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ON GENERALIZED CLIFFORD ALGEBRAS AND SPIN LATTICE SYSTEMS*

A.K. Kwaśniewski

Institute of Computer Science, Białystok University
Sosnowa 64, 15-887 Białystok, Poland
and Higher School of Mathematics and Applied Computer Science
Czysta 11, 15-463 Białystok, Poland
e-mail: kwandr@noc.uw.edu.pl

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The incessantly growing area of applications of Clifford algebras and naturalness of their use in formulating problems for direct calculation entitles one to call them Clifford numbers. The generalized "universal" Clifford numbers are here introduced via k-ubic form Q_k replacing quadratic one in familiar construction of an appropriate ideal of tensor algebra. One of the epimorphic images of universal algebras $k - C_n \cong T(V)/I(Q_k)$ is the algebra $Cl_n^{(k)}$ with n generators and these are the algebras to be used here. Because generalized Clifford algebras $Cl_n^{(k)}$ possess inherent $Z_k \oplus Z_k \oplus \Lambda \oplus Z_k$ grading — this makes them an efficient apparatus to deal with spin lattice systems. This efficiency is illustrated here by derivation of two major observations. Namely — partition functions for vector and planar Potts models and other model with Z_n invariant Hamiltonian are polynomials in generalized hyperbolic functions of the n-th order. Secondly, the problem of algorithmic calculation of the partition function for any vector Potts model as treated here is reduced to the calculation of $\mathrm{Tr}(\gamma_{i_1} ... \gamma_{i_s})$, where γ's are the generators of the generalized Clifford algebra. Finally the expression for $\mathrm{Tr}(\gamma_{i_1} ... \gamma_{i_s})$, for arbitrary collection of such γ matrices is derived.

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1. Introduction

The problem of eventual calculation of the partition function for any vector Potts model is treated in two major steps. At first it is reduced to the calculation of $\mathrm{Tr}(\gamma_{i_1} ... \gamma_{i_s})$, where γ's are the generators of the generalized

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Clifford algebra. Then — following [4] the expression for $\mathrm{Tr}(\gamma_{i_1}...\gamma_{i_s})$, for arbitrary collection of such γ matrices is derived. As a result we arrive at the **first general statement**: the knowledge of an algorithm for calculating the expression for $\mathrm{Tr}(\gamma_{i_1}...\gamma_{i_s})$ assures in principle the possibility of calculation of partition functions in all models for which the transfer matrix is an element from generalized Clifford algebra *i.e.* those with Z_n invariant Hamiltonian. The method — successfully experienced for Z_2 case — becomes complicated for Z_n, $n > 2$, however algorithms are under controll specifically due to the knowledge of corresponding algebra properties and those of generalized hyperbolic functions $\{h_s(z)\}_{s\in Z'_n}$ ([5,6]); $Z'_n = \{0,1,2,...,n-1\}$.

The **second general statement** [6] might be also the reason for temporal complacency. Namely, as observed and stated in [6] (see also related [7–9,11,12]) — partition functions for vector and planar Potts models and other models with Z_n invariant Hamiltonian are just polynomials in these $\{h_s(z)\}_{s\in Z'_n}$ generalized hyperbolic functions of the n-th order. Hence an effort to guess the thermodynamics of the system, though considered as tantalous, looks perhaps slightly more a reasonable, tangible task if — for example — assisted by computer simulations and calculations [12]. Our note is organized as follows:

After just a crumb of history preliminaries follow. These are to present in brief indispensable knowledge on (a) $\{h_s(z)\}_{s\in Z'_n}$ generalized hyperbolic functions of the n-th order and on (b) generalized Clifford algebras. (For more on (a) and (b) one is invited to visit the Appendix.) After that, follows a presentation of reasoning leading us to the first and the second general statements above. For extensive literature on hyperbolic functions $\{h_s(z)\}_{s\in Z'_n}$ of the n-th order see [13, 14] and also [10] for hyperbolic mappings of the n-th order. As for generalized Clifford algebras see [3,4] and [10] for extensive literature on the subject. The appearing soon notion of Pfaffian is recalled in the Appendix.

2. Preliminaries and formulation of the main statements

We start — as announced — with a crumb of history, however preceded by an indispensable notation while formulating the general problem. Let us then define the family of states for Z_n vector Potts model on a $p \times q$ torus lattice (p rows, q columns) to be the set S of $(p \times q)$ matrices with entries from Z_n.

$$S = \{(s_{i,k} = (p \times q); \quad s_{i,k} \in Z_n\}; \quad Z_n = (\omega^l)_{l=0}^{n-1} \omega = \exp\left\{\frac{2\pi i}{n}\right\}. \quad (2.1)$$

The total energy E is then given by:

$$-\frac{E[(s_{i,k})]}{kT} = a\sum_{i,k=1}^{p,q}(s_{i,k}^{-1}s_{i,k+1}+s_{i,k+1}^{-1}s_{i,k}) + b\sum_{i,k=1}^{p,q}\left(s_{i,k}^{-1}s_{i+1,k}+s_{i+1,k}^{-1}s_{i,k}\right)$$

and the partition function reads as follows:

$$Z = \sum_{(s_{i,k})\in S}\exp\left\{-\frac{E[\{s_{i,k}\}]}{kT}\right\}. \tag{2.2}$$

We write sometime $Z \equiv Z_N$ whenever it is important to indicate that this is a partition function for the toroidal grid with N sites. For $n = 2$ we shall arrive at Ising model.
The partition function could be written in terms of transfer matrix and for that purpose we introduce the standard notation:

$$\begin{aligned} \vec{s}\cdot\vec{s}\,' &= \sum_{i=1}^{p} s_i s_i',\ \vec{s}_k = \begin{pmatrix} s_{1,k} \\ s_{2,k} \\ \vdots \\ s_{p,k}\end{pmatrix},\ \vec{s}_k^{\,*} = \begin{pmatrix} s_{1,k}^* \\ s_{2,k}^* \\ \vdots \\ s_{p,k}^*\end{pmatrix} \\ (s_{i,k}) &= (\vec{s}_1,\vec{s}_2,\ldots,\vec{s}_q)\,. \end{aligned} \tag{2.3}$$

With the notation (2.3) adopted, the partition function Z may be now rewritten in a form

$$Z = \sum_{\vec{s}_1,\vec{s}_2,\ldots,\vec{s}_q}\exp\left\{a\sum_{k=1}^{q}(\vec{s}_k^{\,*}\cdot\vec{s}_{k+1}+\vec{s}_{k+1}^{\,*}\cdot\vec{s}_k) + b\sum_{k=1}^{q}(\vec{s}_k^{\,*}\cdot\Sigma_1\vec{s}_k+\vec{s}_k\cdot\Sigma_1\vec{s}_k^{\,*})\right\}, \tag{2.4}$$

after the natural periodicity conditions have been imposed *i.e.*

$$\vec{s}_{q+1} = \vec{s}_1\,, \quad (\vec{s}_k)_1 = (\vec{s}_k)_{p+1}; \quad k = 1,\ldots,q\,, \tag{2.5}$$

where $(\vec{x})_i$ denotes the i-th component of $\vec{x}$. Periodicity conditions warrant that we are dealing with the model on $p\times q$ torus lattice. The matrix Σ_1 in (2.4) is a $p\times q$ generalized Pauli matrix with matrix elements $\delta_{i+1,j}$, where $i,j \in Z_p' = \{0,1,\ldots,p-1\}$ and "+" is understood as denoting the Z_p' group action on indices of $\vec{s}_k$ vectors via addition mod p. We introduce also the σ_1 generalized Pauli matrix, which is one of the three $\sigma_1,\sigma_2,\sigma_3$ — playing the same role in representing $C_{2p}^{(n)}$ generalized Clifford algebras (see Morris [3]) as the "usual" ones in representing the known $C_{2p}^{(2)}$ Clifford algebras via tensor products of σ matrices [15].

$C_{2p}^{(n)}$ generalized Clifford algebra is defined [16] to be generated by $\gamma_1, ..., \gamma_{2p}$ matrices satisfying:

$$\gamma_i\gamma_j = \omega\gamma_j\gamma_i, \quad i < j, \ \gamma_i^n = \mathbf{1}, \quad i,j = 1,2,...,2p, \quad \omega = \exp\frac{2\pi i}{n}. \tag{2.6}$$

The very algebra has — up to equivalence — only one irreducible and faithful representation, and its generators can be represented as tensor products of generalized Pauli matrices:

$$\sigma_1 = (\delta_{i+1,j}), \quad \sigma_2 = (\omega^i\delta_{i+1,j}), \quad \sigma_3 = (\omega^i\delta_{i,j}), \tag{2.7}$$

where $i, j \in Z'_n = \{0, 1, ..., n-1\}$ — the additive cyclic group.
It is now obvious that Z may be represented as

$$Z = \mathrm{Tr}\, M^q \tag{2.8}$$

as we have

$$Z = \sum_{\vec{s}_1,...,\vec{s}_q} M(\vec{s}_1,\vec{s}_2)M(\vec{s}_2,\vec{s}_3)...M(\vec{s}_q,\vec{s}_1),$$

where matrix elements of the transfer matrix M are given by:

$$M(\vec{s},\vec{s}') = \exp\{2b\,\mathrm{Re}\,(\vec{s}^* \cdot \Sigma_1\vec{s})\}\exp\{2a\,\mathrm{Re}\,(\vec{s}^* \cdot \vec{s}')\}. \tag{2.9}$$

It is convenient to consider the matrix M as a product

$$M = BA,$$

where the corresponding matrix elements are identified as

$$\begin{aligned} A(\vec{s}'',\vec{s}') &= \exp\{2a\,\mathrm{Re}\,(\vec{s}''^* \cdot \vec{s}')\}, \\ B(\vec{s},\vec{s}'') &= \exp\{2b\,\mathrm{Re}\,(\vec{s}^* \cdot \Sigma_1\vec{s})\}\delta(\vec{s},\vec{s}''). \end{aligned} \tag{2.10}$$

As all these A, B, M matrices are multi-indexed it is obvious that they might be represented either as tensor products of $(n \times n)$ matrices (p times) or as $(n^p \times n^p)$ matrices.

Of course, for $n = 2$ we shall arrive at Ising model on $p \times q$ torus lattice. Calculations that might be carried out now for the Z_n vector Potts model simplify tremendously in the case of $n = 2$ *i.e.* for the Ising model and there lead to the known Onsager–Kaufman expression for complete partition function ([14, 15]) which after carrying out the thermodynamic limit goes into the Onsager formula. The method we choose is an appropriate generalization of the one used in [14] which consists there in reducing the problem of finding of the partition function for the Ising model to calculation of $\overline{\mathrm{Tr}}(P_1, ..., P_s)$, where $\overline{\mathrm{Tr}}$ is the normalized trace while P's are linear combinations of γ matrices — generators of usual Clifford algebra naturally assigned to the lattice.

Then the observation that $\overline{\mathrm{Tr}}(P_1, ..., P_{2s})$ is just a Pfaffian [14] of an antisymmetric matrix formed with scalar products of P's leads one to calculation of the determinant from this very matrix. Parallely, one may show ([5, 12]) that in the Ising case of [14] *i.e.* in the Z_2 group case

$$M^q = (B_- A)^q V_0 + B_+ A)^q V_1) \,, \tag{2.11}$$

where

$$\begin{aligned} B_- &= \exp\left\{2bi\left(\sum_{\alpha=1}^{p-1} \bar{\gamma}_\alpha \gamma_{\alpha+1} - \bar{\gamma}_p \gamma_1\right)\right\} \,, \\ B_+ &= \exp\left\{2bi\left(\sum_{\alpha'=1}^{p-1} \bar{\gamma}_{\alpha'} \gamma_{\alpha'+1} + \bar{\gamma}_p \gamma_1\right)\right\} \,, \end{aligned}$$

$$V_0 = \tfrac{1}{2}(\mathbf{1} + U), \quad V_1 = \tfrac{1}{2}12(\mathbf{1} - U) \,. \tag{2.12}$$

$$U = \prod_{k=1}^{p} \gamma_k \bar{\gamma}_k \tag{2.13}$$

and of course $U^2 = \boldsymbol{I}$ as we are now temporarily inspecting the Ising case *i.e.* Z_2 group case. The formula (2.11) coincides then with the one known for Ising model [14] apart from the obvious and insignificant scaling of constants a and b by factor 2. And here comes the **first main statement** valid — as we shall see — for arbitrary $n > 1$. Namely, M^q is a polynomial in $\gamma_1, ...\gamma_{2p}$ matrices. Indeed — for that to see it is enough to take into account (2.11), the defining property (2.6) of $\gamma_1, ...\gamma_{2p}$ matrices and the fact established in [5–7] that

$$A = \otimes^p \hat{a} \,, \tag{2.14}$$

i.e. A is the p-th tensor power of the $(n \times n)$ "interaction matrix" $\hat{a}$, which has the form of a circulant matrix $W[\sigma_1]$:

$$\hat{a} = (\hat{a}_{I,J}) = \left(\exp\left\{2a \,\mathrm{Re}\left(\omega^{J-I}\right)\right\}\right) = \sum_{l=0}^{n-1} \lambda_l \sigma_1^l \equiv W[\sigma_1] \,, \tag{2.15}$$

where $I, J \in Z'_n = \{0, 1, 2, ..., n-1\}$ and

$$\lambda_l = \exp\{2a \,\mathrm{Re}(\omega^l)\} \,, \quad l \in Z'_n \,. \tag{2.16}$$

(For more details consult [5].) Using the formula (2.11), the notion of Pfaffian and its relation to determinant the author of [14] reobtained the complete partition function leading to the famous Onsager formula. The matrices V_0, V_1, from (2.11) have also simple form and thus Tr M^q disentangles for $n = 2$ to be the sum only *four summands of the Pfaffian type i.e.* $\mathrm{Tr}(P_1 P_2 ... P_s)$. These four arising Pfaffians contribute to the partition function to give [14]:

$$\begin{aligned} Z &= 2^{pq-1} \\ &\times \left\{ \prod_{k,l=1}^{p,q} \left[\cosh 2a' \cosh 2b' - \sinh 2a' \cos \frac{\pi}{q}(2l+1) - \sinh 2b' \cos \frac{\pi}{p}(2k+1) \right]^{\frac{1}{2}} \right. \\ &+ \prod_{k,l=1}^{p,q} \left[\cosh 2a' \cosh 2b' - \sinh 2a' \cos \frac{\pi}{q}(2l+1) - \sinh 2b' \cos \frac{2\pi k}{p} \right]^{\frac{1}{2}} \\ &+ \prod_{k,l=1}^{p,q} \left[\cosh 2a' \cosh 2b' - \sinh 2a' \cos \frac{2\pi l}{q} - \sinh 2b' \cos \frac{\pi}{p}(2k+1) \right]^{\frac{1}{2}} \\ &\left. - \sigma \prod_{k,l=1}^{p,q} \left[\cosh 2a' \cosh 2b' - \sinh 2a' \cos \frac{2\pi l}{q} - \sinh 2b' \cos \frac{2\pi k}{p} \right]^{\frac{1}{2}} \right\}, \end{aligned} \tag{2.17}$$

where σ denotes the sign of $T - T_c$ and $a' = 2a$, $b' = 2b$. Both the square root and the σ-sign have appeared here because of the use of $Pf(A)^2 = \det A$ relation, where A is antisymmetric matrix and Pf denotes Pfaffian mapping [13].

Here $\cosh \equiv h_0$ and $\sinh \equiv h_1$; $0, 1 \in Z_2'$ are hyperbolic functions of the second order to be replaced by $\{h_s(z)\}_{s \in Z_n'}$ hyperbolic functions of the n-th order in the Z_n' case.

The role of familiar $\gamma_1, ..., \gamma_{2p}$ matrices represented via tensor products of σ matrices [14] in the customary $C_{2p}^{(2)}$ Clifford algebra case is now to be taken over by generalized $\gamma_1, ..., \gamma_{2p}$ matrices satisfying (2.6) and represented via tensor products of generalized Pauli σ matrices (Morris [4] — see also [3,5–7]). These mathematical simple devices *i.e.* (a) $\{h_s(z)\}_{s \in Z_n'}$ hyperbolic functions of the n-th order and (b)$C_{2p}^{(n)}$ generalized Clifford algebras were expected and in a sense foretold by Baxter in his popular monograph [16]. Here is the opportune, pertinent and well-timed quotation from it:

The only hope that occurs to me is just as Onsager (1944) and Kaufman (1949) originally solved the zero-field Ising model by using the algebra of spinor operators, so there may be similar algebraic methods for solving the eight-vertex and Potts models.

We would like to stress that **both** $C_{2p}^{(n)}$ (5–7,17) and $\{h_s(z)\}_{s \in Z_n'}$ [4-6] inventions appreciably contribute in full of meaning to the development of Potts-like models investigation.

The use of $\{h_s(z)\}_{s\in Z'_n}$ hyperbolic functions of the n-th order as indicated in [5] leads to the formulation of our **second main statement**: partition functions are polynomials in $\{h_s(z)\}_{s\in Z'_n}$ generalized hyperbolic functions of the n-th order in the case of vector and planar Potts models and other models with Z_n invariant Hamiltonian and with duality property. Let us explain it now briefly. There are two steps to arrive at the second main statement. The first one consists of the simple observation that whenever we have an element U of an associative algebra with unity, say — matrix U such that $U^n = I$; $U^k \neq I$ $0 < k < n$ then

$$\exp\{zU\} = \sum_{k\in Z_n} U^k \sum_{r\geq 0} \frac{z^{nr+k}}{(nr+k)!} = \sum_{k\in Z_n} h_k(z) U^k \,. \tag{2.18}$$

If also V is such that $V^n = I$; $V^k \neq I$ $0 < k < n$ then obviously we have

$$\mathrm{Tr}\{\exp[xU]\exp[yV]\} = \sum_{k,l\in Z_n} h_k(x) h_l(y) \mathrm{Tr}(U^k V^l) \,. \tag{2.19}$$

The second step is to realize that after the dual parameter a^* has been introduced ([7, 8]) the partition function for the toroidal grid with N sites $Z \equiv Z_N$ has that of (2.19) form:

$$Z_N = [\det \hat{a}(a)]^{\frac{N}{2}} \mathrm{Tr}\{\exp[a^*(\sigma_1 + \sigma_1^+)]\exp[b(\sigma_3 + \sigma_3^+)]\}^N \,. \tag{2.20}$$

where $\sigma_1, \sigma_2, \sigma_3$ defined by (2.7) are generalized Pauli matrices — playing the same role in representing $C_{2p}^{(n)}$ generalized Clifford algebras (see Morris [4]) as the "usual" ones in representing the known $C_{2p}^{(2)}$ Clifford algebras via tensor products of σ matrices [15]. The "interaction matrix" $\hat{a}$, which has the form of a circulant matrix $W[\sigma_1]$ is defined by (2.15) and (2.16) and the dual parameter a^* is any fixed solution of the equation [7, 8]

$$[\det \hat{a}(a^*)] = n^n [\det \hat{a}(a)]^{-1}. \tag{2.21}$$

3. The structure of transfer matrix and a trace formula

In order to see that A and B matrices from (2.10) are just some elements of $C_{2p}^{(n)}$ we shall express them in terms of operators X_k and Z_k ; $k = 1, 2, ..., p$ *i.e.* matrices typical for tensor product representation of generalized Clifford algebras via generalized Pauli matrices (see (A.3)). Naturally $M = BA$ (as in Ising case) where A and B are expressed below in terms

$$\begin{aligned} X_k &= I\otimes ...\otimes I\otimes\sigma_1\otimes I\otimes ...\otimes I \quad (p-\text{factors})\,, \\ Z_k &= I\otimes ...\otimes I\otimes\sigma_3\otimes I\otimes ...\otimes I \quad (p-\text{factors})\,, \end{aligned} \tag{3.1}$$

where σ_1 and σ_3 are situated on the k-th site, counting from the left-hand side.

The matrix A may be therefore now rewritten as a product of $(n^p \times n^p)$ matrices

$$A = \prod_{k=1}^{p} W[X_k]\,, \quad \text{where} \quad W[X_k] = \sum_{l=0}^{n-1} \lambda_l X_k^l\,. \tag{3.2}$$

Similarly, for the matrix B we derive:

$$B = \exp\left\{ b \sum_{k=1}^{p} \left(Z_k^{-1} Z_{k+1} + Z_{k+1}^{-1} Z_k \right) \right\}\,, \tag{3.3}$$

where $Z_{p+1} = Z_1$. The formula (3.3) follows from the simple observation that matrix elements of $Z_k^{-1}Z_{k+1} + Z_{k+1}^{-1}Z_k$ (multi-indexed by $\vec{s}$ and $\vec{s}''$) give exactly ln of the corresponding term of (2.10) expression for B. The δ function arises due to the fact that $\sigma_3 = (\delta_{I,J}\omega_I)$ and the exponentiation of matrix elements is easy because B is simply proportional to unit matrix. Once A and B have been represented as in (3.2) and (3.3) it is easy to express them in terms of generalized γ matrices. Introducing then the tensor product representation (A.3) we get:

$$\begin{aligned} X_k &= \omega^{n-1}\gamma_k^{n-1}\bar{\gamma}_k \\ Z_k^{-1}Z_{k+1} &= \bar{\gamma}_k^{n-1}\gamma_{k+1}\,, \quad \text{for odd } n \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} X_k &= \xi\omega^{n-1}\gamma_k^{n-1}\bar{\gamma}_k \\ Z_k^{-1}Z_{k+1} &= \xi\bar{\gamma}_k^{n-1}\gamma_{k+1} \quad \text{for even } n\,, \end{aligned} \tag{3.5}$$

where $k = 1, 2, ..., p-1$ and $\xi^2 = \omega$.

The corresponding expressions on the boundaries read:

$$Z_p^{-1}Z_1 = U\bar{\gamma}_p^{n-1}\gamma_1 \quad \text{for odd } n \tag{3.6}$$

and

$$Z_p^{-1}Z_1 = \xi^{-1}\bar{\gamma}_p^{n-1}\gamma_1 \quad \text{for even } n\,, \tag{3.7}$$

where

$$\omega \cdot U = \otimes^p \sigma_1\,. \tag{3.8}$$

For the proof of (3.4)–(3.7) use (A.6) and (A.7) from the Appendix.

From now on we shall proceed with formulas for n being odd, loosing nothing from generality of considerations while corresponding formulas for the case of n even are easily derivable from those for the odd case. This in mind we get

$$A = \prod_{k=1}^{p} W[\omega^{-1}\gamma_k^{n-1}\bar{\gamma}_k], \tag{3.9}$$

$$B = \exp\left\{ b\sum_{k=1}^{p}(\bar{\gamma}_k^{n-1}\gamma_{k+1} + \gamma_{k+1}^{n-1}\bar{\gamma}_k) \right\} \times \exp\left\{ bU\bar{\gamma}_p^{n-1}\gamma_1 + bU^{-1}\gamma_1^{n-1}\bar{\gamma}_p \right\} . \tag{3.10}$$

Our first goal is then achieved if one notes that

$$U = \prod_{k=1}^{p} \gamma_k^{n-1}\bar{\gamma}_k \, , \tag{3.11}$$

i.e. the transfer matrix M is now expressed in terms of generalized γ matrices. NOTE: it is rather trivial and important to note that $U^n{=}1, Z_k^n{=}1$, $X_k^n = 1$ with obvious implication of the same property for the n-th order polynomials in (3.9) and (3.10) — see (2.18) and (2.19). Now it is quite clear that one may reduce the $\mathrm{Tr}\, M^q$ problem to calculation of $\mathrm{Tr}(\gamma_{i_1}\gamma_{i_2}...\gamma_{i_s})$ for any collections of γ's.

(Note that for $n = 2$ the way to get the complete partition function is shorter as there, it is enough to reduce the $\mathrm{Tr}\, M^q$ problem to calculation of $\mathrm{Tr}(P_1 P_2 ... P_s)$ where P's are linear combinations of γ's. Hence the number of necessary summations is much, much smaller than in the case $n > 2$, where it is rather useless to try to represent A and B matrices in that convenient form.)

Hence now and to this end the main goal of this section is to provide formula for $\mathrm{Tr}(\gamma_{i_1}\gamma_{i_2}...\gamma_{i_s})$ for any collections of γ's. We shall quote it after [5]. Note! By definition, in this section Tr map is normalized *i.e.* $\mathrm{Tr}\, I = 1$. The derivation has the form of a sequence of observations.

Observation 1.

Let $k \neq n \, mod \, n$, $k \in N$; then $\mathrm{Tr}(\gamma_{i_1}\gamma_{i_2}...\gamma_{i_k}) = 0$

Proof: The same as for usual Clifford algebras. Use the matrix U defined by (3.11).

Observation 2.

$\mathrm{Tr}(\gamma_{i_1}\gamma_{i_2}...\gamma_{i_{kn}}) \neq 0$ iff there exists permutation $\delta \in S_{kn}$, such that

$$i_{\sigma(1)} = i_{\sigma(2)} = ... = i_{\sigma(n)} \, , \; i_{\sigma(n+1)} = ... = i_{\sigma(2n)}, \dots , i_{\sigma(kn-n+1)} = ... = i_{\sigma(kn)} \, .$$

Proof: The proof follows from observation that due to (A.1) if no n-tuple of the same γ's exists then $\mathrm{Tr}(...) = 0$. Other steps of the proof are reduced to this first one.

It is therefore trivial to note, but important to realize, that:

Observation 3.

$\mathrm{Tr}(\gamma_{i_1}\gamma_{i_2}...\gamma_{i_k}) = 0$ or $\mathrm{Tr}(\gamma_{i_1}\gamma_{i_2}...\gamma_{i_k}) \in Z_n$ — the multiplicative group of n-th roots of unity.

In Lemma 3 the letter k denotes again an arbitrary integer while in all preceding lemmas, and in the following, $i_1, i_2, ..., i_k$ run from 1 to number of generators of the given algebra. This number was chosen to be even, however note that the "odd case" problem is reduced to this very one due to the properties of generalized Clifford algebra representations (Morris [4]).

The major problem now is to determine this value "0 or ω^l" $l \in Z_n'$ for arbitrary set of indices $i_1, i_2, ..., i_k$. In order to do that define a signum like function K (unfortunately it is an epimorphism only for $n = 2$) — as follows:

$$K : S_p \to Z_n \; ; \;\; \Theta_{\sigma(1)}\Theta_{\sigma(2)}...\Theta_{\sigma(p)} = K(\sigma)\Theta_1\Theta_2...\Theta_p \; , \tag{3.12}$$

where Θ's satisfy (A.1) except for the condition $\gamma_i^n = 1$, which is now replaced by $\Theta_i^2 = 1$.

This definition being adapted, it is now not very difficult to see that:

Observation 4.

$\mathrm{Tr}(\gamma_{i_1}...\gamma_{i_{pn}}) = K(\Sigma)K(\sigma)$, for

(a) $i_{\sigma(1)} = ... = i_{\sigma(n)}, \; ... \; , i_{\sigma(pn-n+1)} = ... = i_{\sigma(pn)}$ and

(b) $i_{\tilde{\sigma}(n)} < i_{\tilde{\sigma}(2n)} < i_{\tilde{\sigma}(pn)}$,

where $\tilde{\sigma} \equiv \Sigma \circ \sigma$, while Σ is a permutation of the elements $\{n, 2n, ..., pn\}$. (The group of Σ's is naturally identified with an appropriate subgroup of S_{pn}.)

Proof: The proof relies on observation that these are only **different** n-tuples, which are "rigidly" shifted ones trough the others, *i.e.* there is no permutation within any given n-tuple.

The generalization of the Observation 4 to the arbitrary case of some of the n-tuples being equal is straightforward. (The necessary change of conditions (a) and (b) is obvious.)

This in mind and from other preceding observations we finally get the

Trace Formula:

$$\mathrm{Tr}(\gamma_{i_1}\gamma_{i_2}...\gamma_{i_{pn}}) = \sum_{\sigma \in S_{pn}}{}' \sum_{\vec{p}} \sum_{\Sigma \in S_{\vec{p}}} K(\Sigma)K(\sigma) \times \delta(i_{\tilde{\sigma}(1)}, ..., i_{\tilde{\sigma}(p_1 n)})$$

$$\times \delta(i_{\tilde{\sigma}(p_1 n+1)}, ..., i_{\tilde{\sigma}([p_1+p_2]n)}) \times ... \times \delta(i_{\tilde{\sigma}(pn-p_l n+1)}, ..., i_{\tilde{\sigma}(pn)}) \tag{3.13}$$

with the notation following notation generalizing the "Z_2–Pfaffian case":

Notation: $\vec{p} = (p_1, p_2, ..., p_l)$, $p_i \geq 1$, $\sum_{i=1}^{l} p_i = p$, $\tilde{\sigma} = \Sigma \circ \sigma$, and $S_{\vec{p}}$ is a subgroup of S_{pn} isomorphic to the group of all block matrices obtained via permutations of "block columns" of the matrix

$$\begin{pmatrix} I_{p_1^n} & & & \\ & I_{p_2^n} & & \\ & & \ddots & \\ & & & I_{p_l^n} \end{pmatrix}, \text{ where } I_k \text{ is the } (k \times k) \text{ unit matrix.}$$

δ — here denotes the multi-indexed Kronecker delta *i.e.* it assigns zero unless all its arguments are equal and in this very case $\delta(...) = 1$. The sum Σ' is meant to take into account **only** those permutations that do satisfy the conditions:

(a) $\sigma(1) < \sigma(2) < ... < \sigma(p_1 n)$, ... , $\sigma(pn - p_l n + 1) < ... < \sigma(pn)$, and

(b) $\sigma(l) < \sigma(p_1 n + 1) < ... < \sigma(pn - p_l n + 1)$.

Comments:

(1) For the case of $n = 2$ the theorem gives us the Pfaffian of the product $\gamma_{i_1}, ..., \gamma_{i_{p^2}}$, as in the case, (and only! for $n = 2$) $K(\Sigma) = 1$ and we are left, as a result with only Σ' sum, while Kronecker deltas become functions of the same number of indices i_j.

(2) The theorem may be of significant help in computer aided calculating of $\mathrm{Tr}\, M^q$ because any element of generalized Clifford algebra is a polynomial in γ's satisfying (A.1) and quite a lot of additional information on the structure of $M^q \in C_{2p}^{(n)}$ is available ([5–8,12]).

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Appendix

1. According to the principal theorem of abstract algebra there exists unique algebra n-C_k for which the diagram below is commutative what means that $\alpha = \sigma \circ \alpha_0$. Such type objects are called in abstract algebra universal. The "usual" Clifford algebras 2-C_k are defined that way for example in [2]. Here our universal n-C_k Clifford algebras are defined accordingly via the following commutative diagram [3]

$$\begin{tikzcd} \varsigma \arrow[rr, "\alpha_0"] \arrow[dr, "\alpha"'] & & n\text{-}C_k \arrow[dl, "\sigma"] \\ & A & \end{tikzcd}$$

where ς is a complex vector space of $dim\, \varsigma = k$, n-C_k and A are associative algebras, $\sigma \in \mathrm{Hom}(n - C_k; A)$ while α_0, α are monomorphisms with the property: $[\alpha_0(x)]^n = Q_n(x)\mathbf{1}$, $[\alpha_0(x)]^n = Q_n(x)\mathbf{1}$. Here Q_n denotes n-ubic form [3]. Let $\{\gamma_i\}_{i=1}^k$ be α_0 – images of the vector space ς basis.
Universal n-C_k Clifford algebra is generated by the generators $\{\gamma_i\}_{i=1}^k$ subjected to the following relations

$$\{\gamma_{i_1}, \gamma_{i_2}, ..., \gamma_{i_n}\} = \delta(i_1, ..., i_n) \;\; i_1, ..., i_n = 1, 2, ..., k\,, \tag{A.1}$$

where the "n-anticommutator" [3] is just symmetrizer operator

$$\{a_1, a_2, ..., a_n\} = \sum_{\sigma \in S_n} a_{\sigma(1)} ... a_{\sigma(n)}$$

with S_n denoting the group of permutations and

$$\delta(i_1, ..., i_n) = \begin{cases} 1 & \text{for} \quad i_1 = \ldots = i_n \\ 0 & \text{otherwise} \end{cases}$$

$C_k^{(n)}$ generalized Clifford algebra [3], [4] generated by $\gamma_1, ..., \gamma_k$ matrices satisfying:

$$\gamma_i \gamma_j = \omega \gamma_j \gamma_i, \quad i < j, \quad \gamma_i^n = \mathbf{1}, \quad i, j = 1, 2, ..., k. \tag{A.2}$$

$C_k^{(n)}$ ***is the epimorphic image*** of the universal n-C_k algebra [3].

Namely, one may verify that matrices satisfying (A.2) also satisfy (A.1) and it is easily seen that both sets of commutation relations coincide if and only if $n = 2$ as then $\omega^{-1} = \omega$. $C_{2p}^{(n)}$ generalized Clifford algebra used in this note is generated by $\gamma_1, ..., \gamma_{2p}$ matrices satisfying:

$$\gamma_i \gamma_j = \omega \gamma_j \gamma_i, \; i < j, \; \gamma_i^n = \mathbf{1}, \; i, j = 1, 2, ..., 2p\,. \tag{A.3}$$

The $C_{2p}^{(n)}$ algebra has — up to equivalence — only one irreducible and faithful representation, and its generators can be represented as tensor products of generalized Pauli matrices:

$$\sigma_1 = (\delta_{i+1,j}), \quad \sigma_2 = (\omega^i \delta_{i+1,j}), \quad \sigma_3 = (\omega^i \delta_{i,j})\,, \tag{A.4}$$

where $i, j \in Z_n' = \{0, 1, ..., n-1\}$ — the additive cyclic group.
One easily checks, that $\{\sigma_i\}_1^3$ do satisfy (A.3) for n being odd.
Let I denotes since now the unit $(n \times n)$ matrix and let

$$\begin{aligned}
\gamma_1 &= \sigma_3 \otimes I \otimes I \otimes \ldots \otimes I \otimes I\,, \\
\gamma_2 &= \sigma_1 \otimes \sigma_3 \otimes I \otimes \ldots \otimes I \otimes I\,, \\
&\vdots \\
\gamma_p &= \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \ldots \otimes \sigma_1 \otimes \sigma_3\,, \\
\bar{\gamma_1} &= \sigma_2 \otimes I \otimes I \otimes \ldots \otimes I \otimes I\,, \\
\bar{\gamma_2} &= \sigma_1 \otimes \sigma_2 \otimes I \otimes \ldots \otimes I \otimes I\,, \\
&\vdots \\
\bar{\gamma_p} &= \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \ldots \otimes \sigma_1 \otimes \sigma_2\,,
\end{aligned} \tag{A.5}$$

then $\gamma_i, \bar{\gamma_j}$ $i,j = 1,\ldots,p$ do satisfy (A.2) with ω replaced by ω^{-1}, hence (A.5) are generators of the algebra isomorphic to $C_{2p}^{(n)}$ (isomorphism is given by $\sigma_1 \leftrightarrow \sigma_3$ in (A.5)). (This very (A.5) representation was chosen for technical reason — we get, for example, in calculations of section II, the matrix U without coefficients *etc.*). It is also to be noted that for n being odd

$$\sigma_3 = \sigma_1^{n-1}\sigma_2\,. \tag{A.6}$$

The case of n being even leads to similar representation with σ_1 unchanged but σ_2 and σ_3 now equal to:

$$\sigma_2 = (\xi^i \delta_{i+1,j}),\ \sigma_3 = \xi\sigma_1^{n-1}\sigma_2\,, \tag{A.7}$$

where ξ is a primitive $2n$-th root of unity such that $\xi^2 = \omega$.
(A.5) (with these appropriate for case $n = 2\nu$ generalized Pauli matrices) reproduces the same type representation of $C_{2p}^{(n)}$ as the one for the case $n = 2\nu + 1$.One easily proves that

$$\begin{array}{l} \sigma_3^{n-1}\sigma_2 = \omega\sigma_1 \\ \sigma_2^{n-1}\sigma_1 = \sigma_3^{-1} \end{array} \ \text{for } n = 2\nu + 1 \tag{A.8}$$

and

$$\begin{array}{l} \sigma_3^{n-1}\sigma_2 = \xi^{-1}\omega\sigma_1 \\ \sigma_2^{n-1}\sigma_1 = \sigma_3^{-1} \end{array} \ \text{for } n = 2\nu\,. \tag{A.9}$$

$C_k^{(n)}$ algebras had already appeared unrecognized in [18, 19].

2. In this part of the Appendix we recall the definition of hyperbolic functions of n-th order (for extensive references and many identities see: [10, 13, 20] and references therein). Let x be any element of an associative, finite dimensional algebra with unity I. Then

$$\begin{aligned} \exp\{x\} &= \sum_{i=0}^{n-1} h_i(x)\,, \quad \text{where} \\ h_i(x) &= \sum_{k=0}^{\infty} \frac{x^{nk+i}}{(nk+i)!}\,, \quad i = 0, \ldots, n-1\,. \end{aligned} \tag{A.10}$$

One expresses hyperbolic functions of n-th order h_i; $i \in Z'_n$ in terms expotentials due to

$$f_i(\omega x) = \omega^i f_i(x), \quad i = 0, 1, \ldots, n-1\,; \quad \omega = \exp\left\{\frac{2\pi i}{n}\right\}. \tag{A.11}$$

The (A.11) reveals the Z_n symmetry properties of these generalized “cosh” functions and we get from this set of relations

$$f_i(x) = \frac{1}{n}\sum_{k=0}^{n-1} \omega^{-ki} \exp\left\{\omega^k x\right\}, \quad i = 0, 1, \ldots, n-1\,. \tag{A.12}$$

For considerations of the section II we need the following observation:
Let U be an element of an associative, finite dimensional algebra with unity I and in addition let $U^n = \mathbf{1}$ and $U^K \neq \mathbf{1}$ for $0 < k < n$. Then V_s ; $s \in Z'_n$ defined as follows [5, 13, 20]

$$V_s = \frac{1}{n}\sum_{i=0}^{n-1} \omega^{-si} U^i\,, \quad s \in Z'_n \tag{A.13}$$

form the family of mutually orthogonal projection operators. In particular and specifically one notices that for $V_0 \equiv V$ the following is true:
Observation: Let U be as above. Then V defined as follows $V = \frac{1}{n}\sum_{i=0}^{n-1} U^i$

$$V^n = V\,. \tag{A.14}$$

Proof: For the proof, just note that for some a_i's $V^n = \sum_{i=0}^{n-1} a_i U^i$ and both sides of this identity equation must be symmetric in U^i monomials. One concludes therefore that $a_i = a_j$, $i, j = 0, 1, \ldots, n-1$ hence — counting the number of all arising summands — one arrives at the conclusion of the *Observation*.

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