TWO-LOOP AMPLITUDES FOR $e^+e^- \to q\bar{q}g$: THE n_f-CONTRIBUTION*

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We discuss the calculation of the n_f-contributions to the two-loop amplitude for $e^+e^- \to qg\bar{q}$. The calculation uses an efficient method based on nested sums. The result is presented in terms of multiple polylogarithms with simple arguments, which allow for analytic continuation in a straightforward manner.

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1. Introduction

Searches for new physics in particle physics rely to a large extend on our ability to constrain the parameters of the standard model. For instance, the strong coupling constant α_s can be measured by using the data for $e^+e^- \to$ 3-jets. At present, the error on the extraction of α_s from this measurement is dominated by theoretical uncertainties [1], most prominently, by the truncation of the perturbative expansion at a fixed order.

The perturbative QCD calculation of $e^+e^- \to$ 3-jets at Next-to-Next-to-Leading Order (NNLO) requires the tree-level amplitudes for $e^+e^- \to$ 5 partons [2], the one-loop amplitudes for $e^+e^- \to$ 4 partons [3,4] as well as the two-loop amplitude for $e^+e^- \to q\bar{q}g$ together with the one-loop amplitude $e^+e^- \to q\bar{q}g$ to order ε^2 in the parameter of dimensional regularization.

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The helicity averaged squared matrix elements at the two-loop level for $e^+e^- \to q\bar{q}g$ have recently been given [5]. In contrast, having the two-loop amplitude available, one keeps the full correlation between the incoming e^+e^- and the outgoing parton's spins and momenta. Thus, one can study oriented event-shape observables. In addition, one has the option to investigate event-shape observables in polarized e^+e^--annihilation at a future linear e^+e^--collider TESLA.

2. Calculation

We are interested in the following reaction

$$e^+ + e^- \to q + g + \bar{q}\,, \tag{1}$$

which we consider in the form, $0 \to q(p_1) + g(p_2) + \bar{q}(p_3) + e^-(p_4) + e^+(p_5)$, with all particles in the final state, to be consistent with earlier work [3]. The kinematical invariants for this reaction are denoted by

$$s_{ij} = (p_i + p_j)^2\,, \quad s_{ijk} = (p_i + p_j + p_k)^2\,, \quad s = s_{123}\,, \tag{2}$$

and it is convenient to introduce the dimensionless quantities

$$x_1 = \frac{s_{12}}{s_{123}}, \quad x_2 = \frac{s_{23}}{s_{123}}. \tag{3}$$

Working in a helicity basis, it suffices to consider the pure photon exchange amplitude $\mathcal{A}_\gamma$ as it allows the reconstruction of the full amplitude with Z-boson exchange by adjusting the couplings. Furthermore, the complete information about $\mathcal{A}_\gamma$ is given by just one independent helicity amplitude, which we take to be $A_\gamma(1^+,2^+,3^-,4^+,5^-)$. All other helicity configurations can be obtained from parity and charge conjugation.

We can write $A_\gamma(1^+,2^+,3^-,4^+,5^-)$ in terms of coefficients c_2, c_4, c_6 and c_{12} for the various independent spinor structure as

$$\begin{aligned} A_\gamma(1^+,2^+,3^-,4^+,5^-) \;=\; & \frac{i}{\sqrt{2}}\frac{[12]}{s^3} \\ & \times \left\{ s\langle 35\rangle[42]\left[(1-x_1)\left(c_2 + \frac{2}{x_2}c_6 - c_{12}\right) + (1-x_2)\left(c_4 - c_{12}\right) + 2c_{12}\right]\right. \\ & \left. -\langle 31\rangle[12]\left[[43]\langle 35\rangle\left(c_2 + \frac{2}{x_2}c_6 - c_{12}\right) + [41]\langle 15\rangle\left(c_4 - c_{12}\right)\right]\right\}\,, \end{aligned} \tag{4}$$

where we have introduced the short-hand notation for spinors of definite helicity, $|i\pm\rangle = |p_i\pm\rangle = u_\pm(p_i) = v_\mp(p_i)$, $\langle i\pm| = \langle p_i\pm| = \bar{u}_\pm(p_i) = \bar{v}_\mp(p_i)$, and for the spinor products $\langle pq\rangle = \langle p-|q+\rangle$ and $[pq] = \langle p+|q-\rangle$.

The coefficients c_i depend on the x_1 and x_2 of Eq. (3) and can be calculated in conventional dimensional regularization. To that end, we proceed as follows [6, 7]. In a first step, with the help of Schwinger parameters [8], we map the tensor integrals to combinations of scalar integrals in various dimensions and with various powers ν_i of the propagators. For every basic topology, these scalar integrals can be written as nested sums involving Γ-functions. The evaluation of the nested sums proceeds systematically with the help of the algorithms of [6], which rely on the algebraic properties of the so called Z-sums,

$$Z(n; m_1, \dots, m_k; x_1, \dots, x_k) = \sum_{n \geq i_1 > i_2 > \dots > i_k > 0} \frac{x_1^{i_1}}{{i_1}^{m_1}} \dots \frac{x_k^{i_k}}{{i_k}^{m_k}} . \quad (5)$$

By means of recursion the algorithms allow to solve the nested sums in terms of a given basis in Z-sums to any order in ε. Z-sums can be viewed as generalizations of harmonic sums [9] and an important subset of Z-sums are multiple polylogarithms [10],

$$\mathrm{Li}_{m_k, \dots, m_1}(x_k, \dots, x_1) = Z(\infty; m_1, \dots, m_k; x_1, \dots, x_k). \quad (6)$$

All algorithms for this procedure have been implemented in FORM [11] and in the GiNaC framework [12, 13]. In this way, we could calculate all loop integrals contributing to the one- and two-loop virtual amplitudes very efficiently in terms of multiple polylogarithms.

The perturbative expansion in α_{s} of the functions c_i is defined through

$$c_i = \sqrt{4\pi\alpha_{\mathrm{s}}} \left(c_i^{(0)} + \left(\frac{\alpha_{\mathrm{s}}}{2\pi}\right) c_i^{(1)} + \left(\frac{\alpha_{\mathrm{s}}}{2\pi}\right)^2 c_i^{(2)} + O(\alpha_{\mathrm{s}}^3) \right) . \quad (7)$$

Then, after ultraviolet renormalization, the infrared pole structure of the renormalized coefficients c_i^{ren} agrees with the prediction made by Catani [14] using an infrared factorization formula. We use this formula to organize the finite part into terms arising from the expansion of the pole coefficients and a finite remainder,

$$c_i^{(2),\mathrm{fin}} = c_i^{(1),\mathrm{ren}} - \boldsymbol{I}^{(1)}(\varepsilon) c_i^{(1),\mathrm{ren}} - \boldsymbol{I}^{(2)}(\varepsilon) c_i^{(0)} , \quad (8)$$

for $i = \{2, 4, 6, 12\}$, and with the one- and two-loop insertion operators $\boldsymbol{I}^{(1)}(\varepsilon)$ and $\boldsymbol{I}^{(2)}(\varepsilon)$ given in [14].

As an example, we present our result for $n_f N$-contribution to the finite part $c_{12}^{(2),\mathrm{fin}}$ at two loops,

$$
\begin{aligned}
c_{12}^{(2),\mathrm{fin}}(x_1,x_2) &= n_f N\Bigg(3\frac{\ln(x_1)}{(x_1+x_2)^2}+\frac{1}{4}\frac{\ln(x_2)^2-2\mathrm{Li}_2(1-x_2)}{x_1(1-x_2)}\\
&+\frac{1}{12}\frac{\zeta(2)}{(1-x_2)x_1}-\frac{1}{18}\frac{13x_1^2+36x_1-10x_1x_2-18x_2+31x_2^2}{(x_1+x_2)^2x_1(1-x_2)}\ln(x_2)\\
&+\frac{x_1^2-x_2^2-2x_1+4x_2}{(x_1+x_2)^4}R_1(x_1,x_2)-\frac{1}{12}\frac{R(x_1,x_2)}{x_1(x_1+x_2)^2}\Bigg[5x_2+42x_1+5\\
&-\frac{(1+x_1)^2}{1-x_2}-4\frac{1-3x_1+3x_1^2}{1-x_1-x_2}-72\frac{x_1^2}{x_1+x_2}\Bigg]+\Bigg[\frac{1}{12}\frac{1}{x_1(1-x_2)}+\frac{6}{(x_1+x_2)^3}\\
&-\frac{1+2x_1}{x_1(x_1+x_2)^2}\Bigg](\mathrm{Li}_2(1-x_2)-\mathrm{Li}_2(1-x_1))-\frac{1}{(x_1+x_2)x_1}\Bigg)\\
&-\frac{1}{2}I\pi n_f N\frac{\ln(x_2)}{x_1(1-x_2)}\,. \qquad (9)
\end{aligned}
$$

We have introduced the function $R(x_1,x_2)$, which is well known from [15],

$$
\begin{aligned}
R(x_1,x_2) &= \qquad (10)\\
&\left(\frac{1}{2}\ln(x_1)\ln(x_2)-\ln(x_1)\ln(1-x_1)+\frac{1}{2}\zeta(2)-\mathrm{Li}_2(x_1)\right)+(x_1\leftrightarrow x_2)\,.
\end{aligned}
$$

In addition, it is convenient, to define the symmetric function $R_1(x_1,x_2)$, which contains a particular combination of multiple polylogarithms [10],

$$
\begin{aligned}
R_1(x_1,x_2) &= \Bigg(\ln(x_1)\mathrm{Li}_{1,1}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right)-\frac{1}{2}\zeta(2)\ln(1-x_1-x_2)\\
&+\mathrm{Li}_3(x_1+x_2)-\ln(x_1)\mathrm{Li}_2(x_1+x_2)-\frac{1}{2}\ln(x_1)\ln(x_2)\ln(1-x_1-x_2)\\
&-\mathrm{Li}_{1,2}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right)-\mathrm{Li}_{2,1}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right)\Bigg)+(x_1\leftrightarrow x_2)\,. \qquad (11)
\end{aligned}
$$

We have made the following checks on our result. As remarked, the infrared poles agree with the structure predicted by Catani [14]. This provides a strong check of the complete pole structure of our result. In addition, we have tested various relations between the c_i. For instance, the combination x_1c_6 is symmetric under exchange of x_1 with x_2. Finally, we could compare with the result for the squared matrix elements, *i.e.* the interference of the two-loop amplitude with the Born amplitude, and the interference of the one-loop amplitude with itself. The results of [5] are given in terms of one- and two-dimensional harmonic polylogarithms, which form a subset of the multiple polylogarithms [10]. Thus, we have performed the comparison analytically and we agree with the results of [5].

3. Conclusions

Our result represents one contribution to the full next-to-next-to-leading order calculation of $e^+e^- \to$ 3-jets. It has been obtained by means of an efficient method based on nested sums and is expressed in terms of multiple polylogarithms with simple arguments. As a consequence, our result can be continued analytically and applies also to $(2+1)$-jet production in deep-inelastic scattering or to the production of a massive vector boson in hadron-hadron collisions. At the same time, it provides an important cross check on the results for the squared matrix elements [5] with a completely independent method.

After the results of Section 2 had been presented at this conference, Garland *et al.* published results for the complete two-loop amplitude for $e^+e^- \to q\bar{q}g$. Our results are in agreement with Ref. [16].

REFERENCES

[1] G. Dissertori, `hep-ex/0105070`.

[2] F.A. Berends, W.T. Giele, H. Kuijf, *Nucl. Phys.* **B321**, 39 (1989); K. Hagiwara, D. Zeppenfeld, *Nucl. Phys.* **B313**, 560 (1989).

[3] Z. Bern, L. Dixon, D.A. Kosower, S. Weinzierl, *Nucl. Phys.* **B489**, 3 (1997); Z. Bern, L. Dixon, D.A. Kosower, *Nucl. Phys.* **B513**, 3 (1998).

[4] E.W.N. Glover, D.J. Miller, *Phys. Lett.* **B396**, 257 (1997); J.M. Campbell, E.W.N. Glover, D.J. Miller, *Phys. Lett.* **B409**, 503 (1997).

[5] L.W. Garland, T. Gehrmann, E.W.N. Glover, A. Koukoutsakis, E. Remiddi, *Nucl. Phys.* **B627**, 107 (2002).

[6] S. Moch, P. Uwer, S. Weinzierl, *J. Math. Phys.* **43**, 3363 (2002).

[7] S. Moch, P. Uwer, S. Weinzierl, talk given at the Europhysics Conference on HEP, Budapest, July 2001, `hep-ph/0110407`.

[8] O.V. Tarasov, *Phys. Rev.* **D54**, 6479 (1996); *Nucl. Phys.* **B502**, 455 (1997); C. Anastasiou, E.W.N. Glover, C. Oleari, *Nucl. Phys.* **B575**, 416 (2000), Erratum *Nucl. Phys.* **B585**, 763 (2000).

[9] J.A.M. Vermaseren, *Int. J. Mod. Phys.* **A14**, 2037 (1999); J. Blümlein, S. Kurth, *Phys. Rev.* **D60**, 014018 (1999); `hep-ph/9810241`.

[10] A.B. Goncharov, *Math. Res. Lett.* **5**, 497 (1998); J.M. Borwein, D.M. Bradley, D.J. Broadhurst, P. Lisonek, `math.CA/9910045`; E. Remiddi, J.A.M. Vermaseren, *Int. J. Mod. Phys.* **A15**, 725 (2000).

[11] J.A.M. Vermaseren, `math-ph/0010025`.

[12] C. Bauer, A. Frink, R. Kreckel, `cs/0004015`.

[13] S. Weinzierl, *Comput. Phys. Commun.* **145**, 357 (2002).

[14] S. Catani, *Phys. Lett.* **B427**, 161 (1998).

[15] R.K. Ellis, D.A. Ross, A.E. Terrano, *Nucl. Phys.* **B178**, 421 (1981).

[16] L.W. Garland, T. Gehrmann, E.W.N. Glover, A. Koukoutsakis, E. Remiddi, `hep-ph/0206067`.