random displacements) over all randomly accessible velocities $V(t+\triangle t) = v'$ at a time $t+\triangle t$, $\triangle t > 0$. That determines the forward drift of the process:

$$b(v, t) = \lim_{\triangle t \downarrow 0}\left[\int v' p(v, v', \triangle t)dv' - v\right] = -\beta v \tag{16}$$

and thus provides us with an information about the mean tendency of the dynamics on small (but not too small if compared to the relaxation time β^{-1}) scales. Analogously, we can derive a diffusion function for the Ornstein–Uhlenbeck process which is constant and equals $2D\beta^2$.

To arrive at local conservation laws, an additional input of an initial probability density $\rho_0(v)$ of V_0 is necessary. To that end, one may choose an asymptotic stationary (invariant, Maxwell–Boltzmann) density of the Ornstein–Uhlenbeck process: $\rho_0(v) = (\frac{m}{2\pi kT})^{1/2} \exp[\frac{mv^2}{2kT}]$.

2. From Poisson probability law to jump-type processes

A random variable X taking discrete values $0, y, 2y, 3y, \ldots$, with $y > 0$ is said to have Poisson distribution $\mathcal{P}(\lambda), \lambda \geq 0$ with jump size y, if the probability of $X = ky$ is given by $P(X = ky) = \frac{\lambda^k}{k!}\exp(-\lambda)$. The characteristic function of $\mathcal{P}(\lambda)$ reads:

$$E\left[\exp(ipX)\right] = \exp[\lambda(\mathrm{e}^{ipy} - 1)] = \sum_0^\infty \mathrm{e}^{-\lambda}\frac{\lambda^k}{k!}\mathrm{e}^{ik(py)} = \sum_0^\infty \mathrm{e}^{ik(py)}P(X = ky) \tag{17}$$

and the first moment of the probability measure equals $E[X] = \lambda$. Notice that $P(X = 0) = \exp(-\lambda)$, hence the numerical value of $\lambda \geq 0$ fixes the no-jump probability. For the Poisson random variable with values $b + ky$, $k = 0, 1, \ldots$ we would get $E[\exp(ipX)] = \exp[ibp + \lambda(\mathrm{e}^{ipy} - 1)]$.

If we consider n independent random variables $X_j, 1 \leq j \leq n$ such that X_j has Poisson distribution $\mathcal{P}(\lambda_j)$ with jump size y_j, then a new variable

X can be introduced by means of the distribution of $X_1 + \ldots + X_n$ whose characteristic function reads

$$E[\exp(ipX)] = \exp\left[\sum_{j=1}^{n} \lambda_j \left(\mathrm{e}^{ipy_j} - 1\right)\right]. \tag{18}$$

The exponent in the above might include an additional term $ip\sum_1^n b_j$ if nonrandom shifts of each jump ky_j by b_j were allowed.

We can admit not only jumps of fixed magnitudes $y_1, \ldots, y_n$ but also jumps covering an arbitrary range in R_+. Let the distribution function of the magnitude of the jump be $P(x < y) = \mu(y)$. In this case we set

$$E[\exp(ipX)] = \exp\left[\int_{R_+} (\mathrm{e}^{ipy} - 1)\mu(dy)\right] \tag{19}$$

assuming that the integral in the exponent exists. (Notice that the previous formula is recovered, if we choose $\mu(dy) = \sum_{j=1}^n \lambda_j \delta(y - y_j)dy$.)

For any Borel set $A \subset R$ bounded away from the origin, the random variable X_A representing jumps bounded by A, gives rise to a characteristic exponent $\int_A (\mathrm{e}^{ipy} - 1)d\mu(y)$, and the expected number $E_A[X]$ of jumps of size bounded by A is equal to $\mu(A)$. We can interpret that in terms of jumps of different sizes that are mutually independent. Jumps whose size is bounded by $[y, y + \triangle y)$, $\triangle y \ll 1$, do contribute a Poisson component with exponent function approximately equal to $(\mathrm{e}^{ipy} - 1)\mu([y, y + \triangle y))$.

Let us consider an expression for a characteristic function of a probability measure of a certain random variable X that is given in the form $E[\exp(ipX)] = \exp[-F(p)]$ where $p \in R^1$ and for $-\infty < p < +\infty$, $F = F(p)$ is a real valued, bounded from below, locally integrable function. If $F(p)$ satisfies the celebrated Lévy–Khinchin formula, then the pertinent measure is positive and we may introduce positivity preserving semigroups, together with the induced (Markovian) stochastic processes. Let us concentrate our attention on the integral part of the Lévy–Khinchin formula, which is responsible for arbitrary stochastic jump features. In that case, $F(p)$ takes the form:

$$F(p) = -\int_{-\infty}^{+\infty} \left[\exp(ipy) - 1 - \frac{ipy}{1 + y^2}\right] \nu(dy)\,, \tag{20}$$

where $\nu(dy)$ stands for the so-called Lévy measure on R^1.

A generic feature of jump-type processes is that they admit jumps of arbitrarily small size (without any lower bound) and some care is necessary when evaluating contributions from a close neighbourhood of the origin. One

obvious way to bypass this problem amounts to an ε-cutoff which allows to neglect "too small" jumps. Namely, for a probability law with the characteristic exponent $-F(p)$, we can consider its restriction to upward jumps of size exceeding a given lower bound like *e.g.* to all $y > \varepsilon > 0$:

$$\phi_\varepsilon^+(p) = \int\limits_{y>\varepsilon} \left[\mathrm{e}^{ipy} - 1 - \frac{ipy}{1+y^2} \right] \nu(dy) = \int\limits_{y>\varepsilon} [\mathrm{e}^{ipy} - 1] d\nu(y) - ipb_\varepsilon^+ ,$$

$$b_\varepsilon^+ = \int\limits_{y>\varepsilon} \frac{y}{1+y^2} \nu(dy) . \tag{21}$$

Clearly, we deal here with a random variable of the type considered before, and we can try to isolate contributions from jumps of the size $[y, y + \triangle y)$ by coarse-graining a Borel set A of interest. A formal exploitation of

$$\nu(dy) = \sum_{j=1}^{n} \lambda_j \delta(y - y_j) dy \tag{22}$$

gives rise to

$$E[\exp(ipX)] = \exp \left[\sum_{j=1}^{n} \left[\lambda_j (\mathrm{e}^{ipy_j} - 1) - ip \frac{\lambda_j y_j}{1+y_j^2} \right] \right] . \tag{23}$$

Further specializing the problem we shall consider Lévy measures that obey the spatial reflection property $\nu(-dy) = -\nu(dy)$. Then, we can readily extend our discussion to jumps of all sorts in R^1, *i.e.* y can take values in both R_+ and R_-, with the only restriction to be observed that $|y| > \varepsilon > 0$. Notice that we shall deal with two distinct types of jumps, either positive or negative, with no common jump point for them. This fact means that they are independent components of the more general random variable:

$$\phi_\varepsilon(p) = \int\limits_{|y|>\varepsilon} [\mathrm{e}^{ipy} - 1] \nu(dy) - ipb_\varepsilon , \tag{24}$$

where (choose $\nu(dy) = \frac{dy}{\pi y^2}$ for concreteness) the deterministic term identically vanishes in view of

$$b_\varepsilon = b_\varepsilon^+ + b_\varepsilon^- = \int\limits_{y>\varepsilon} \frac{y}{1+y^2} \nu(dy) + \int\limits_{y<-\varepsilon} \frac{y}{1+y^2} \nu(dy) \equiv 0 . \tag{25}$$

In previous steps we have indirectly exploited a defining property of infinitely divisible probability laws: if $\exp\phi(p)$ is a characteristic function of a given probability distribution, then $[\exp\phi(p)]^t = \exp[t\phi(p)],\ t>0$ is likewise a characteristic function of an infinitely divisible probability law again. This feature extends our discussion to the stochastic jump and jump-type processes (time homogeneous with independent increments). Obviously, for such processes $E[\exp(ipX(t))] = \exp[t\phi(p)]$ while $E_A[X(t)] = t\nu(A)$, and our previous arguments retain their validity with respect to

$$E[\exp(ipX(t))]_\varepsilon = \exp[t\phi_\varepsilon(p)] = \exp\left[t\int\limits_{|y|>\varepsilon}\left(\mathrm{e}^{ipy}-1\right)\nu(dy)\right]. \tag{26}$$

Coming back to the Lévy–Khinchin formula, let us replace a function $F(p)$ by an operator acting in a suitable domain according to the recipe: $F(p)\to\hat{H}=F(\hat{p})$ where $\hat{p}=-i\nabla$. We easily learn that for times $t\geq 0$ there holds

$$[\exp(-t\hat{H})]f(x) = [\exp(-tF(p))\hat{f}(p)]^\vee(x)\,, \tag{27}$$

where the superscript $\vee$ denotes the inverse Fourier transform and $\hat{f}$ stands for the Fourier transform of a function f.

If we set $p_t = \frac{1}{\sqrt{2\pi}}[\exp(-tF(p))]^\vee$, then the action of $\exp(-t\hat{H})$ can be given in terms of a convolution: $\exp(-t\hat{H})f = f * p_t$, where $(f*g)(x) := \int\limits_R g(x-z)f(z)dz$. Clearly, there holds:

$$\begin{aligned}\partial_t\rho(x,t) &= -(\hat{H}\rho)(x,t) \Longrightarrow\\ \partial\rho_\varepsilon(A,t) &= \int\limits_R dx\left[\int\limits_{|y|>\varepsilon}[\chi_A(x+y)-\chi_A(x)]\nu(dy)\right]\rho_\varepsilon(x,t)\end{aligned} \tag{28}$$

which displays a generic master equation form. Indeed, we have here $\partial_t\rho_\varepsilon(A,t) = \int\limits_R q_\varepsilon(x,t,A)\rho_\varepsilon(x,t)dx$ where q_ε is interpreted as the jump intensity.

Let us however emphasize that the above simplification occurs only in the $|y|>\varepsilon>0$ jumping size regime. The real rôle of two integral terms in the expression for b_ε is to compensate the divergent contributions from the Lévy measure when the principal value integral $\varepsilon\to 0$ limit is considered; then the *standard* jump process theory does not literally apply since arbitrarily small jumps are admitted. Anyway, those two terms are irrelevant if we assume an $\varepsilon>0$ cutoff, irrespective of how small the chosen (and fixed) ε is.

The best known example of the stable probability law that is compatible with the above definitions is provided by the classic Cauchy density which

will be our reference probability law in below. Let us focus our attention on the case which is specified by $F(p) = |p|$. The corresponding semigroup generator $\hat{H} = |\nabla|$ is a pseudodifferential operator. The associated kernel p_t is a transition density of the jump-type Cauchy process, which is a solution of a pseudodifferential Fokker–Planck equation:

$$\partial_t \rho(x,t) = -|\nabla|\rho(x,t). \tag{29}$$

The pertinent probability density reads $\rho(x,t) = \frac{1}{\pi}\frac{t}{t^2+x^2}$ and the corresponding space-time homogeneous transition density (*e.g.* the semigroup kernel function) is:

$$\rho(x,t) = \frac{1}{\pi}\frac{t}{t^2+x^2} \Longrightarrow p(y,s,x,t) = \frac{1}{\pi}\frac{t-s}{(t-s)^2+(x-y)^2}\,, \quad 0<s<t$$

$$\langle \exp[ipX(t)] \rangle := \int\limits_R \exp(ipx)\rho(x,t)dx = \exp(-|p|t)\,. \tag{30}$$

The characteristic function of $p(y,s,x,t)$ for y,s fixed, reads $\exp[ipy - |p|(t-s)]$, and the Lévy measure needed to evaluate the Lévy–Khinchin integral reads: $\nu(dy) := \lim_{t\downarrow 0}[\frac{1}{t}p(0,0,y,t)]dy = \frac{dy}{\pi y^2}$.

The Cauchy process belongs to the category of jump-type processes, where apart from the long jumps-tail (no fixed bound can be imposed on their length) which is the reason of the nonexistence of moments of the probability measure, sample paths of the process may have an infinite number of jumps of arbitrarily small size. By general arguments, pertaining to the space $D_E[0,\infty)$ of right continuous functions with left limits (cadlag), both in the finite and infinite time interval the number of jumps is at most countable.

An introduction of the ε-cutoff which eliminates small jumps is fed up by the physical intuition. An approximation of the jump-type process should in principle be possible in terms of more traditional jump processes which involve a finite number of jumps in a finite time interval. This radical approximation assumption is usually achieved by giving a characterisation of the affiliated Markovian jump-type processes in terms of approximating (convergent) families of so-called *step* processes. The step processes are not yet the jump processes of the standard daily experimental evidence. They need to have no accumulation points of jumps in a finite time interval: in that case the number of jumps is finite on each finite time interval and between jumps the sample path is constant.

Let us recall that the operator $|\nabla|$ acts as follows:

$$|\nabla|f(x) = -\frac{1}{\pi}\int\limits_R \left[f(x+y) - f(x) - \frac{y\nabla f(x)}{1+y^2}\right]\frac{dy}{y^2}\,. \tag{31}$$

By turning back to the pseudodifferential Fokker–Planck equation with the ε-cutoff implemented, let us introduce an operator $|\nabla|_\varepsilon$:

$$|\nabla|_\varepsilon f(x) = -\frac{1}{\pi} \int\limits_{|y|>\varepsilon} [f(x+y) - f(x)] \frac{dy}{y^2}. \tag{32}$$

It suffices to replace f by ρ_ε to arrive at the right-hand-side of the previously defined Fokker–Planck equation for the step process approximant of the Cauchy process. Its generator is just $|\nabla|_\varepsilon$.

While mitigating the "arbitrarily small jumps" problem, the ε-cutoff does not remove all obstacles related to the Cauchy process. Indeed, at the first glance the situation looks deceivingly simple, because on a finite time interval there can be at most finitely many points $t \in [0,T]$ at which the jump size exceeds a given positive number. In view of that, $\sup_{t\in[0,T]}|X_t^\varepsilon| < \infty$ where X_t^ε stands for the ε-bounded Cauchy process (the same argument extends to the unrestricted Cauchy process X_t). However, there must be *no fixed* upper bound for the size of jumps (except for being finite), since a stochastically continuous process with independent increments having, with probability 1, no jumps of size exceeding a certain constant δ, would possess all moments. That is certainly not the case for the Cauchy process, which is known not to have any moments.

Hence imposing or not imposing an upper bound on the jump size (call it a δ cutoff) is another critical issue that hampers a reliable approximation of the jump — type process in terms of experimentally verifiable jump processes whose jump size is bounded both from below and from above. This derives in part from to the resolution limitations of realistic experimental arrangements (any experimental data collection and any computer simulation/experimentation have built-in lower and upper jump size bounds), and in part from the fact that all observations are carried in finite time on systems of finite spatial extension. Let us point out that by imposing both ε and δ cutoffs on the Cauchy process, we would reduce the problem to the standard jump process.

Obstacles related to heavy tails of the probability distribution can be visualized by making computer experiments for the converging (in fact, diverging) variance test. Namely, given a sample of jumps determined by the Cauchy distribution: $X_1, X_2, \ldots X_n$, with $1 \leq i \leq n$, one can form a statistics based on first i "observations" and ask for the behaviour of the averages with respect to i. First we need the i-th mean value: $\overline{X} = \frac{1}{i}\sum_{k=1}^{i} X_i$ and next the i-th variance: $S_i^2 = \frac{1}{i-1}\sum_{k=1}^{i}(X_k - \overline{X}_i)^2$. Plots of $\overline{X}_i$ and S_i^2 against i would show up a fairly irregular behaviour and definite *non-convergence* signatures for large values of $i \leq n$.

3. Local characteristics of the Ornstein–Uhlenbeck–Cauchy process

Although we directly refer to the specific Cauchy stochastic process, in fact we stay in a much broader setting of so-called Lévy flights. Generically, the variance and higher cumulants of those processes are infinite (nonexistent). There is also physically more singular subclass of such processes for which even the first moment (mean value) is nonexistent. That is true for the Cauchy process. Thus we need to relax the limitations of the standard Gaussian paradigm: we face here a fundamental problem of establishing other means (than variances and mean values) to characterize statistical properties of Lévy processes.

Specifically, if a usual statistical analysis is performed on any experimentally available set of frequency data, there is no obvious method to extract a reliable information about tendencies (local mean values) of the random dynamics. Nonexistence of mean values and higher moments may also be interpreted as the nonexistence of observable (*e.g.* mean, like drifts or local currents) regularities of the dynamics. Moreover, as we have learned before, the jump-type processes usually admit arbitrarily small jumps (with no lower bound) and finite, but arbitrarily large jump sizes (with no upper bound). Any laboratory experiment or computer simulation would involve both the lower (coarse-graining) and upper bound on the jump size. Mathematically, that sets (as suitable) the framework of standard jump processes for which the central limit theorem is known to hold true in its Gaussian version (even if we account for a possible abnormally slow convergence to a Gaussian). Therefore, there is no clear-cut procedures allowing to attribute an unambiguous statistical interpretation in terms of Lévy processes to given phenomenological data. Moreover, no realistic formulation of a fluctuation-dissipation theorem is possible in that case (nonexistence of variances) which pushes us away from any conceivable thermal equilibrium framework.

The starting point for Ornstein and Uhlenbeck (*cf.* the previous section) was the dissipative Langevin equation

$$\frac{dV}{dt} = -\lambda V(t) + A(t)\,, \tag{33}$$

where $V(t)$ is a random variable describing the velocity of a particle, $\lambda > 0$ is a dissipation constant, and $A(t)$ is another random variable whose probabilistic features are determined by the probability distribution of $V(t)$, which is assumed to satisfy a concrete law when $t \to \infty$. Because $V(t)$ may have no time derivative, the Langevin equation was soon replaced by another one, the stochastic differential equation, namely

$$dV(t) = -\lambda V(t)dt + dB(t)\,, \quad V(0) = v_0 \tag{34}$$

which received a rigorous interpretation within the framework of stochastic analysis. In the case when the probability distribution of $V(t)$, $t \to \infty$, is the Maxwell one, $B(t)$ must be a Gaussian (in fact Wiener, $B_t = W_t$) process, and then we end up with a classical Ornstein–Uhlenbeck process.

Now, we shall discuss properties of the process $V(t)$, in the case when $B = (B(t))_{t \geq 0}$ is the Cauchy process instead of the traditional Wiener one. By straightforward integration we obtain that for $t \geq s$

$$V(t) = \mathrm{e}^{-\lambda(t-s)} V(s) + \mathrm{e}^{-\lambda t} \int_s^t \mathrm{e}^{\lambda \tau} dB(\tau)\,. \tag{35}$$

There are a number of (equivalent) procedures to deduce a probability density of the process $V(t)$ from the Cauchy increments statistics. One may follow a direct probabilistic route which, upon assuming that a characteristic function of the Cauchy probability measure reads

$$E[\mathrm{e}^{ipB(t)}] = \mathrm{e}^{-t\psi(p)} \tag{36}$$

with the choice of $\psi(p) = \sigma^2 |p|$ (before, we have used $\sigma^2 = 1$), leads to the transition density:

$$p_{t-s}(u,v) = P[V(t) = u | V(s) = v] = \frac{1}{\pi} \frac{\sigma^2(t-s)}{(u - v\mathrm{e}^{-\lambda(t-s)})^2 + \sigma^4(t-s)}\,, \tag{37}$$

where $\sigma^2(t-s) = \frac{\sigma^2}{\lambda}(1 - \mathrm{e}^{-\lambda(t-s)})$.

Since $V(0) = v_0$, the probability density of $V(t)$ is given by

$$P[V(t) = v] = \frac{1}{\pi} \frac{\sigma^2(t)}{(v - v_0 \mathrm{e}^{-\lambda t})^2 + \sigma^4(t)}\,.$$

Now, we shall demonstrate an important property (mentioned before in connection with the Ornstein–Uhlenbeck process) of the so-called stochastic continuity, which is a necessary condition to give a stochastic process an unambiguous status. Namely, we need to show that for any $\varepsilon > 0$ the following equation is satisfied

$$\lim_{t \to s} P[|V(t) - V(s)| \geq \varepsilon] = 0\,.$$

This equation is equivalent to

$$\lim_{t \to 0} \int_{|u-v| \geq \varepsilon} p_t(u,v) du = 0\,.$$

Because of

$$\int\limits_{|u-v|\geq\varepsilon} p_t(u|v)du = 1 - \frac{1}{\pi}\left[\arctan\frac{\varepsilon + v(1-\mathrm{e}^{-\lambda t})}{\sigma^2(t)} + \arctan\frac{\varepsilon - v(1-\mathrm{e}^{-\lambda t})}{\sigma^2(t)}\right]$$

and remembering that $\sigma^2(t) = \frac{\sigma^2}{\lambda}(1-\mathrm{e}^{-\lambda t})$, the stochastic continuity property does follow.

The nonexistence of moments of the probability measure in case of the Cauchy process leads to straightforward difficulties, since the standard local characteristics of the diffusion-type process like the drift and the diffusion function (or coefficient) seem to be excluded in the present case. However, for the considered Ornstein–Uhlenbeck–Cauchy process, the notion of the forward drift of the process proves to make sense.

Since we know the Markov transition function $p_{t-s}(u,v)$, $t \geq s$, for the process V_t, we can exploit our experience with diffusion processes and say that the process V_t has a drift (in fact, forward drift) if the following limit

$$\lim_{t\to s}\frac{1}{t-s}\int\limits_{|u-v|\leq\delta}(u-v)p_{t-s}(u,v)du \tag{38}$$

does not depend on the choice of $\delta > 0$. If so, then its value depending only on (v,s) we denote by $b(v,s)$ and call it the drift coefficient. Clearly, if p is homogeneous in time, then the drift coefficient depends only on the variable v. Let us emphasize that in the above definition we do not require the process V_t to have finite moments.

We claim that the jump-type Markov process $V(t)$ has a (forward) drift which reads $b(v) = -\lambda v$. Indeed, by first evaluating the indefinite integral

$$I = \frac{1}{\pi}\int (u-v)\frac{\sigma^2(t)du}{(u-v\mathrm{e}^{-\lambda t})^2 + \sigma^4(t)}$$

and substituting $z = u - v\mathrm{e}^{-\lambda t}$, we get

$$\begin{aligned}\frac{\sigma^2(t)}{\pi}\quad &\int\frac{zdz}{z^2+\sigma^4(t)} + \frac{v}{\pi}\left(\mathrm{e}^{-\lambda t}-1\right)\int\frac{\sigma^2(t)dz}{z^2+\sigma^4(t)}\\ &= \frac{\sigma^2(t)}{2\pi}\log(z^2+\sigma^4(t)) + \frac{v}{\pi}\left(\mathrm{e}^{-\lambda t}-1\right)\arctan\left(\frac{z}{\sigma^2(t)}\right).\end{aligned}$$

Hence

$$I = \frac{\sigma^2(t)}{2\pi}\log\left[\left(u-v\mathrm{e}^{-\lambda t}\right)^2 + \sigma^4(t)\right] + \frac{v}{\pi}\left(\mathrm{e}^{-\lambda t}-1\right)\arctan\left[\frac{u-v\mathrm{e}^{-\lambda t}}{\sigma^2(t)}\right]$$

and consequently the limit

$$\lim_{t\to 0}\frac{1}{t}I\Big|_{u=v-\varepsilon}^{u=v+\varepsilon} = \lim_{t\to 0}\frac{1}{t}\frac{\sigma^2(t)}{2\pi}$$
$$\times\left(\log[(v+\varepsilon-v\mathrm{e}^{-\lambda t})^2+\sigma^4(t)]-\log[(v-\varepsilon-v\mathrm{e}^{-\lambda t})^2+\sigma^4(t)]\right)$$
$$+\lim_{t\to 0}\frac{1}{t}\frac{v}{\pi}(\mathrm{e}^{-\lambda t}-1)\left(\arctan\left[\frac{v+\varepsilon-v\mathrm{e}^{-\lambda t}}{\sigma^2(t)}\right]-\arctan\left[\frac{v-\varepsilon-v\mathrm{e}^{-\lambda t}}{\sigma^2(t)}\right]\right)$$
$$=0-\lambda\frac{v}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=-\lambda v$$

exists and is ε-independent. This is the forward drift of the process $V(t)$ which proves a consistency of the derived transition probability density with the stochastic differential equation for the process $V_t = V(t)$.

It is well known that for Markovian diffusion processes all local characteristics of motion (conditional expectation values that yield drifts and variances) can be derived from transition probability densities, supplemented (if needed) by the density of the process. We have demonstrated that, in the non-Gaussian context, the nonexistence of moments does not necessarily imply the nonexistence of local characteristics (drifts) of the process.

However, the situation becomes uncomfortable once we attempt to evaluate another local moment. Namely, in the present case there holds:

$$\lim_{t\to s^+}\frac{1}{t-s}\int\limits_{|v-u|\le\delta}(u-v)^2 p_t(u,v)du=\frac{2\sigma^2}{\pi}\delta \tag{39}$$

i.e. an explicit cutoff δ (upper bound on the size of jumps) persists in this formula and there is no way to remove that jump size restriction from the formalism, unless we wish to get the divergent integral.

This property is a clear indication that a convergence to a Gaussian might always be expected if the Ornstein–Uhlenbeck–Cauchy process is approximated (we disregard an issue of how good that approximation is) by means of jump processes with an upper and lower bound on the jump size. In that case both the mean and variance would exist for the approximating process. In particular, the central limit theorem would work as usual for the (ε,δ)-jump process approximation of the Cauchy process B_t.

The author receives financial support from the the Polish State Committee for Scientific Research (KBN) grant No. 2 P03B 086 16.

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