CHARGED BROWNIAN PARTICLE IN A MAGNETIC FIELD*

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(Received October 30, 2000)

We derive explicit forms of Markovian transition probability densities for the velocity space and phase-space Brownian motion of a charged particle in a constant magnetic field.

PACS numbers: 02.50.Ga

1. Motivation

An old-fashioned problem of the Brownian motion of a charged particle in a constant magnetic field has originated from studies of the diffusion of plasma across a magnetic field [1, 2] and nowadays, together with a free Brownian motion example, stands for a textbook illustration of how transport and auto-correlation functions should be computed in generic situations governed by the Langevin equation, *cf.* [3] but also [4, 5]. To our knowledge, except for the paper [2], no attempt was made in the literature to give a complete characterization of the pertinent stochastic process. However a striking peculiarity of Ref. [2] is that the Brownian motion in a magnetic field is there described in terms of *operator-valued* probability distributions that involve fractional powers of matrices. In consequence, we have no clean relationship with the standard formalism of Kramers–Smoluchowski equations, nor ways to stay in conformity with the standard wisdom about probabilistic procedures valid in case of the free Brownian

* Presented at the XIII Marian Smoluchowski Symposium on Statistical Physics, Zakopane, Poland, September 10–17, 2000.

motion (Ornstein–Uhlenbeck process), *cf.* [6–8]. Therefore, we address an issue of the Brownian motion of a charged particle in a magnetic field anew, to unravel its features of a fully-fledged stochastic diffusion process.

2. Velocity-space diffusion process

The standard analysis of the Brownian motion of a free particle employs the Langevin equation $\frac{d\vec{u}}{dt} = -\beta\vec{u} + \vec{A}(t)$, where $\vec{u}$ denotes the velocity of the particle and the influence of the surrounding medium on the motion (random acceleration) of the particle is modelled by means of two independent contributions. A systematic part $-\beta\vec{u}$ represents a dynamical friction. The remaining fluctuating part $\vec{A}(t)$ is supposed to display a statistics of the familiar white noise: *(i)* $\vec{A}(t)$ is independent of $\vec{u}$, *(ii)* $\langle A_i(s)\rangle = 0$ and $\langle A_i(s)\,A_j(s')\rangle = 2q\delta_{ij}\delta(s-s')$ for $i,j = 1,2,3$, where $q = \frac{k_{\rm B}T}{m}\beta$ is a physical parameter. The Ornstein–Uhlenbeck stochastic process comes out on that conceptual basis.

The linear friction model can be adopted to the case of diffusion of charged particles in the presence of a constant magnetic field which acts upon particles via the Lorentz force. The Langevin equation for that motion reads:

$$\frac{d\vec{u}}{dt} = -\beta\vec{u} + \frac{q_e}{mc}\vec{u}\times\vec{B} + \vec{A}(t)\,, \tag{1}$$

where q_e denotes an electric charge of the particle of mass m.

Let us assume for simplicity that the constant magnetic field $\vec{B}$ is directed along the z-axis of a Cartesian reference frame: $\vec{B} = (0,0,B)$ and $B = \text{const}$. In this case Eq. (1) takes the form

$$\frac{d\vec{u}}{dt} = -\Lambda\vec{u} + \vec{A}(t)\,, \tag{2}$$

where

$$\Lambda = \begin{pmatrix} \beta & -\omega_c & 0 \\ \omega_c & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}, \tag{3}$$

and $\omega_c = \frac{q_e B}{mc}$ denotes the Larmor frequency. Assuming the Langevin equation to be (at least formally) solvable, we can infer a probability density $P(\vec{u},t|\vec{u}_0)$, $t>0$, conditioned by the the initial velocity data choice $\vec{u} = \vec{u}_0$ at $t = 0$. Physical circumstances of the problem enforce a demand:

(i) $P(\vec{u},t|\vec{u}_0) \to \delta^3(\vec{u}-\vec{u}_0)$ as $t \to 0$ and

(ii) $P(\vec{u},t|\vec{u}_0) \to \left(\frac{m}{2\pi k_{\rm B}T}\right)^{3/2} \exp\left(-\frac{m|\vec{u}_0|^2}{2k_{\rm B}T}\right)$ as $t\to\infty$.

A formal solution of Eq. (2) reads:

$$\vec{u}\,(t) - \mathrm{e}^{-\Lambda t}\vec{u}_0 = \int_0^t \mathrm{e}^{-\Lambda(t-s)}\vec{A}\,(s)\,ds\,. \tag{4}$$

By taking into account that

$$\mathrm{e}^{-\Lambda t} = \mathrm{e}^{-\beta t}\begin{pmatrix} \cos\omega_c t & \sin\omega_c t & 0 \\ -\sin\omega_c t & \cos\omega_c t & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{e}^{-\beta t}U\,(t)\,, \tag{5}$$

we can rewrite (4) as follows

$$\vec{u}\,(t) - \mathrm{e}^{-\beta t}U\,(t)\,\vec{u}_0 = \int_0^t \mathrm{e}^{-\beta(t-s)}U\,(t-s)\,\vec{A}\,(s)\,ds\,. \tag{6}$$

Statistical properties of $\vec{u}\,(t) - \mathrm{e}^{-\Lambda t}\vec{u}_0$ are identical with those of $\vec{A}\,(s)\,ds$. In consequence, the problem of deducing a probability density $P\,(\vec{u}, t|\vec{u}_0)$ is equivalent to deriving the probability distribution of the random vector

$$\vec{S} = \int_0^t \psi\,(s)\,\vec{A}\,(s)\,ds\,, \tag{7}$$

where $\psi\,(s) = \mathrm{e}^{-\Lambda(t-s)} = \mathrm{e}^{-\beta(t-s)}U\,(t-s)$.

The white noise term $\vec{A}\,(s)$ in view of the integration with respect to time is amenable to a more rigorous analysis that invokes the Wiener process increments and their statistics, [9]. Let us divide the time integration interval into a large number of small subintervals Δt. We adjust them suitably to assure that $\psi\,(t)$ is effectively constant on each subinterval $(j\Delta t, (j+1)\,\Delta t)$ and equal $\psi\,(j\Delta t)$. As a result we obtain the expression

$$\vec{S} = \sum_{j=0}^{N-1} \psi\,(j\Delta t) \int_{j\Delta t}^{(j+1)\Delta t} \vec{A}\,(s)\,ds\,. \tag{8}$$

Here $\vec{B}\,(\Delta t) = \int_{j\Delta t}^{(j+1)\Delta t} \vec{A}\,(s)\,ds$ stands for the above-mentioned Wiener process increment. Physically, $\vec{B}\,(\Delta t)$ represents the *net* acceleration which a Brownian particle may suffer (in fact accumulates) during an interval of time Δt. Equation (8) becomes

$$\vec{S} = \sum_{j=0}^{N-1} \psi\left(j\Delta t\right) \vec{B}\left(\Delta t\right) = \sum_{j=0}^{N-1} \vec{s}_j \, , \tag{9}$$

where we introduce $\vec{s}_j = \psi\left(j\Delta t\right) \vec{B}\left(\Delta t\right) = \psi_j \vec{B}\left(\Delta t\right)$.

The Wiener process argument [7, 8] allows us to infer the probability distribution of $\vec{s}_j$. It is enough to employ the fact that the distribution of $\vec{B}\left(\Delta t\right)$ is Gaussian with mean zero and variance $q = \frac{k_{\mathrm{B}}T}{m}\beta$. Then

$$w\left[\vec{B}\left(\Delta t\right)\right] = \left(\frac{1}{4\pi q\Delta t}\right)^{3/2} \exp\left(-\frac{\left|\vec{B}\left(\Delta t\right)\right|^2}{4q\Delta t}\right) \tag{10}$$

and in view of $\vec{s}_j = \psi_j \vec{B}\left(\Delta t\right)$ by performing the change of variables in (10) we get

$$\widetilde{w}\left[\vec{s}_j\right] = \det\left[\psi_j^{-1}\right] w\left[\psi_j^{-1}\vec{s}_j\right] = \frac{1}{\det\psi_j} w\left[\psi_j^{-1}\vec{s}_j\right] \, . \tag{11}$$

Since $\det\psi\left(s\right) = \mathrm{e}^{-3\beta(t-s)}$ and $\psi^{-1}\left(s\right) = U\left[-\left(t-s\right)\right]\mathrm{e}^{\beta(t-s)}$ we obtain

$$\widetilde{w}\left[\vec{s}_j\right] = \left(\frac{1}{4\pi q\Delta t}\right)^{3/2} \frac{1}{\mathrm{e}^{-3\beta(t-j\Delta t)}} \exp\left(-\frac{\left|\mathrm{e}^{\beta(t-j\Delta t)}U\left[-\left(t-j\Delta t\right)\right]\vec{s}_j\right|^2}{4q\Delta t}\right) \tag{12}$$

and finally

$$\widetilde{w}\left[\vec{s}_j\right] = \left(\frac{1}{4\pi q\Delta t}\frac{1}{\mathrm{e}^{-2\beta(t-j\Delta t)}}\right)^{3/2} \exp\left(-\frac{\left|\vec{s}_j\right|^2}{4q\Delta t\mathrm{e}^{-2\beta(t-j\Delta t)}}\right) \, . \tag{13}$$

Clearly, $\vec{s}_j$ are mutually independent random variables whose distribution is Gaussian with mean zero and variance $\sigma_j^2 = 2q\Delta t\mathrm{e}^{-2\beta(t-j\Delta t)}$. Hence, the probability distribution of $\vec{S} = \sum_{j=0}^{N-1}\vec{s}_j$ is again Gaussian with mean zero. Its variance equals the sum of variances of $\vec{s}_j$ *i.e.* $\sigma^2 = \sum_j \sigma_j^2 = 2q\sum_j \Delta t\mathrm{e}^{-2\beta(t-j\Delta t)}$. After taking the limit $N \to \infty$ $(\Delta t \to 0)$ we arrive at

$$\sigma^2 = 2q\int_0^t ds\mathrm{e}^{-2\beta(t-s)} = \frac{k_{\mathrm{B}}T}{m}\left(1-\mathrm{e}^{-2\beta t}\right) \, . \tag{14}$$

Because of $\vec{S} = \vec{u}\,(t) - \mathrm{e}^{-\Lambda t}\vec{u}_0$ the transition probability density of the Brownian particle velocity, conditioned by the initial data $\vec{u}_0$ at $t_0 = 0$ reads

$$P\left(\vec{u}, t|\vec{u}_0\right) = \left(\frac{1}{2\pi\frac{k_{\mathrm{B}}T}{m}\left(1-\mathrm{e}^{-2\beta t}\right)}\right)^{3/2} \exp\left(-\frac{\left|\vec{u}-\mathrm{e}^{-\Lambda t}\vec{u}_0\right|^2}{2\frac{k_{\mathrm{B}}T}{m}\left(1-\mathrm{e}^{-2\beta t}\right)}\right). \quad (15)$$

The process is Markovian and time-homogeneous, hence the above formula can be trivially extended to encompass the case of arbitrary $t_0 \neq 0$: $P\left(\vec{u}, t|\vec{u}_0, t_0\right)$ arises by substituting everywhere $t - t_0$ instead of t.

Physical arguments (*cf.* demand *(ii)* preceding Eq. (4)) refer to an asymptotic probability distribution (invariant measure density) $P(u)$ of the random variable $\vec{u}$ in the Maxwell–Boltzmann form

$$P\left(\vec{u}\right) = \left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2} \exp\left(-\frac{m\left|\vec{u}\right|^2}{2k_{\mathrm{B}}T}\right). \quad (16)$$

This time-independent probability density together with the time-homogeneous transition density (15) uniquely determine a stationary Markovian stochastic process for which we can evaluate various mean values. Expectation values of velocity components vanish: $\langle u_i\,(t)\rangle = \int_{-\infty}^{\infty} u_i P\left(\vec{u}\right) d\vec{u} = 0$ for $i = 1, 2, 3$. The matrix of the second moments (velocity auto-correlation functions) reads

$$\langle u_i\,(t)\, u_j\,(t_0)\rangle = \int\limits_{-\infty}^{\infty} u_i u_j^0 P\left(\vec{u}, t; \vec{u}_0, t_0\right) d\vec{u}\, d\vec{u}_0\,, \quad (17)$$

where $i, j = 1, 2, 3$ and in view of $P\left(\vec{u}, t; \vec{u}_0, t_0\right) = P\left(\vec{u}, t|\vec{u}_0, t_0\right) P\left(\vec{u}_0\right)$ we arrive at the compact expression

$$\frac{k_{\mathrm{B}}T}{m}\mathrm{e}^{-\Lambda|t-t_0|} = \frac{k_{\mathrm{B}}T}{m}\mathrm{e}^{-\beta|t-t_0|}\begin{pmatrix} \cos\omega_c\,|t-t_0| & \sin\omega_c\,|t-t_0| & 0 \\ -\sin\omega_c\,|t-t_0| & \cos\omega_c\,|t-t_0| & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (18)$$

In particular, the auto-correlation function (second moment) of the x-component of velocity equals

$$\langle u_1\,(t)\, u_1\,(t_0)\rangle = \frac{k_{\mathrm{B}}T}{m}\mathrm{e}^{-\beta|t-t_0|}\cos\omega_c\,|t-t_0| \quad (19)$$

in agreement with white noise calculations of Refs. [1] and [3], *cf.* Chap. 11, formula (11.25). The so-called running diffusion coefficient arises here via straightforward integration of the function $R_{11}(\tau) = \langle u_1(t)u_1(t_0)\rangle$, where $\tau = t - t_0 > 0$:

$$D_1(t) = \int_0^t \langle u_1(0)u_1(\tau)\rangle d\tau = \frac{k_{\mathrm{B}}T}{m}\frac{\beta + [\omega_c \sin(\omega_c t) - \beta\cos(\omega_c t)]\exp(-\beta t)}{\beta^2 + {\omega_c}^2} \tag{20}$$

with an obvious asymptotics (the same for $D_2(t)$): $D_B = \lim_{t\to\infty} D_1(t) = \frac{k_{\mathrm{B}}T}{m}\frac{\beta}{\beta^2+{\omega_c}^2}$ and the large friction (ω_c fixed and bounded) version $D = \frac{k_{\mathrm{B}}T}{m\beta}$.

3. Spatial process — dynamics in the plane

The cylindrical symmetry of the problem allows us to consider separately processes running on the XY plane and along the Z-axis (where the free Brownian motion takes place). We shall confine further attention to the two-dimensional XY-plane problem. Henceforth, each vector will carry two components which correspond to the x and y coordinates respectively. We will directly refer to the vector and matrix quantities introduced in the previous section, but while keeping the same notation, we shall simply disregard their z-coordinate contributions.

We define the spatial displacement $\vec{r}$ of the Brownian particle as follows

$$\vec{r} - \vec{r}_0 = \int_0^t \vec{u}\,(\eta)\, dn\,, \tag{21}$$

where $\vec{u}\,(t)$ is given by Eq. (2) (except for disregarding the third coordinate).

Our aim is to derive the probability distribution of $\vec{r}$ at time t provided that the particle position and velocity were equal $\vec{r}_0$ and $\vec{u}_0$ respectively, at time $t_0 = 0$. To that end we shall mimic procedures of the previous section. In view of:

$$\vec{r} - \vec{r}_0 - \int_0^t \mathrm{e}^{-\Lambda\eta}\vec{u}_0 = \int_0^t d\eta \int_0^\eta ds \mathrm{e}^{-\Lambda(\eta-s)}\vec{A}\,(s)\ , \tag{22}$$

we have

$$\vec{r} - \vec{r}_0 - \Lambda^{-1}\left(1 - \mathrm{e}^{-\Lambda t}\right)\vec{u}_0 = \int_0^t \Lambda^{-1}\left(1 - \mathrm{e}^{\Lambda(s-t)}\right)\vec{A}\,(s)\, ds\,, \tag{23}$$

where

$$\Lambda^{-1} = \frac{1}{\beta^2 + \omega_c^2}\begin{pmatrix} \beta & \omega_c \\ -\omega_c & \beta \end{pmatrix} \tag{24}$$

is the inverse of the matrix Λ (regarded as a rank two sub-matrix of that originally introduced in Eq. (3)). Let us define two auxiliary matrices

$$\begin{aligned} \Omega &\equiv \Lambda^{-1}\left(1-\mathrm{e}^{-\Lambda t}\right) , \\ \phi(s) &\equiv \Lambda^{-1}\left(1-\mathrm{e}^{\Lambda(s-t)}\right) . \end{aligned} \tag{25}$$

Because of:

$$\mathrm{e}^{-\Lambda t} = \exp\left\{-t\begin{pmatrix}\beta & -\omega_c \\ \omega_c & \beta\end{pmatrix}\right\} = \mathrm{e}^{-\beta t}\begin{pmatrix}\cos\omega_c t & \sin\omega_c t \\ -\sin\omega_c t & \cos\omega_c t\end{pmatrix} = \mathrm{e}^{-\beta t} U(t) , \tag{26}$$

we can represent matrices Ω, $\phi(s)$ in more detailed form. We have:

$$\Omega = \frac{1}{\beta^2+\omega_c^2}\left\{\begin{pmatrix}\beta & \omega_c \\ -\omega_c & \beta\end{pmatrix} - \mathrm{e}^{-\beta t}\begin{pmatrix}\beta & \omega_c \\ -\omega_c & \beta\end{pmatrix}\begin{pmatrix}\cos\omega_c t & \sin\omega_c t \\ -\sin\omega_c t & \cos\omega_c t\end{pmatrix}\right\} \tag{27}$$

and

$$\begin{aligned} \phi(s) &= \Lambda^{-1}\left(1-\mathrm{e}^{-\beta(t-s)}U(t-s)\right) \\ &= \frac{1}{\beta^2+\omega_c^2}\begin{pmatrix}\beta & \omega_c \\ -\omega_c & \beta\end{pmatrix}\begin{pmatrix}1-\mathrm{e}^{\beta(s-t)}\cos\omega_c(s-t) & -\mathrm{e}^{\beta(s-t)}\sin\omega_c(s-t) \\ \mathrm{e}^{\beta(s-t)}\sin\omega_c(s-t) & 1-\mathrm{e}^{\beta(s-t)}\cos\omega_c(s-t)\end{pmatrix}. \end{aligned} \tag{28}$$

Next steps imitate procedures of the previous section. Thus, we seek for the probability distribution of the random (planar) vector

$$\vec{R} = \int_0^t \phi(s)\,\vec{A}(s)\,ds ,$$

where $\vec{R} = \vec{r} - \vec{r}_0 - \Omega\vec{u}_0$.

Dividing the time interval $(0,t)$ into small subintervals to assure that $\phi(s)$ can be regarded constant over the time span $(j\Delta t,(j+1)\Delta t)$ and equal $\phi(j\Delta t)$, we obtain

$$\vec{R} = \sum_{j=0}^{N-1}\phi(j\Delta t)\int_{j\Delta t}^{(j+1)\Delta t}\vec{A}(s)\,ds = \sum_{j=0}^{N-1}\phi(j\Delta t)\,\vec{B}(\Delta t) = \sum_{j=0}^{N-1}\vec{r}_j , \tag{29}$$

where $\vec{r}_j = \phi(j\Delta t)\,\vec{B}(\Delta t) = \phi_j\vec{B}(\Delta t)$.

By invoking the probability distribution (10) we perform an appropriate change of variables: $\vec{r}_j = \phi_j \vec{B}\left(\Delta t\right)$ to yield a probability distribution of $\vec{r}_j$

$$\widetilde{w}\left[\vec{r}_j\right] = \det\left[\phi_j^{-1}\right] w\left[\phi_j^{-1}\vec{r}_j\right] = \frac{1}{\det\phi_j} w\left[\phi_j^{-1}\vec{r}_j\right] . \tag{30}$$

Presently (not to be confused with previous steps (11)–(15)) we have

$$\det\phi\left(s\right) = \frac{1}{\beta^2+\omega_c^2}\left(1+\mathrm{e}^{2\beta(s-t)} - 2\mathrm{e}^{\beta(s-t)}\cos\omega_c\left(s-t\right)\right) \tag{31}$$

and

$$\phi^{-1}\left(s\right) = \frac{1}{1+\mathrm{e}^{2\beta(s-t)} - 2\mathrm{e}^{\beta(s-t)}\cos\omega_c\left(s-t\right)}\left[1-\mathrm{e}^{\beta(s-t)}U\left(-\left(s-t\right)\right)\right]\Lambda . \tag{32}$$

So, the inverse of the matrix ϕ_j has the form:

$$\phi_j^{-1} = \frac{\widetilde{A}_j}{\gamma_j} , \tag{33}$$

where

$$\widetilde{A}_j = \begin{pmatrix} 1-\mathrm{e}^{\beta(j\Delta t-t)}\cos\omega_c\left(j\Delta t-t\right) & \mathrm{e}^{\beta(j\Delta t-t)}\sin\omega_c\left(j\Delta t-t\right) \\ -\mathrm{e}^{\beta(j\Delta t-t)}\sin\omega_c\left(j\Delta t-t\right) & 1-\mathrm{e}^{\beta(j\Delta t-t)}\cos\omega_c\left(j\Delta t-t\right) \end{pmatrix} \times\begin{pmatrix} \beta & -\omega_c \\ \omega_c & \beta \end{pmatrix} \tag{34}$$

and

$$\gamma_j = 1+\mathrm{e}^{2\beta(j\Delta t-t)} - 2\mathrm{e}^{\beta(j\Delta t-t)}\cos\omega_c\left(j\Delta t-t\right) . \tag{35}$$

There holds:

$$\det\phi_j^{-1} = \left(\det\phi_j\right)^{-1} = \left(\beta^2+\omega_c^2\right)\frac{1}{\gamma_j} \tag{36}$$

and as a consequence we arrive at the following probability distribution of $\vec{r}_j$

$$\widetilde{w}\left[\vec{r}_j\right] = \frac{1}{\frac{1}{\beta^2+\omega_c^2}\gamma_j}\left(\frac{1}{4\pi q\Delta t}\right)\exp\left\{\frac{\left|\widetilde{A}_j\begin{pmatrix} r_j^x \\ r_j^y\end{pmatrix}\right|^2}{\gamma_j^2 4q\Delta t}\right\} . \tag{37}$$

In view of

$$\left|\widetilde{A}_j\begin{pmatrix} r_j^x \\ r_j^x\end{pmatrix}\right|^2 = \left(\beta^2+\omega_c^2\right)\gamma_j\left[\left(r_j^x\right)^2 + \left(r_j^y\right)^2\right] \tag{38}$$

that finally leads to

$$\widetilde{w}\left[\vec{r}_j\right] = \left(\frac{\beta^2+\omega_c^2}{4\pi q\Delta t\gamma_j}\right)\exp\left\{-\frac{\left(\beta^2+\omega_c^2\right)\,|\vec{r}_j|^2}{4q\Delta t\gamma_j}\right\}. \tag{39}$$

Since this probability distribution is Gaussian with mean zero and variance $\sigma_j^2 = 2q\Delta t\frac{1}{\beta^2+\omega_c^2}\gamma_j$, the random vector $\vec{R}$ as a sum of independent random variables $\vec{r}_j$ has the distribution

$$w\left(\vec{R}\right) = \frac{1}{2\pi\sum_j\sigma_j^2}\exp\left(-\frac{R_x^2+R_y^2}{2\sum_j\sigma_j^2}\right). \tag{40}$$

$$\sigma^2 = \sum_j\sigma_j^2 = 2q\sum_j\Delta t\frac{1}{\beta^2+\omega_c^2}\gamma_j\,. \tag{41}$$

In the limit of $\Delta t\to 0$ we arrive at the integral

$$\sigma^2 = 2q\frac{1}{\beta^2+\omega_c^2}\int_0^t\gamma\left(s\right)ds \tag{42}$$

with $\int_0^t\gamma\left(s\right)ds = t+\varTheta$, where

$$\varTheta = \varTheta(t) = \frac{1}{2\beta}\left(1-\mathrm{e}^{-2\beta t}\right) - 2\frac{1}{\beta^2+\omega_c^2}\left[\beta+\left(\omega_c\sin\omega_c t-\beta\cos\omega_c t\right)\mathrm{e}^{-\beta t}\right]. \tag{43}$$

That gives rise to an ultimate form of the transition probability density of the spatial displacement process:

$$P\left(\vec{r},t|\vec{r}_0,t_0=0,\vec{u}_0\right) = \frac{1}{4\pi\frac{k_{\mathrm{B}}T}{m}\frac{\beta}{\beta^2+\omega_c^2}\left(t+\varTheta\right)}\exp\left(-\frac{|\vec{r}-\vec{r}_0-\varOmega\vec{u}_0|^2}{4\frac{k_{\mathrm{B}}T}{m}\frac{\beta}{\beta^2+\omega_c^2}\left(t+\varTheta\right)}\right) \tag{44}$$

with $\varOmega=\varOmega(t)$ defined in Eqs. (25), (27). Notice that an asymptotic diffusion coefficient $D_B = D\frac{\beta^2}{\beta^2+\omega_c^2}$ encodes an attenuation signature for the spatial dispersion (when ω_c grows up at β fixed).

The spatial displacement process governed by the above transition probability density surely is *not* Markovian. That can be checked by inspection: the Chapman–Kolmogorov identity is not valid, like in the standard free Brownian motion example where the Ornstein–Uhlenbeck process induced (sole) spatial dynamics is non-Markovian as well.

4. Phase-space process

4.1. Axial direction

We take advantage of the cylindrical symmetry of our problem, and consider separately the (free) Brownian dynamics in the direction parallel to the magnetic field vector, *e.g.* along the Z-axis.

That amounts to the familiar Ornstein-Uhlenbeck process in its extended phase-space form. In the absence of external forces, the kinetic (Kramers–Fokker–Planck equation) reads:

$$\partial_t W + u\nabla_z W = \beta \nabla_u (Wu) + q\triangle_u W \,, \tag{45}$$

where $q = D\beta^2$. Here β is the friction coefficient, D will be identified later with the spatial diffusion constant, and (as before) we set $D = k_{\mathrm{B}}T/m\beta$ in conformity with the Einstein fluctuation-dissipation identity. The joint probability distribution (in fact, density) $W(z,u,t)$ for a freely moving Brownian particle which at $t=0$ initiates its motion at x_0 with an arbitrary initial velocity u_0 can be given in the form of the maximally symmetric displacement probability law:

$$W(z,u,t) = W(R,S) = \left[4\pi^2(FG-H^2)\right]^{-1/2} \exp\left\{ -\frac{GR^2 - HRS + FS^2}{2(FG-H^2)} \right\}, \tag{46}$$

where

$$R = z - u_0(1-\mathrm{e}^{-\beta t})\beta^{-1}, \quad S = u - u_0 \mathrm{e}^{-\beta t}$$

while

$$F = \frac{D}{\beta}(2\beta t - 3 + 4\mathrm{e}^{-\beta t} - \mathrm{e}^{-2\beta t})\, G = D\beta(1-\mathrm{e}^{-2\beta t})$$

and

$$H = D(1-\mathrm{e}^{-\beta t})^2 .$$

4.2. Planar process

Now we shall consider Brownian dynamics in the direction perpendicular to the magnetic field $\vec{B}$, hence (while in terms of configuration-space variables) we address an issue of the planar dynamics. We are interested in the complete phase-space process, hence we need to specify the transition probability density $P(\vec{r},\vec{u},t|\vec{r}_0,\vec{u}_0,t_0=0)$ of the Markov process conditioned by the initial data $\vec{u}=\vec{u}_0$ and $\vec{r}=\vec{r}_0$ at time $t_0=0$. That is equivalent to deducing the joint probability distribution $W(\vec{S},\vec{R})$ of random vectors $\vec{S}$ and $\vec{R}$, previously defined to appear in the form $\vec{S} = \vec{u}(t) - \mathrm{e}^{-\Lambda t}\vec{u}_0$ and $\vec{R} = \vec{r} - \vec{r}_0 - \Omega\vec{u}_0$, respectively. Let us stress that presently all vectors are regarded as two-dimensional versions (the third component being simply disregarded) of the original random variables we have discussed so far in Sections 2 and 3.

Vectors $\vec{S}$ and $\vec{R}$ both share a Gaussian distribution with mean zero. Consequently, the joint distribution $W(\vec{S},\vec{R})$ is determined by the matrix of variances and covariances: $C = (c_{ij}) = (\langle x_i x_j \rangle)$, where we abbreviate four phase-space variables in a single notion of $x = (S_1, S_2, R_1, R_2)$ and label components of x by $i, j = 1, 2, 3, 4$. In terms of $\vec{R}$ and $\vec{S}$ the covariance matrix C reads:

$$C = \begin{pmatrix} \langle S_1 S_1 \rangle & \langle S_1 S_2 \rangle & \langle S_1 R_1 \rangle & \langle S_1 R_2 \rangle \\ \langle S_2 S_1 \rangle & \langle S_2 S_2 \rangle & \langle S_2 R_1 \rangle & \langle S_2 R_2 \rangle \\ \langle R_1 S_1 \rangle & \langle R_1 S_2 \rangle & \langle R_1 R_1 \rangle & \langle R_1 R_2 \rangle \\ \langle R_2 S_1 \rangle & \langle R_2 S_2 \rangle & \langle R_2 R_1 \rangle & \langle R_2 R_2 \rangle \end{pmatrix} . \tag{47}$$

The joint probability distribution of $\vec{S}$ and $\vec{R}$ is given by

$$W\left(\vec{S},\vec{R}\right) = W\left(\vec{x}\right) = \frac{1}{4\pi^2}\left(\frac{1}{\det C}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\sum_{i,j} c_{ij}^{-1} x_i x_j\right) , \tag{48}$$

where c_{ij}^{-1}denotes the component of the inverse matrix C^{-1}.

The probability distributions of $\vec{S}$ and $\vec{R}$, which were established in the previous sections, determine a number of expectation values:

$$g \equiv \langle S_1 S_1 \rangle = \langle S_2 S_2 \rangle = \frac{k_{\mathrm{B}} T}{m}\left(1 - \mathrm{e}^{-2\beta t}\right) \tag{49}$$

while $\langle S_1 S_2 \rangle = \langle S_2 S_1 \rangle = 0$. Furthermore:

$$f \equiv \langle R_1 R_1 \rangle = \langle R_2 R_2 \rangle = 2\frac{k_{\mathrm{B}} T}{m}\frac{\beta}{\beta^2 + \omega_c^2}(t + \Theta) = 2D_{\mathrm{B}}(t + \Theta) . \tag{50}$$

In addition we have $\langle R_1 R_2 \rangle = \langle R_2 R_1 \rangle = 0$. As a consequence, we are left with only four non-vanishing components of the covariance matrix C: $c_{13} = c_{31} = \langle S_1 R_1 \rangle$, $c_{14} = c_{41} = \langle S_1 R_2 \rangle$, $c_{23} = c_{32} = \langle S_2 R_1 \rangle$, $c_{24} = c_{42} = \langle S_2 R_2 \rangle$ which need a closer examination.

We can obtain those covariances by exploiting a dependence of the random quantities $\vec{S}$ and $\vec{R}$ on the white-noise term $\vec{A}(s)$ whose statistical properties are known. There follows:

$$S_1 = \int_0^t ds \mathrm{e}^{-\beta(t-s)}\left[\cos\omega_c (t-s) A_1(s) + \sin\omega_c (t-s) A_2(s)\right] ,$$

$$S_2 = \int_0^t ds \mathrm{e}^{-\beta(t-s)}\left[-\sin\omega_c (t-s) A_1(s) + \cos\omega_c (t-s) A_2(s)\right] , \tag{51}$$

$$R_1 = \int_0^t ds \frac{1}{\beta^2+\omega_c^2}\Big[\beta\left(1-\mathrm{e}^{-\beta(t-s)}\cos\omega_c(t-s)\right)$$
$$+\omega_c\mathrm{e}^{-\beta(t-s)}\sin\omega_c(t-s)\Big]A_1(s)$$
$$+\int_0^t ds \frac{1}{\beta^2+\omega_c^2}\Big[-\beta\mathrm{e}^{-\beta(t-s)}\sin\omega_c(t-s)$$
$$+\omega_c\left(1-\mathrm{e}^{-\beta(t-s)}\cos\omega_c(t-s)\right)\Big]A_2(s)\ ,$$
$$R_2 = \int_0^t ds \frac{1}{\beta^2+\omega_c^2}\Big[-\omega_c\left(1-\mathrm{e}^{-\beta(t-s)}\cos\omega_c(t-s)\right)$$
$$+\beta\mathrm{e}^{-\beta(t-s)}\sin\omega_c(t-s)\Big]A_1(s)$$
$$+\int_0^t ds \frac{1}{\beta^2+\omega_c^2}\Big[\omega_c\mathrm{e}^{-\beta(t-s)}\sin\omega_c(t-s)$$
$$+\beta\left(1-\mathrm{e}^{-\beta(t-s)}\cos\omega_c(t-s)\right)\Big]A_2(s)\ .$$

Multiplying together suitable components of vectors $\vec{S}$ and $\vec{R}$ and taking averages of those products in conformity with the rules $\langle A_i(s)\rangle = 0$ and $\langle A_i(s)A_j(s')\rangle = 2q\delta_{ij}\delta(s-s')$, where $q=\frac{k_{\mathrm{B}}T}{m}\beta$, $i,j=1,2,3$, we arrive at:

$$h\equiv\langle R_1S_1\rangle=\langle R_2S_2\rangle=2q\frac{1}{\beta^2+\omega_c^2}\int_0^t ds\mathrm{e}^{-\beta(t-s)}[\beta\cos\omega_c(t-s)$$
$$+\omega_c\sin\omega_c(t-s)-\beta\mathrm{e}^{-\beta(t-s)}]=q\frac{1}{\beta^2+\omega_c^2}\left(1-2\mathrm{e}^{-\beta t}\cos\omega_c t+\mathrm{e}^{-2\beta t}\right) \quad (52)$$

and

$$k\equiv\langle R_1S_2\rangle=-\langle R_2S_1\rangle=2q\frac{1}{\beta^2+\omega_c^2}\int_0^t ds\mathrm{e}^{-\beta(t-s)}[-\beta\sin\omega_c(t-s)$$
$$+\omega_c\cos\omega_c(t-s)-\omega_c\mathrm{e}^{-\beta(t-s)}]$$
$$=q\frac{1}{\beta^2+\omega_c^2}\left[2\mathrm{e}^{-\beta t}\sin\omega_c t-\frac{\omega_c}{\beta}\left(1-\mathrm{e}^{-2\beta t}\right)\right]. \quad (53)$$

The covariance matrix $C=(c_{ij})$ has thus the form

$$C=\begin{pmatrix} g & 0 & h & -k \\ 0 & g & k & h \\ h & k & f & 0 \\ -k & h & 0 & f \end{pmatrix} \tag{54}$$

while its inverse C^{-1} reads as follows:

$$C^{-1}=\frac{1}{\det C}\left(fg-h^2-k^2\right)\begin{pmatrix} f & 0 & -h & k \\ 0 & f & -k & -h \\ -h & -k & g & 0 \\ k & -h & 0 & g \end{pmatrix}, \tag{55}$$

where $\det C=\left(fg-h^2-k^2\right)^2$. The joint probability distribution of $\vec{S}$ and $\vec{R}$ can be ultimately written in the form:

$$W\left(\vec{S},\vec{R}\right)=$$
$$\frac{1}{4\pi^2\left(fg-h^2-k^2\right)}\exp\left(-\frac{f\left|\vec{S}\right|^2+g\left|\vec{R}\right|^2-2h\vec{S}\cdot\vec{R}+2k\left(\vec{S}\times\vec{R}\right)_{i=3}}{2\left(fg-h^2-k^2\right)}\right). \tag{56}$$

In the above, all vector entries are two-dimensional. The specific $i=3$ vector product coordinate in the exponent is simply an abbreviation for the (ordinary R^3-vector product) procedure that involves merely first two components of three-dimensional vectors (the third is then arbitrary and irrelevant), hence effectively involves our two-dimensional $\vec{R}$ and $\vec{S}$.

One of the authors (P. G.) receives financial support from the Polish State Committee for Scientific Research (KBN) grant No. 2 PO3B 086 16.

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