ON SIMPLE AND SUBTLE PROPERTIES OF NEUTRINOS* **

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Neutrino flavors, light and heavy, are discussed in their interdependence within the minimal unifying gauge group of SO10. The general situation which excludes the existence of an *exact* symmetry from which *exactly* vanishing light neutrino masses would follow is discussed. *Subtle* and *simple* consequences for 'low–low' oscillation phenomena are presented in a general framework.

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1. SO10 extension of the standard model: $(\nu\mathcal{N})^{\dot{\gamma}}_{(J)}$

In the title above the label $^{\dot{\gamma}}$ denotes, for $\dot{\gamma} = 1, 2$, the two left-chiral components of the spin 1/2 fields ν and $\mathcal{N}$, respectively.

The label (J), for $J = 1, 2, 3$, denotes family number. So we define the (minimal) fermionic extension of the three standard model fermion families to three families of 16-representations of SO10 :

$$\{f\}^{\dot{\gamma}}_{(J)} = \left\{ \begin{array}{l|l} u^{1,\ 2,\ 3},\ \nu_e & \mathcal{N}_e\ ,\ \hat{u}^{1,\ 2,\ 3} \\ d^{1,\ 2,\ 3},\ e^- & \hat{e}^+\ ,\ \hat{d}^{1,\ 2,\ 3} \end{array} \right\}^{\dot{\gamma}}_{(J)} . \tag{1}$$

Under the standard model gauge group $SU3_c \times SU2_L \times U1_{\mathcal{Y}}$ the fermion flavors in Eq. (1) transform as

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$$\left\{ \begin{array}{c} u^{1,\ 2,\ 3} \\ \\ d^{1,\ 2,\ 3} \end{array} \right\} = \left(3,\ 2,\ \tfrac{1}{6}\right);\ \left\{ \begin{array}{c} \nu_e \\ \\ \mathrm{e}^- \end{array} \right\} = \left(1,\ 2,\ -\tfrac{1}{2}\right),$$
$$\mathcal{N}_e = (1,\ 1,\ 0)\,;\ \hat{\mathrm{e}}^+ = (1,\ 1,\ 1)\,,$$
$$\left\{\hat{u}^{1,\ 2,\ 3}\right\} = \left(\overline{3},\ 1,\ -\tfrac{2}{3}\right);\ \left\{\hat{d}^{1,\ 2,\ 3}\right\} = \left(\overline{3},\ 1,\ \tfrac{1}{3}\right). \tag{2}$$

We ask as a guiding question whether the associated B–L (Baryon number–Lepton number) charge with quantum numbers

$$\left\{ \begin{array}{c} \tfrac{1}{3},\ \tfrac{1}{3},\ \tfrac{1}{3},\ -1 \,\big|\ +1, -\tfrac{1}{3},\ -\tfrac{1}{3},\ -\tfrac{1}{3} \\ \tfrac{1}{3},\ \tfrac{1}{3},\ \tfrac{1}{3},\ -1 \,\big|\ +1\,,\ -\tfrac{1}{3},\ -\tfrac{1}{3},\ -\tfrac{1}{3} \end{array} \right\} \tag{3}$$

can be conserved — if not gauged — in the limit $m_{\mathcal{N}(J)} \to \infty$. The limit of infinite mass for the (standard model gauge group) singlet neutrino flavors $\mathcal{N}_{e(j)}$ implies a Majorana mass matrix of the form

$$\mathcal{H}_m = \tfrac{1}{2}\, m_{\mathcal{N}JJ'} \mathcal{N}_{\dot{\gamma}J} \mathcal{N}^{\dot{\gamma}}_{J'} + \text{h.c.},$$
$$m_{\mathcal{N}JJ'} = m_{\mathcal{N}J'J} \to M_{JJ'}\,; \quad M \to \infty\,. \tag{4}$$

Apart from being symmetric the mass matrix $m_{\mathcal{N}}$ in Eq. (4) is complex general and the physical masses in the infinite mass limit involve the eigenvalues of the associated Hermitian matrices $(mm^\dagger)^{1/2}$ and (equivalently) $(m^\dagger m)^{1/2}$. Including the $\mathcal{N}_{(J)}$ flavors (before taking the infinite mass limit) the B–L current is of vectorial form. In a chiral basis this amounts to a matching number of positive and negative eigenvalues of the B–L charge as exhibited in Eq. (3) :

$$j_\mu^{\mathrm{B-L}\,(16)} = \sum_J \sum_r^{16} (\mathrm{B–L})_r\, f^{*\beta}_{r(J)} \sigma_{\mu\beta\dot{\gamma}} f^{\dot{\gamma}}_{r(J)}. \tag{5}$$

For finite masses $m_{\mathcal{N}J}$ the B–L current is broken by these masses, but in the infinite mass limit the remaining (45) flavors are constrained to form a chiral current:

$$j_\mu^{\mathrm{B-L}\,(15)} = \sum_J \sum_r^{15} (\mathrm{B–L})_r\, f^{*\beta}_{r(J)} \sigma_{\mu\beta\dot{\gamma}} f^{\dot{\gamma}}_{r(J)}. \tag{6}$$

This reduced chiral current ($j_\mu^{\mathrm{B-L}(15)}$ in Eq. (6)) develops a gravitational anomaly and thus fails to be conserved in a gravitational environment:

$$D^\mu j_\mu^{\mathrm{B-L}\,(15)} = 3c_1\,(\mathrm{spin}) p_1\,(R) = 3\mathcal{A}_1\,(R)\,,$$
$$c_1\,(\mathrm{spin}) = -\tfrac{1}{24}\,. \tag{7}$$

In Eq. (7) D_μ denotes the covariant derivative with respect to the vierbein. Since the current is equivalent to an antisymmetric 3-form (in four dimensions) the divergence does not involve the vierbein or the metric and thus the right hand side of Eq. (7) necessarily defines a topological 4-form. This 4-form defines, apart from the overall factor 3, the Hirzebruch–Atiyah index form. The integral of this form yields the chiral topological invariant of the (full) Dirac operator pertaining to the curvature 2-form of the metric (or vierbein).

It is also referred to as $\mathcal{A}$ genus in the (mathematical) literature and extends to any (Euclidean (compact)) space with dimension dividable by 4: $d_n = 4n$. The coefficient c_n(spin) gives the relation between the Hirzebruch–Atiyah and Pontryagin classes in $d_n = 4n$ dimensions. The latter are generically (for all n simultaneously) defined through the relations

$$\begin{aligned} \overline{R}^a_b \; &= \; \tfrac{1}{2\pi}\,\tfrac{1}{2} dx^\mu \,\wedge\, dx^\nu \left(R^a_b\right)_{\mu\nu} \;, \\ \mathrm{Det}\left(1 - \lambda \overline{R}\right) \; &= \; \sum_n \lambda^{2n} p_n\left(R\right) , \\ p_1 \; &= \; \tfrac{1}{16\pi^2} R^a_{b\mu\nu} \widetilde{R}^{b\mu\nu}_a \,; \quad \widetilde{R}^{b\,\mu\,\nu}_a = \tfrac{1}{2} \varepsilon^{\mu\nu\sigma\tau} \left(R^a_b\right)_{\sigma\tau} \;, \\ p_2 \; &= \; -\tfrac{1}{4} \mathrm{tr}\, \overline{R}^4 + \tfrac{1}{8} \left(\mathrm{tr}\, \overline{R}^2\right)^2 \;; \quad \cdots \end{aligned} \tag{8}$$

with

$$\begin{aligned} \mathcal{A}(R;\lambda) \; &= \; \mathrm{Det}\left(\frac{\lambda \, \frac{1}{2} \overline{R}}{\sin\left(\lambda\, \frac{1}{2} \overline{R}\right)} \right) = \sum_n \lambda^{2n}\, c_n(R) \\ c_1 \; &= \; -\tfrac{1}{24}\, p_1 \,; \quad c_2 = \frac{1}{2^7\, 3^2\, 5} \left(-4p_2 + 7p_1^2\right) \;; \; \cdots \,. \end{aligned} \tag{9}$$

— The classes (Hirzebruch–Atiyah and Pontryagin classes) are *simple.*

— The relations are *obvious* but not clearly simple.

— The consequence is *subtle* : There does not exist any exact symmetry, which is *necessary* to maintain exactly massless neutrino flavors. This is so even if the overall (6 chiral flavor) neutrino mass matrix would exactly conserve B–L.

In order to have exact B–L symmetry the six chiral neutrino flavors have to combine into triply doubled (Dirac) *pairs* with equal overall B–L quantum numbers :

$$\begin{aligned} \left\{\nu_A\right\}_{(J)} &= \left(\begin{array}{c} \varepsilon_{\alpha\beta} \left(\mathcal{N}^{\dot\beta}\right)^{*(J)} \\ \nu^{\dot\gamma}_{(J)} \end{array} \right) ; \quad A = 1, \cdots, 4 \,, \\ A = 1,2 &\leftrightarrow \alpha = 1,2 \,; \quad A = 3,4 \leftrightarrow \dot\gamma = 1,2 \end{aligned} \tag{10}$$

with a B–L conserving mass matrix of the form equivalent to the case of (electrically) charged flavors:

$$\begin{aligned} \mathcal{H}_m^{\nu,\mathcal{N}} &= m_{JJ'}^{\nu,\mathcal{N}}\, \overline{\nu}_J \tfrac{1}{2}(1+\gamma_{5\,\mathrm{L}})\nu_{J'} + \text{h.c.}\,, \\ \gamma_{5\,\mathrm{L}} &= -\gamma_{5\,\mathrm{R}} = i\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3\,. \end{aligned} \tag{11}$$

The reduced mass matrix $m_{JJ'}^{\nu,\mathcal{N}}$ is an unrestricted complex 3 by 3 matrix. The physical masses are the eigenvalues of the Hermitian (nonnegative) combination $\left(m^{\nu,\mathcal{N}} m^{\dagger\nu,\mathcal{N}}\right)^{1/2}$. While in the above situation B–L is exactly conserved and therefore not gauged, three arbitrary neutrino masses are allowed.

Using the (Dirac doubled) field variables $\{\nu_A\}_{(J)}$ as defined in Eqs. (10) and (11) the conserved B–L current is given by

$$j_\mu^{\mathrm{B-L}\,(16)} = \sum_J \left(\begin{array}{l} \frac{1}{3}\sum_{q=u,\ d\ ,\ c} \overline{q}_{(J)}^c \gamma_\mu q_{(J)}^c \\ -\overline{\nu}_{(J)}\gamma_\mu\,\nu_{(J)} \\ -\overline{e}_{(J)}\gamma_\mu e_{(J)} \end{array} \right). \tag{12}$$

A *simpler* form of the *subtle* consequence just outlined is: there is no neutrino flavor with exactly vanishing mass [1] [1].

It is mandatory to mention here, which type of sometimes tacitly assumed consequence(s) do *not* follow from this consequence but rather constitute just a logical *possibility*:

— Neutrino flavors come in (Dirac) doubled (B–L) conserving and mass degenerate pairs.

— The mixing matrix of the observed three light neutrino flavors is exactly unitary.

2. The actual SO10 scenario

Any extension of the Standard Model, which on the level of the gauge group contains SO10 (irrespective of whether this extension contains supersymmetry or not) leads to several *simple* consequences:

— B–L as well as all other leptonic numbers are broken (spontaneously).

— The minimum of neutrino flavors is six, and any mass degeneracy of these *six* flavors is *accidental*.

[1] The diploma thesis cited contains common work with P.M.

— The mixing matrix of any three out of the six neutrino flavors is *not* unitary.

— The number of real CP violating parameters, associated *just* with the mass matrix of the minimally six neutrino flavors is 15. These are all observable, albeit with variable sensitivity at (relatively) low energies.

A simple phenomenological excursion

Let us 'fix ideas' with respect to the three light neutrino flavors, and restricting the *assumed* heavy ones to three, *assuming* hierarchical masses [2][2]:

$$\begin{aligned}
\left\{ m^{\nu, \mathcal{N}} \right\} &= \left\{ m_{1,2,3},\, M_{\ 1,2,3} \right\} , \\
M_{1,2,3} &\gg 1 \text{ GeV} ; \quad m_{1,2,3} \ll 1 \text{ GeV} , \\
m_3 &\gg m_2 > \text{or } \gg m_1 ; \quad \Delta_{ij}^2 = m_i^2 - m_j^2 , \\
\Delta_{32}^2 &= 3 \times 10^{-3} \text{ eV}^2 ; \text{ to be modified eventually} , \\
\Delta_{32} &= 0.055 \text{ eV} = 632^\circ \text{ K} , \\
\Delta_{21}^2 &= 3.5 \times 10^{-5} \text{ eV}^2 ; \text{ to be modified eventually} , \\
\Delta_{21} &= 0.0059 \text{ eV} = 68.5^\circ \text{ K} .
\end{aligned} \tag{13}$$

This yields, reemphasizing the elimination *by assumption* of a common mass for the three light neutrino flavors much larger than the mass differences

$$\begin{aligned}
m_3 &\simeq 0.055 \text{ eV} = 632^\circ \text{ K} , \\
m_2 &\simeq 0.0059 \text{ eV} = 68.5^\circ \text{ K} , \\
m_1 &\gtrless 2^\circ \text{ K} .
\end{aligned} \tag{14}$$

Then the mass eigenstates 3 and 2 constitute hot dark matter at the time of nucleosynthesis but cold today, whereas the lightest neutrino flavor 1 may be still in relativistic (mean) motion today. The general $(\nu\,, \mathcal{N})$ mass term is of the form

$$\begin{aligned}
\mathcal{H}_m^{\nu, \mathcal{N}} &= \tfrac{1}{2}\, \nu_i^{\dot{\delta}} \varepsilon_{\dot{\gamma}\dot{\delta}} \mathcal{M}_{ik} \nu_k^{\dot{\gamma}} + \text{h.c.} ; \quad \mathcal{M}_{ik} = \mathcal{M}_{ki} , \\
i, k &= 1, \cdots, 3 + N_h ; \; r = k - 3 = 1, \cdots, N_h ; \quad \varepsilon_{\dot{\gamma}\dot{\delta}} \mathcal{N}_r^{\dot{\delta}} = \mathcal{N}_{\dot{\gamma} r} , \\
\mathcal{M} &= \begin{pmatrix} 0 & \mu^T \\ \mu & M \end{pmatrix} .
\end{aligned} \tag{15}$$

In Eq. (15) we generalize to $N_h \geq 3$ heavy neutrino flavors, while the minimal SO10 scenario has $N_h = 3$.

[2] The cited paper gives an extensive phenomenological review.

The mass matrix $\mathcal{M}$ is compatible with the so extended standard model renormalizability conditions, provided the $N_H \times 3$ matrix μ and its transpose are generated through Yukawa couplings of one or several electroweak scalar doublets.

$$\begin{aligned} -\mathcal{L}_{\mathrm{Y}} &= g_{ki}^{(\alpha)} \mathcal{N}_{\dot{\gamma}\, k-3} \left(\phi^{0}_{(\alpha)} \, , \; \phi^{-}_{(\alpha)} \right)^{*} \left(\begin{array}{c} \nu^{i\,\dot{\gamma}} \\ \mathrm{e}^{-i\,\dot{\gamma}} \end{array} \right) + \text{h.c.} \, , \\ i &= 1,2,3 \, ; \quad k = 4, \cdots , 3 + N_h \, ; \quad \alpha = 1, \cdots , n_{sc} \, , \\ \mu_{ki} &= g_{ki}^{(\alpha)} \, v^{ch}_{(\alpha)} \, ; \quad v^{ch}_{(\alpha)} = \langle \Omega | \, \phi^{0}_{(\alpha)} \, | \Omega \rangle^{*} \end{aligned} \tag{16}$$

In Eq. (16) α numbers scalar doublets. For one doublet in the SM or two doublets in the MSSM the vacuum expected values are

$$\begin{aligned} &\mathrm{SM}: \quad n_{sc} = 1 \, ; \quad v^{ch}_{(1)} \to \frac{1}{\sqrt{2}} v \, ; \quad v = \left(G_F \sqrt{2} \right)^{-1/2} \, , \\ &\mathrm{MSSM}: \quad n_{sc} = 2 \, ; \left(\begin{array}{c} v^{ch}_{(1)} \to \frac{1}{\sqrt{2}} \, v^{u} \\ \\ v^{ch}_{(2)} \to 0 \end{array} \right) ; \; v^{u} = \sin \beta v \, . \end{aligned} \tag{17}$$

Generic estimate of light versus heavy flavors

A *simple* consequence of the structure of $\mathcal{M}$ as defined in Eq. (15) concerns the absolute value of the determinant, given the *assumption* that each (complex) eigenvalue of the $N_h \times N_h$ matrix M is much larger in absolute value than (the absolute value of) any given element of the $N_h \times 3$ matrix μ. It then follows for the (complex) determinant of $\mathcal{M}$:

$$\begin{aligned} \mathrm{Det}\, \mathcal{M} &= \mathrm{Det} \mathcal{M}^{'} \, ; \quad \mathcal{M}^{'} = \left(\begin{array}{cc} 0 & \mu^{'\,T} \\ \mu^{'} & M^{'} \end{array} \right) \\ \mu^{'} &= \left(\begin{array}{c|ccc} k = 4 & \hat{\mu}_{11} & \hat{\mu}_{12} & \hat{\mu}_{13} \\ k = 5 & \hat{\mu}_{21} & \hat{\mu}_{22} & \hat{\mu}_{23} \\ k = 6 & \hat{\mu}_{31} & \hat{\mu}_{32} & \hat{\mu}_{33} \\ k = 7 & & 0 & \\ \cdots & & 0 & \\ k = N_h & & 0 & \end{array} \right) , \\ M^{'} &= \left(\begin{array}{cc} 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & \hat{M} \end{array} \right) ; \quad \hat{M} = \hat{M}_{\kappa\lambda} \, , \\ k, l &= \kappa + 3, \lambda + 3 = 7, \cdots , N_h \end{aligned} \tag{18}$$

In Eq. (18) $\hat{\mu}$ is formed from any subset of three out of the N_h (three component-) row vectors of μ with maximal rank, which we generically assume to be maximal, *i.e.* three. For the case of minimal heavy neutrino flavors $N_h = 3$, we have *simply* $\hat{\mu} = \mu$ and $\hat{M} \to 1$.

Modulo *simple* permutations according to the subset of three row vectors in μ chosen to form $\hat{\mu}$, the $N_h - 3 \times N_h - 3$ matrix $\hat{M}$ is of the form

$$\hat{M}_{\kappa\lambda} = \left[\begin{array}{l} M_{\kappa\lambda} + \left(L_\kappa^{(i)} M_{(i)\lambda} + \kappa \leftrightarrow \lambda \right) \\ + L_\kappa^{(i)} M_{(i)(j)} L_\lambda^{(j)} \end{array} \right],$$

$$|L_\kappa^{(i)}| = O(1); \quad (i),(j) = 1,2,3. \tag{19}$$

Except for *non-generic* matrices μ and M the eigenvalues of $\hat{M}$ ($N_h - 3$ in number) are of the same order of magnitude that those of $\mathcal{M}$ (N_h in number).

From the structure of $\mathcal{M}'$ in Eq. (18) it follows

$$\begin{aligned} |\mathrm{Det}\,\mathcal{M}| &= \left(\prod_i^3 m_i \right) \left(\prod_J^{N_h} M_J \right) \\ &= |\mathrm{Det}_{3\times 3}\,\hat{\mu}|^2 |\mathrm{Det}_{N_h-3\times N_h-3}\,\hat{M}|\,. \end{aligned} \tag{20}$$

In Eq. (20) (extending Eq. (13)) $M_1, \cdots, M_{N_h}$ denote the masses of the heavy flavors. The above *simple* relations are known as 'sea–saw' mechanism [3].

Introducing the geometric mean (nonnegative) masses

$$\prod_i^3 m_i = m_{\mathrm{light}}^3; \quad \prod_J^{N_h} M_J = M_{\mathrm{heavy}}^{N_h}$$

$$|\mathrm{Det}_{3\times 3}\hat{\mu}| = \overline{\mu}^3; \quad |\mathrm{Det}_{N_h-3\times N_h-3}\hat{M}| = \overline{M}^{N_h-3} \tag{21}$$

the relation in Eq. (20) takes the form

$$m_{\mathrm{light}}\, M_{\mathrm{heavy}} = \overline{\mu}^2 \left(\frac{\overline{M}}{M_{\mathrm{heavy}}} \right)^{N_h/3-1}. \tag{22}$$

The estimate of the Yukawa induced (doublet-singlet) mass $\overline{\mu}$ is derived from the *simple* $16 \times 16 \times 10$ SO10 mass relation evoluted for quark flavors to a unification mass $M_{\mathrm{unif}} \sim 10^{16}$ GeV:

$$\overline{\mu} = C_\nu \tfrac{1}{3} (m_u\, m_c\, m_t)^{1/3} \sim C_\nu\ 0.35\ \mathrm{GeV}; \quad C_\nu = O(1)\,. \tag{23}$$

Let us measure the geometric mean light neutrino mass $m_{\rm light}$ in units of 10^{-2} eV. Then the estimate in Eq. (22) takes the form

$$m_{\rm light} \;=\; K_{\rm light} 10^{-11}\,{\rm GeV} \to M_{\rm heavy} \sim 1.2 \times 10^{10}\,{\rm GeV} \\ \times \left(\frac{C_\nu}{K_{\rm light}}\right) \left(\frac{\overline{M}}{M_{\rm heavy}}\right)^{N_h/3-1} . \tag{24}$$

For given light neutrino mass (in geometric mean) and given $\overline{\mu}$, $M_{\rm heavy}$ can be reduced through the ratio $\overline{M}/M_{\rm heavy}$ even considerably if there are many heavy neutrino flavors beyond the minimal three. However, this emerges as the *simple* consequence of the light neutrino flavors mixing predominantly to the heaviest three, while only little to the *lighter* $N_h - 3$ ones.

I wish to emphasize here that any mass relations inside and outside of a unifying gauge group out of the increasing sequence SO10$< E6 < \ldots < E8^p$ cannot explain the pattern of light *and* heavy neutrino flavors. *E.g.* the mass matrix M remains totally unconstrained within the SM, yet within SO10 it has the quantum numbers of the 126 irreducible representation. If there exist elementary scalars transforming as this 126 (complex) representation the induced structure of M is intrinsically tied to the symmetries remaining after the breakdown of SO10 gauge invariance, beyond the symmetries of the SM. If these symmetries are associated with an $N = 1$ or 2 susy structure again no clear mass relations among the known fermion families arise naturally.

On the other hand the mass matrix M can be induced through finite loop effects involving *e.g.* the square of scalar 16 representations, but again no direct structure reflecting this situation on the known fermion families can be derived. On the other hand the generic ratio

$$\frac{m_{\rm light}}{M_{\rm heavy}} \sim 10^{-21} \leftrightarrow M_{\rm heavy} \sim 10^7 \;{\rm TeV} \tag{25}$$

is a *subtle* measure of lepton number violation at electroweak and lower energies, as well as of associated CP violation. The generic mass scale $M_{\rm heavy}$ is well above the assumed SUSY scale — by seven orders of magnitude — if the latter is assumed to be 1 TeV. From this a dangerous enhancement of both CP-violation beyond the CKM matrix and lepton number violation by these seven orders of magnitude indirectly affect SUSY induced contributions to all electric dipole moments (transition dipole moments for neutrinos) including the neutron, and directly the charged leptons, as well as to lepton number violating process like $\mu \to e\,\gamma$ and $\mu \to 3\,e$.

This in my opinion augments the necessity to look for a theoretically viable explanation of mass and mixing structures but it must be sought along hitherto unexplored paths. Further new phenomena beyond the SM hopefully will give some clues.

3. The Majorana equations for $\nu\,\mathcal{N}$ flavors

In the following we restrict, for simplicity, the discussion to the three heavy neutrino flavors . Then the field equations for the associated chiral fields $\nu_j^{\dot\gamma}$, $\nu^*_{\alpha\,j}\, j = 1, \ldots, 6$ take the form

$$
\begin{aligned}
\nu_{j+3}^{\dot\gamma} &= \mathcal{N}_j^{\dot\gamma}\,; \quad j = 1,2,3\,, \\
\nu^*_{\alpha j} &= \varepsilon_{\alpha\beta}\left(\nu_j^{\dot\beta}\right)^*, \\
iD^{\dot\gamma\alpha}\nu^*_\alpha &= \mathcal{M}\,\nu^{\dot\gamma}\,, \\
iD_{\alpha\dot\gamma}\nu^{\dot\gamma} &= \mathcal{M}^\dagger\,\nu^*_\alpha\,, \\
\mathcal{M}^\dagger &= \overline{\mathcal{M}}\,; \quad \mathcal{M} = \mathcal{M}^T\,.
\end{aligned}
\tag{26}
$$

In Eq. (26) the covariant derivatives are restricted to gravity.

Diagonalization to mass eigenstates $\nu \to \hat\nu$

It is *simple* but somewhat involved [4] to unitarily diagonalize the 6 by 6 matrix $\mathcal{M}$

$$
\begin{aligned}
\mathcal{M} &= U\mathcal{M}_D U^T\,; \quad U = \begin{pmatrix} U_{(11)} & U_{(12)} \\ U_{(21)} & U_{(22)} \end{pmatrix}, \\
\mathcal{M}_D &= \mathrm{diag}\left(m_{1,2,3}; M_{1,2,3}\right).
\end{aligned}
\tag{27}
$$

In Eq. (27) we break up the 6 by 6 matrix U into 3 by 3 blocks, since neutrino's prepared in a system which does not allow for the production of heavy neutrino flavors involves only the *mass eigenfields* $\hat\nu_k\,;\; k = 1,2,3,$ whereas again at electroweak energy scales only (predominantly) the modes of $\nu_j\,;\; j = 1,2,3$ are produced through the electroweak interactions and are inducing a reaction downstream the oscillation path. We refer to this as 'low–low' oscillation physics. Eq. (26) becomes

$$
\begin{aligned}
iD^{\dot\gamma\alpha}\left(U^T\nu\right)^*_\alpha &= \mathcal{M}_D\left(U^T\nu\right)^{\dot\gamma}, \\
iD_{\alpha\dot\gamma}\left(U^T\nu\right)^{\dot\gamma} &= \mathcal{M}_D\left(U^T\nu\right)^*_\alpha\,, \\
\hat\nu_n &= U_{mn}\nu_m\,; \quad m,n = 1,\cdots,6
\end{aligned}
$$

and for $k,l = 1,2,3$:

$$\begin{aligned}
\nu_k &= \left(\overline{U}_{(11)}\right)_{kl}\, \hat{\nu}_l + \left(\overline{U}_{(12)}\right)_{kl} \hat{\mathcal{N}}_l, \\
\mathcal{N}_k &= \left(\overline{U}_{(21)}\right)_{kl} \nu_l + \left(\overline{U}_{(22)}\right)_{kl} \hat{\mathcal{N}}_l, \\
\nu_k^* &= \left(U_{(11)}\right)_{kl} \hat{\nu}_l^* + \left(U_{(12)}\right)_{kl} \hat{\mathcal{N}}_l^*, \\
\mathcal{N}_k^* &= \left(U_{(21)}\right)_{kl} \nu_l^* + \left(U_{(22)}\right)_{kl} \hat{\mathcal{N}}_l^*.
\end{aligned} \tag{28}$$

'low–low' oscillation states and amplitudes

For 'low–low' oscillations only the 3 by 3 matrices $U_{(11)}$ and $\overline{U}_{(11)}$ are operative. First we consider production of a normalized state (without subscript (0)) at $t = 0$ of ν type k:

$$\begin{aligned}
\left(U_{(11)}\right)_{kl} &= \mathcal{A}_{kl}\, ; \quad \mathcal{A} = \tau U_0\, , \\
\left|\vec{p}; t = 0; k\right\rangle_{(0)} &= \mathcal{A}_{kl} \left|\vec{p}; m_l; 0\right\rangle\, , \\
\left|\vec{p}; t = 0; k\right\rangle_{(0)} &= N_k \left|\vec{p}; t = 0; k\right\rangle\, , \\
N_k^2 &= \left(\mathcal{A}\mathcal{A}^\dagger\right)_{kk} = \tau_{kk}^2 \to \left|\vec{p}; t = 0; k\right\rangle \\
&= \varrho_k \mathcal{A}_{kl} \left|\vec{p}; m_l; 0\right\rangle\, ; \quad \varrho_k = \frac{1}{N_k}\, .
\end{aligned} \tag{29}$$

In Eq. (29) we decomposed the matrix $U_{(11)} = \mathcal{A}$ into the product of a Hermitian positive matrix τ and a unitary (3 by 3) matrix U_0. The deviation of τ with $0 \leq \tau \leq 1$ from unity is of the generic order $m_{\text{light}}/M_{\text{heavy}} \sim 10^{-21}$, yet it is one of the *subtle* properties of neutrino flavor mixing that this deviation is at the very origin of the light neutrino masses. This is only so, if we *assume* the generation of these masses through mixing with heavy flavors, as we do here.

subtle things:

— the normalization $\varrho_k \mathcal{A}_{kl}$ can be enforced by properly tagging ν_k upon production irrespective of decay rates, *e.g.* for $\pi^+ \to \underline{\mu^+} + \nu_\mu + n\gamma$;

— spin states, probabilities for helicities

$$\begin{aligned}
&-1 : 1 - \frac{1 - v}{2}\, , \\
&+1 : \frac{1 - v}{2} \sim \frac{m_\nu^2}{4E^2}\, .
\end{aligned}$$

The precise probability of the 'wrong' helicity state does depend on the nature of production and decay amplitudes.

— to go from neutrino flavors to antineutrino flavors amounts to exchange the helicities and to substitute $\mathcal{A} \to \overline{\mathcal{A}}$.

Oscillation amplitudes

We denote the amplitude of transition from $t = 0$ to t and from initially produced neutrino flavor k to neutrino flavor k' by $T_{k'\leftarrow k}$, neglecting the 'wrong' helicity states, and likewise by $AT_{k'\leftarrow k}$ the corresponding antineutrino amplitude.

$$\begin{aligned}
T_{k'\leftarrow k}(t) &= \varrho_{k'}\varrho_k \left(\mathcal{A}U_{\rm diag}(t)\mathcal{A}^\dagger\right)_{k'k} ,\\
AT_{k'\leftarrow k}(t) &= \varrho_{k'}\varrho_k \left(\overline{\mathcal{A}}U_{\rm diag}(t)\overline{\mathcal{A}}^\dagger\right)_{k'k} ,\\
U_{\rm diag}(t) &= {\rm diag}\left({\rm e}^{-iE_1t}, {\rm e}^{-iE_2t}, {\rm e}^{-iE_3t}\right) ,\\
E_j &= \sqrt{p^2+m_j^2} \sim p + \tfrac{1}{2}\frac{m_j^2}{p} .
\end{aligned} \tag{30}$$

The amplitudes in Eq. (30) describe oscillations in vacuo. Matter effects modify $\mathcal{A}$ in an energy dependent way. CPT invariance is manifest, whereas T invariance requires $\mathcal{A}$ to be real :

$$\begin{aligned}
& CPT : (AT_{k\leftarrow k'}(-t))^* = T_{k'\leftarrow k}(t) ,\\
& T : (T_{k\leftarrow k'}(-t))^* = T_{k'\leftarrow k}(t) ,\\
\text{and}\quad T \leftrightarrow AT ,\quad & \\
& \rightarrow \mathcal{A} = \overline{\mathcal{A}} .
\end{aligned} \tag{31}$$

Counting phases

We generalize again the counting of imaginary parameters or equivalently of complex phase factors to $3 \rightarrow N_h$ heavy neutrino flavors with $n = 3 + N_h$. The number of independent imaginary parameters in the mass matrix $\mathcal{M}$ of the form given in Eq. (15) is

$$\#\,\varphi(\mathcal{M}) = \tfrac{1}{2}n(n+1) - 6 . \tag{32}$$

The number of phases is 15 for 6 neutrino flavors.

4. Conclusions and outlook

— The major experimentally accessible features: $m_{1,2,3}$ and U_0 as defined in Eq. (29), *i.e.* light neutrino mass and mixing hopefully will establish the specific structure of $\nu - \mathcal{N}$ dynamics.

— The *subtle* and small effects: 1) $\mathcal{A} = \tau U_0$ with $\tau \neq 1$ and 2) lepton flavor violation in conjunction with the observation of 'wrong' neutrino helicities or neutrinoless double β decay are in generic situations expected to be very small.

— $CP \leftrightarrow T$ violating effects in the ν, $\overline{\nu}$ leptonic sector are much richer than in the case of quarks and antiquarks. They are tied to the small masses of the light neutrino flavors. Despite this 'low' energy obstruction it seems to me to be worthwhile to look for these effects especially at maximally feasible energies, where light–heavy $\nu \to \mathcal{N}$ transitions begin to play a role.

— "Weniges ist mehr $\cdots$".

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