TOWARDS THE MATRIX MODEL OF M-THEORY ON A LATTICE

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The Wilson discretization of the dimensionally reduced supersymmetric Yang–Mills theory is constructed. This gives a lattice version of the matrix model of M-theory. An SU(2) model is studied numerically in the quenched approximation for $D = 4$. The system shows canonical scaling in the continuum limit. A clear signal for a prototype of the "black hole to strings" phase transition is found. The pseudocritical temperature is determined and the temperature dependence of the total size of the system is measured in both phases. Further applications are outlined.

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1. Introduction and lattice formulation

According to the celebrated Banks, Fishler, Shenker and Susskund (BFSS) hypothesis [1], the matrix quantum mechanics [2] describes qualitatively many properties of the final unifying theory (the M-theory). In particular, confronting its thermodynamics with the predictions of the supergravity models is considered as an important quantitative test of the BFSS hypothesis and more generally the SYM/supergravity duality [3, 4].

The complete solution of the matrix quantum mechanics is not known even though it is much simpler than a conventional field theory. In this paper we construct the Wilson discretization of the model, and study its yet simpler version with the lattice methods. In particular, we find the onset of a prototype black hole to strings phase transition and determine quantitatively some of its properties. Of course, the true black hole phase of the full D0 brane system is much more complex. It is, therefore, quite appealing that the simplified model considered in this exploratory study reveals an important part of the phase structure.

Many other problems can be attacked within the present approach opening a new area of exciting applications. To our knowledge, this is the first study of the M-theory related quantum mechanics and its thermodynamics on a lattice. At the same time we emphasize that the idea of the nonperturbative, numerical or partly analytic, study of the string related matrix models is not new. Several groups have formulated and studied, following the Eguchi and Kawai [5], the zero dimensional (*e.g.* dimensionally reduced to a point) supersymmetric IIB string model [6]. Interestingly, supersymmetry is not broken in this formulation providing one of the attractive features of the whole scheme as opposed to the lattice discretization. The main, impressive goal of the above approach is an attempt to understand the dynamical mechanism of the compactification of higher dimensions (see Refs. [7, 8] for a review, complete references and recent developments). Independently, the Monte Carlo and analytical study of the non-compact SU(N) integrals, relevant for the supersymmetric matrix models, have been reported in Refs. [9].

We begin with the BFSS proposition to use the dimensionally reduced SUSY YM theory in D=10 dimensions as a model for the relevant degrees of freedom of M-theory. For general D the action reads [10]

$$S = \int dt \left(\frac{1}{2} \mathrm{Tr} F_{\mu\nu}(t)^2 + \bar{\Psi}^a(t) \mathcal{D} \Psi^a(t) \right) . \tag{1}$$

In the process of dimensional reduction all fields are assumed to be independent of the space variables x_i, $i = 1 \dots D-1$. Consequently all space derivatives in the field tensor $F_{\mu\nu}$ and in the Dirac operator $\mathcal{D}$ vanish ($\partial_i \to 0$), and the action (1) describes supersymmetric quantum mechanics of $D-1$ bosons and their fermionic partners. The temporal components of the gauge fields are nondynamical and serve to impose Gauss law constraints. The original D dimensional theory is supersymmetric at the classical level only in $D = 2$, 4, 6 and 10 dimensions, where appropriate (Majorana, Weyl or both) conditions are imposed [11]. Fermionic fields $\Psi^a(t)$ belong to the adjoint representation of the gauge group SU(N), $a = 1 \dots N^2 - 1$. Finally, the BFSS proposal requires $N \to \infty$ since this variable corresponds to the eleventh component of the momentum in the infinite momentum frame where the original theory is considered.

We propose to study the above system with methods of the Lattice Field Theory. To this end consider D dimensional hypercubic lattice $N_1 \times \dots \times N_D$ reduced in all space directions to $N_i = 1$, $i = 1 \dots D-1$. Gauge and fermionic variables are assigned to links and sites of the new elongated lattice in the standard manner. The gauge part of the action reads

$$S_{\mathrm{G}} \;=\; -\beta \sum_{m=1}^{N_t} \sum_{\mu>\nu} \frac{1}{N} \mathrm{Re}\,(\mathrm{Tr}\, U_{\mu\nu}(m)), \tag{2}$$

with

$$\beta \;=\; \frac{2N}{a^3 g^2}\,, \tag{3}$$

and

$$\begin{aligned} U_{\mu\nu}(m) \;&=\; U_\nu^\dagger(m) U_\mu^\dagger(m+\nu) U_\nu(m+\mu) U_\mu(m)\,, \\ U_\mu(m) \;&=\; \exp\left(iagA_\mu(am)\right)\,, \end{aligned}$$

where a denotes the lattice constant and g is the gauge coupling in one dimension. The integer time coordinate along the lattice is m. Periodic boundary conditions $U_\mu(m+\nu) = U_\mu(m)$, $\nu = 1 \ldots D-1$, guarantee that Wilson plaquettes $U_{\mu\nu}$ tend, in the classical continuum limit, to the appropriate components $F_{\mu\nu}$ with space derivatives absent. In this formulation the projection on gauge invariant states is naturally implemented.

Discretization of the Dirac operator is analogous to the now standard construction of the supersymmetric Yang–Mills theories on a lattice [12]. We do not address here important, and specific for D=10, questions of Euclidean formulation for the fermionic degrees of freedom already discussed in [9] and Weyl projection on the lattice [13]. Due to the $N_i = 1$ periodicity all hopping terms along the space directions collapse into the diagonal blocks of the fermionic matrix which effectively becomes three-diagonal. This significantly simplifies evaluation of its determinant or pfaffian for Majorana constraint. For example, we have developed an algorithm which reduces the computational effort of the exact evaluation of the pfaffian of the antisymmetric fermionic matrix from $O(V^3)$ to $O(V)$, V being the volume of the system. Even with this improvement, however, lattice simulations with dynamical fermions are much more time consuming than the pure gauge computations.

Therefore, as a first step, we simulate the action (2) in the *quenched* approximation, with the SU(2) gauge group, and for $D = 4$. The reader may well wonder if such a severely reduced system can retain some resemblance to the original one. We follow here the standard pragmatic approach widely accepted for exploratory studies of new constructions [14], also in the supersymmetric models [15]. Instead of dwelling now into arguments [17,18,22,23], we shall discuss these problems together with our results since we belive that they are encouraging even with the assumed simplifications. Needless to say, one should gradually remove the above approximations in the forthcoming computations. This is especially important for studying the low temperature phase where the supersymmetry restoration may be essential.

One of the most exciting features of the new theory is the explanation of the Bekenstein–Hawking entropy puzzle in terms of the microscopic degrees of freedom of the elementary strings/branes [20]. In particular the theory predicts existence of a phase transition at which a black hole "dissolves" into its elementary constituents [3, 21]. Therefore, as a first application of our construction, we study the phase structure of the system (2). Moreover, lattice results on QCD at finite temperature show that the very fact of the *existence* of the phase transition is not sensitive to quenching. Obviously, the sharp transition could occur only at infinite N for this one dimensional system with local interactions. However, similarly to the finite volume effects in statistical physics, we expect a broad crossover for finite and even small N [1]. Subsequent simulations for larger N should provide more information for the quantitative (*e.g.* finite size scaling) analysis.

2. Results

As an order parameter we choose the distribution of the Polyakov line

$$P = \frac{1}{N}\mathrm{Tr}\left(\prod_{m=1}^{N_t} U_D(m)\right). \tag{4}$$

Similarly to lattice QCD, symmetric concentration of the eigenvalues around 0 indicates a low temperature phase (which would have the interpretation of a black hole phase in the full model) where $\langle P\rangle \sim 0$, while clustering around ± 1 (for SU(2)) is characteristic of the high temperature (elementary excitations) phase.

A sample of results for different $N_t (\equiv N_D)$ and β is shown in Fig. 1. Indeed, for each N_t, we see a definite change of the shape with β. This is the first result: the system (2) shows unambiguously the onset of the phase change, even in the quenched approximation and for $N = 2$.

Second, the dependence of the pseudocritical temperature β_c on the time extent N_t is consistent with the continuum limit expectations $T_c \sim (g^2 N)^{1/3}$ [19]. Indeed, the temperature of a system is given by $T = 1/(aN_t)$. Together with Eq. (3) these relations imply $\beta_c \sim N_t^3$. The estimates for β intervals where the change of phases occur are presented in Table I for several lattice sizes N_t. Results of the power law fit are also quoted. A good quality of the fit and the agreement with the canonical exponent, $\gamma = 3$, is encouraging. Simultaneously, we obtain the proportionality coefficient α which translates

[1] In the exactly soluble, for finite $V = L^2$, Ising2 model, a broad enhancement in the specific heat exists already for $L = 2, 3\ldots$, and turns smoothly into a singular peak for $L \to \infty$.

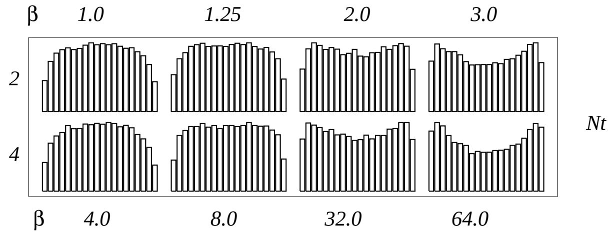

Fig. 1. Distribution of the Polyakov line (4), $-1 < P < 1$, for different β and N_t. Note different β range for different N_t.

into

$$T_c = \left(\frac{\alpha}{2N^2}\right)^{1/3} (g^2N)^{1/3} = (0.28 \pm 0.03)(g^2N)^{1/3}. \tag{5}$$

To summarize this point: the observed dependence of β_c on N_t agrees with the canonical scaling expectations for the one dimensional system, and indicates the finite value of the transition temperature in the continuum. Moreover, the coefficient in the continuum relation (5) has been determined for the first time. Only proportionality of the two scales has been considered until now [3, 19, 21]. Since both, the pseudocritical temperature and α, depend in general on N, it is important to repeat similar analysis for higher gauge groups.

TABLE I

Estimated location of the transition region $\beta_c \in (\beta_{\text{low}}, \beta_{\text{up}})$ for different lattice sizes N_t and results of the power fit.

N_t	β_{low}	β_{up}
2	1.25	1.5
3	3.5	5.0
4	8.0	16.0
5	15.0	40.0
	fit: $\beta_c = \alpha N_t^{\gamma}$	
χ^2/NDF	α	γ
0.55/2	0.17 ± 0.05	3.02 ± 0.33

Concerning the relevance of the small N results, lattice simulations for QCD strongly suggest that indeed 't Hooft choice of the coupling constant, $\lambda = g^2N$, takes into account main large N effects. In fact it was found that even results for SU(2) are not far from those with higher N [17]. This is considered as a hint that even small N calculations may contain useful information [18]. Similar picture follows from the exact solution of the Gross–Witten model for finite N [2]. Of course, other large N features may show up later. Hence the systematic study of the N dependence, augmented by the finite size scaling analysis, should follow present exploratory estimates.

Next we study the temperature dependence of the total size of the system $R^2 = g^2 \sum_a (A_i^a)^2$ [19]. We define for SU(2)

$$\langle R^2 \rangle \equiv \frac{4 - \langle (\mathrm{Tr}\, U_s)^2 \rangle}{a^2}\,, \tag{6}$$

where U_s is any space link. Due to the periodicity ($N_s = 1$) in space (6) is gauge invariant.

One dimensional Yang–Mills coupling g provides a single scale for all continuum observables similarly to Λ_{QCD} in four dimensions. In the following all dimensional quantities quoted in units of $g^{2/3}$ are denoted by a tilde.

Even though the quantum mechanical system (2) is much simpler than the full D-dimensional field theory, extracting the continuum limit of the lattice formulation (2) may be a nontrivial task. For example, the above limit contains the complete information about both the perturbative weak coupling and nonperturbative strong coupling regimes in the continuum. Technically, relation (3) implies that a reasonably small lattice constant, a requires simulation with a very large coupling β. In addition, the one dimensional systems are harder to thermalize. All this poses an interesting challenge in constructing new algorithms suitable for this problem. Some of such algorithms are under development and will be discussed elsewhere. Here we use mostly the standard local Metropolis update. To overcome the critical slowing down we simply increase the number of thermalization and decorrelation sweeps with β, until results become independent of the starting configuration. This turned out to be in accord with the dynamical exponent $z = 2$. For example when running at $\tilde{a} = 1.0$ we used 5000 thermalization and 50 decorrelation sweeps, while for $\tilde{a} = 0.1$ about 10^6 thermalization and 5000 decorrelation sweeps were required. One of the new algorithms mentioned above is the SU(2) heat bath designed for an update of the space–space plaquettes in the action (2) which contain twice the same link. Current version is effective only for $\beta < 64$. Results obtained with the new heat bath and the standard Metropolis agree within statistical errors. To check independently the performance of the Metropolis algorithm

[2] For example, the critical coupling λ_c can be reproduced within 25 % from the first five N's.

for higher β we have also monitored the correlation length in the torelon channel at zero (*i.e.* low) temperature. It reveals the expected canonical scaling with a.

Fig. 2 shows the dependence of $\tilde{R}^2$ on $\tilde{a}$, for several values of the temperature $\tilde{T}$. MC results depend smoothly on a, at fixed T, which confirms the existence of the continuum limit (6). The a dependence is clearly different in low and high temperature regions. For $0.1 < \tilde{T} < 0.3$, $\langle \tilde{R}^2 \rangle$ is practically independent of $\tilde{a}$ and points for different (but small) $\tilde{T}$ collapse on the same line. For higher $\tilde{T}$ quadratic minimum at $\tilde{a} = 0$ develops and shrinks with the further increase of the temperature. For $\tilde{T} > 1.5$ simulations for smaller $\tilde{a}$ are required in order to see this structure and determine the continuum limit. We have also extracted $\langle R^2 \rangle$ from another lattice observable $|\mathrm{Tr}(U_s)|$ with practically the same results.

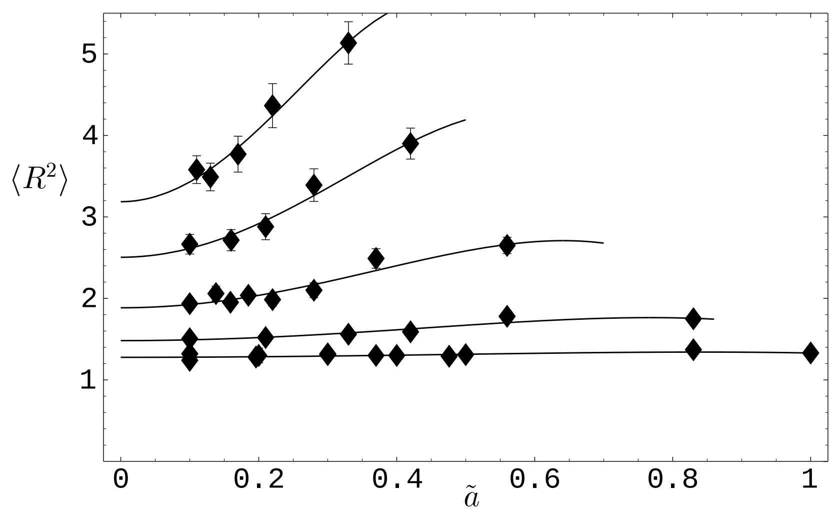

Fig. 2. Dependence of the total size of the system on a, for $\tilde{T} = 0.1, 0.3, 0.6, 0.9,$ 1.2 and 1.5 (upwards) in units of $g^{2/3}$. Quartic fits are represented by the solid lines.

Fig. 3 shows the size of the system extrapolated to $a = 0$ as a function of the temperature. Both quadratic and quartic fits of a dependence were used to perform the extrapolation. We have also checked the stability of quadratic fits with respect to removing one or two data points with smallest a (highest β). Results of the extrapolation were stable with respect to all these variations. Small systematic shifts are included in the errors displayed in Fig. 3. It is known that a typical quantum mechanical system of *finite* number of degrees of freedom does not generate any nontrivial anomalous dimensions or running coupling [23]. This is confirmed, within the available precision, by the quality of our fits which were chosen only on a basis of a simple dimensional considerations.

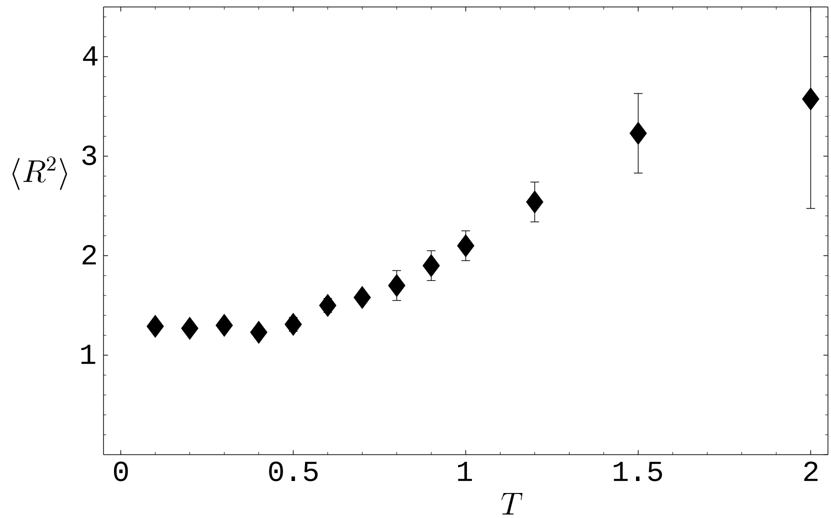

Fig. 3. Size of the system (6) extrapolated to the continuum, as a function of the temperature.

This point is also relevant to the important question of the restoration of the supersymmetry broken by lattice discretization. In fact the problem reduces again to the ability to control the continuum limit where the supersymmetry should be restored [16]. Fortunately, as just explained, the situation is simpler than in the full D dimensional quantum theory with infinite number of degrees of freedom. Both our "experimental" results (*i.e.*, on γ, and successful quadratic extrapolations in Fig. 2) confirm that indeed the continuum limit can be extracted with some confidence. With dynamical fermions the system still remains finite and one does not expect fundamental difference at least in this respect[3].

The location of the transition region in Fig. 3 is in a rough agreement with the estimate (5) of the pseudocritical temperature $\tilde{T}_c = 0.35 \pm 0.04$ [4]. Again, it is evident that the system is indeed different in the two regimes. Moreover, our results agree qualitatively with the analytical prediction obtained by solving a gap equation in the infinite N limit [19]. The latter gives

[3] At the same time other interesting issues (*e.g.*, of the chiral limit) emerge which make unquenched simulations even more challenging, *cf.* the discussion in *Conclusions*.

[4] The pseudocritical temperatures determined from different observables can be different.

a temperature independent constant at low temperatures and the classical $T^{1/2}$ growth for high temperatures. We have also found a reasonable agreement with a simple mean field model for SU(2) with the gauge projection[5]. As expected the model does not have a phase transition, but shows a smooth crossover located as in Fig. 3. The constant value for $\langle R^2 \rangle$ is satisfactorily reproduced in the low temperature, vacuum driven region. At higher temperatures the model predicts intermediate linear, albeit weaker than MC, behavior which asymptotically turns over into $T^{1/2}$ as in the infinite N case.

3. Conclusions

We have constructed the matrix model of M-theory on a lattice in $D =$ 2, 4, 6 and 10 dimensions. The resulting system corresponds to the supersymmetric formulation of Yang–Mills theory on the asymmetric D-dimensional lattice with all $D-1$ space extensions $N_s = 1$. The new construction was tested in the quenched approximation for D=4 and $N = 2$. In particular, we have found the onset of a black hole to strings transition even for the SU(2) gauge group. The pseudocritical temperature was determined. The size of the system was also measured at different temperatures and lattice cut-offs. It shows the expected canonical scaling. After extrapolation to the continuum limit it confirms the existence of the two phases and agrees qualitatively with the mean field calculations.

A host of new applications can follow. On the technical side, new algorithms are required to reduce the critical slowing down at very large values of the lattice coupling. Such studies have already begun. Including dynamical fermions is facilitated by the linear nature of the system and may lead to more efficient fermionic algorithms. Certainly the issue of dynamical fermions is very important especially in the low temperature phase since one expects that supersymmetry should be broken only in a minimal fashion there. With dynamical fermions in $D = 10$ one may have to use the recently proposed chiral formulation [13]. On the other hand for the reduced system the task may be simpler than *e.g.* for QCD. It would also be very interesting to apply analytical methods developed in [24,25]. Incidentally, a merit of the present approach is the possibility to draw from the expertise, techniques and algorithms developed in the lattice community.

A systematic study of the model for higher N would allow finite size analysis and determine more detailed characteristics of the transition. In particular, it would be interesting to check if the "soft" dependence on N observed for $D = 3$ and $D = 4$ SU(N) lattice YM [17], persists in the SYM quantum mechanical model.

[5] To be discussed in detail elsewhere.

Finally, one of the ultimate physical goals would be to study the thermodynamics of the black-hole phase in the full $D = 10$ model and verify existence of the rich phase structure predicted by the string/M theory [3]. This would also provide a possible nontrivial quantitative test of (a version of) the AdS/CFT correspondence at strong coupling not protected by any nonrenormalization theorems [4, 26]. Last but not least, many other problems inspired by the BFSS conjecture can be quantitatively studied.

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