LOCAL INVARIANTS IN EFFECTIVE HYDRODYNAMICS OF TRAPPED DILUTE-GAS BOSE–EINSTEIN CONDENSATES

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In the framework of mean-field approximation the dynamics of Bose–Einstein condensates can be described by the hydrodynamic-like equations. These equations are analyzed here with account of mutual interaction between condensate and non-condensate atoms. The Lagrange invariants and freezing-in invariants of such a system have been found. This allows to get some necessary conditions for creation of an atom laser with controlled parameters of the beam. Particularly, the atom laser beam can carry quite well defined angular momentum. This can be practically realized in the most simple case, when the vorticity of condensate appears to be a freezing-in field. The optimal conditions for a source mode regime are found.

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1. Introduction

One of the youngest, but very rapidly developing field of modern low temperature physics is the problem of trapped Bose gases at very low temperatures, when Bose–Einstein condensation occurs. For recent reviews of this topic in general see Refs. [1–4]. In last few years a considerable progress has been achieved in the understanding of the dynamics of Bose–Einstein Condensates (BEC) both at $T = 0$ [1] and at finite temperatures [2, 5, 10]. However, many questions yet remain in the latter case (see discussion in [11]). To realize the full potential of recent developments in BEC physics, and to analyze adequately the data of the experiments in a quantitatively meaningful manner, a detailed understanding of the BEC dynamics in all its aspects is required.

A detailed investigation of the dynamics (and, particularly, collective processes) of BEC at various temperatures is extremely important in view of the possible applications of unique features of such systems. In all likelihood the most intriguing experimental project associated with trapped atomic gases is the so-called "atom laser" or, in other words, highly coherent atomic beam generator [12, 13] (for a review see also [4]). The proposed experimental configurations should satisfy a number of basic criteria in order to be called an atom laser. Of course, the high phase coherence of atomic beam is required first of all. However, the highly dispersive nature of the BEC suggests that the spatial focusing and stability of cross section of a BEC beam will present possibly more of a problem than encountered in the process of focusing laser light [14]. Very recently the quasi-continuous atom laser has been constructed [15]. Furthermore, the same group demonstrated the successful atom optical manipulation, such as reflection, focusing and the beam storage in a resonator [16] (however, see the earlier successful works [41, 42] based on rather different principle). Thus, another interesting task appears, namely how to create the atom laser with well controlled characteristics. Solution of such a problem can bring atom laser closer to be a useful tool in various potential applications. The present paper solves one of the problems in this direction. Particularly, we consider the possibility for atom laser beam to carry the *predesigned* angular momentum.

The paper is organized as follows. In Section 2 the quasi-hydrodynamic approach to a trapped Bose-gas below the BEC transition temperature is considered. Section 3 is devoted to the description of the method for construction of Lagrange invariants and freezing-in fields by means of the gauge transformation of the BEC quasi-hydrodynamic equations. The applications of such invariants to the creation of atom laser beam with predesigned parameters are considered in Section 4.

2. Quasi-hydrodynamic equations for the condensate atoms

The starting point in description of BEC is the usual Heisenberg equation of motion for the quantum field operator $\hat{\psi}(\boldsymbol{r},t)$

$$\begin{aligned} i\hbar\frac{\partial\hat{\psi}(\boldsymbol{r},t)}{\partial t} &= \left[\hat{\psi}(\boldsymbol{r},t),\hat{\mathcal{H}}\right] \\ &= \left\{-\frac{\hbar^2}{2m}\Delta+U^{(\mathrm{trap})}(\boldsymbol{r})\right\}\hat{\psi}(\boldsymbol{r},t)+g\left|\hat{\psi}(\boldsymbol{r},t)\right|^2\hat{\psi}(\boldsymbol{r},t)\,, \end{aligned} \quad (1)$$

where $U^{(\mathrm{trap})}(\boldsymbol{r})$ is the confining potential, the explicit form of which is not essential for us here. In equation (1) we assumed s-wave short-range interatomic interaction with a strength $g = 4\pi a h^2/m$ (a is the effective scattering length). As usually, set

$$\hat{\psi}(\boldsymbol{r},t) = \Phi(\boldsymbol{r},t) + \tilde{\psi}(\boldsymbol{r},t)\,, \tag{2}$$

where $\Phi(\boldsymbol{r},t) = \langle\hat{\psi}(\boldsymbol{r},t)\rangle$, $\tilde{\psi}(\boldsymbol{r},t)$ is the non-condensate field operator. Taking an average of equation (1) with respect to a broken symmetry non-equilibrium ensemble, we come to the equation for the condensate wave function [10]

$$\begin{aligned} i\hbar\frac{d\Phi(\boldsymbol{r},t)}{dt} &= \left\{-\frac{\hbar^2}{2m}\Delta + U^{(\mathrm{trap})}(\boldsymbol{r}) + gn_c(\boldsymbol{r},t) + 2g\tilde{n}(\boldsymbol{r},t)\right\}\Phi(\boldsymbol{r},t) \\ &\quad + g\tilde{m}(\boldsymbol{r},t)\Phi^*(\boldsymbol{r},t) + g\langle\tilde{\psi}^+(\boldsymbol{r},t)\tilde{\psi}(\boldsymbol{r},t)\tilde{\psi}(\boldsymbol{r},t)\rangle\,, \end{aligned} \tag{3}$$

where $n_c(\boldsymbol{r},t) = |\Phi(\boldsymbol{r},t)|^2$ is the local density of atoms in the condensate,

$$\tilde{n}(\boldsymbol{r},t) = \langle\tilde{\psi}^+(\boldsymbol{r},t)\tilde{\psi}(\boldsymbol{r},t)\rangle \tag{4}$$

is the non-equilibrium non-condensate density. Equation (4) involves also the anomalous non-condensate density $\tilde{m}(\boldsymbol{r},t) = \langle\tilde{\psi}(\boldsymbol{r},t)\tilde{\psi}(\boldsymbol{r},t)\rangle$ and the three-field correlation function $\langle|\tilde{\psi}(\boldsymbol{r},t)|^2\tilde{\psi}(\boldsymbol{r},t)\rangle$. The appearance of the last two terms in (3) is a consequence of Bose broken symmetry in the system.

The earlier approaches to the equation (3) were based on the assumption that all atoms are in the condensate. In this case the so-called Gross–Pitaevskii [17] equation appears:

$$i\hbar\frac{d\Phi(\boldsymbol{r},t)}{dt} = \left\{-\frac{\hbar^2}{2m}\Delta + U^{(\mathrm{trap})}(\boldsymbol{r}) + gn_c(\boldsymbol{r},t)\right\}\Phi(\boldsymbol{r},t)\,, \tag{5}$$

which can be conveniently rewritten in terms of the local condensate density

$$n_c(\boldsymbol{r},t) \;=\; |\Phi(\boldsymbol{r},t)|^2 \tag{6a}$$

and local velocity

$$\boldsymbol{v}_c(\boldsymbol{r},t) \;=\; \frac{\hbar}{m}\nabla\theta(\boldsymbol{r},t)\,. \tag{6b}$$

Here $\theta(\boldsymbol{r},t)$ is the phase of the condensate wave function

$$\Phi(\boldsymbol{r},t) = \sqrt{n_c(\boldsymbol{r},t)}\exp\left(i\theta(\boldsymbol{r},t)\right)\,. \tag{7}$$

It should be noted that the analogous nonlinear Shrödinger equation can be obtained quite rigorously in the case of high density, but weak enough point interaction [18]. However, in that case the physical sense of Φ is not clear enough. So, using (6) and (7), we can present equation (5) as the set of two following equations

$$\frac{\partial n_c}{\partial t} + \nabla(n_c \boldsymbol{v_c}) \; = \; 0\,, \tag{8a}$$

$$\left\{\frac{\partial}{\partial t} + (\boldsymbol{v_c}\boldsymbol{\nabla})\right\}\boldsymbol{v_c} \; = \; -\frac{\nabla\mu_0}{m}\,, \tag{8b}$$

where

$$\mu_0 = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{n_c}}{\sqrt{n_c}} + U^{(\mathrm{trap})}(\boldsymbol{r}) + g n_c(\boldsymbol{r},t)\,.$$

It is remarkable that equations (8) are hydrodynamic looking. This fact is somewhat confusing on the face of it. Really, equations similar to (8) where obtained phenomenologically to describe a superfluid component of liquid helium [19, 20] which is a *strongly interacting* many-particle system. Nevertheless, we should keep in mind that in Bose–Einstein condensed state we deal with the single condensate wave function $\varPhi(\boldsymbol{r},t)$, which allows a strong analogy with the order parameter in super-fluids[1].

The next step is to extend the preceding analysis to finite temperatures where there is a large fraction of atoms outside of the condensate. In this case we need two equations to be used. While the condensate wave function can be described as earlier by the Gross–Pitaevskii equation, the distribution function of the non-condensate atoms obeys a kinetic equation, which must take into account the collisions of both types: non-condensate atoms with each other and their interaction with a condensate. The quite rigorous derivation of the corresponding collision integrals can be found in Appendix A of paper [10]. After some mathematics we come to the corresponding hydrodynamic-like equations

$$\frac{\partial n_c}{\partial t} + \nabla(n_c \boldsymbol{v}_c) \; = \; \varGamma(\boldsymbol{r},t)\,, \tag{9a}$$

$$\left\{\frac{\partial}{\partial t} + (\boldsymbol{v}_c\nabla)\right\}\boldsymbol{v_c} \; = \; -\frac{\nabla\mu}{m}\,, \tag{9b}$$

where

$$\varGamma(\boldsymbol{r},t) = -\int \frac{d^3p}{(2\pi\hbar)^3} J\left[f(\boldsymbol{p},\boldsymbol{r},t)\right]\,. \tag{10}$$

[1] Note, there is no complete microscopic theory of super-fluids until now. We only know that the dynamics of superfluid component in super-fluids can be well described by the hydrodynamic equations similar to (8).

$J\left[f(\boldsymbol{p},\boldsymbol{r},t)\right]$ is the collision integral corresponding to the collisions between condensate and non-condensate atoms, which functionally depends on the distribution function $f(\boldsymbol{p},\boldsymbol{r},t)$ of excited atoms. So, function $\Gamma(\boldsymbol{r},t)$ is the characteristic rate of the atoms exchange between condensate and non-condensate. New chemical potential μ in equation (9b) is now defined by the relation

$$\mu = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{n_c}}{\sqrt{n_c}} + U^{(\mathrm{trap})}(\boldsymbol{r}) + gn_c(\boldsymbol{r},t) + 2g\tilde{n}(\boldsymbol{r},t)\,, \tag{11}$$

where

$$\tilde{n}(\boldsymbol{r},t) = -\int \frac{d^3p}{(2\pi\hbar)^3} f(\boldsymbol{p},\boldsymbol{r},t)$$

is the density of non-condensate atoms. Equations (9) describe the dynamics of BEC like an "effective fluid" with varying density. Term $\Gamma(\boldsymbol{r},t)$ play the role of inhomogeneous and nonstationary source.

3. Hydrodynamic invariants

As we realized in the previous section the evolution of BEC in the frame of reasonable approximations can be described by the equations, which are similar to the hydrodynamic ones. Classical equations of ideal liquid have quite a number of invariants. Except of ordinary integral invariants there are local invariants as well. The Lagrange invariants and freezing-in invariants are most important ones. Lagrange invariants are conserved along the "liquid particles" trajectories, while the freezing-in invariants are used in reference to the fields frozen into a liquid, *i.e.* the corresponding physical quantity (field) vanishes in a frame which moves with the fluid. In papers [21–23] a wide class of invariants was found. Furthermore, the authors of Refs. [21, 22] proposed the method of obtaining new invariants on the basis of already known. The recent paper [24] was devoted to construction of the invariants of superfluid hydrodynamic equations by means of their gauge transformation [25]. This method is very attractive because after the gauge transformation the presence of many additional invariants becomes obvious.

The idea of gauge transformation [24, 25] can be modified to be helpful in our case, *i.e.* BEC dynamics. Really, the local condensate velocity is defined by the equation (6). If we wish the condensate wave function to be single-valued, then the bypassing along vortex line must lead to the change of a phase by the integer of 2π. To be so, we should do a cut. If the leap of phase on the bank of cut is proportional to a new function, say $\boldsymbol{u}_c$, then we can do the following gauge transformation

$$\boldsymbol{v}_c = -\nabla\theta + \nabla\alpha + \boldsymbol{u}_c\,, \tag{12}$$

where α is a gauge function. Gauge of the fields should be done by the equation for $\nabla\theta - \nabla\alpha$ and by the initial conditions. After the substitution of equation (12) into (9b) we get

$$\left\{\frac{\partial}{\partial t} + (\boldsymbol{v}_c\nabla)\right\} u_{ci} = -\frac{\partial}{\partial x_i}\left\{\mu + \frac{\partial}{\partial t}(\alpha - \theta)\right\} - (\boldsymbol{v}_c\nabla)\left[\frac{\partial\alpha}{\partial x_i} - \frac{\partial\theta}{\partial x_i}\right] - u_{cj}\frac{\partial v_{cj}}{\partial x_i} + \left\{v_{cj} + \frac{\partial}{\partial x_j}(\theta - \alpha)\right\}\frac{\partial v_{cj}}{\partial x_i}\,. \tag{13}$$

It can be easily tested that if the gauge function obeys the following equation

$$\left\{\frac{\partial}{\partial t} + (\boldsymbol{v}_c\nabla)\right\}(\theta - \alpha) = \mu - \frac{1}{2}v_c^2\,, \tag{14}$$

then equation (13) becomes

$$\left\{\frac{\partial}{\partial t} + (\boldsymbol{v}_c\nabla)\right\} u_{ci} = -u_{cj}\frac{\partial v_{cj}}{\partial x_i}\,. \tag{15}$$

The gauge of field $\boldsymbol{u}_c$ is determined by equation (14) and by the initial condition for θ or $\boldsymbol{u}_c$. The scalar product of the field $\boldsymbol{u}_c$ and the flux line element $\delta\boldsymbol{l}$ behaves like a mass element, which is conserved along any trajectory [24]. Direct test shows that

$$\left\{\frac{\partial}{\partial t} + (\boldsymbol{v}_c\nabla)\right\}(\boldsymbol{u}_c\delta\boldsymbol{l}) = 0\,. \tag{16}$$

So, the quantity $\boldsymbol{u}_c\delta\boldsymbol{l}$ is the Lagrange invariant. It should be noted that in this case the vorticity $\boldsymbol{w} = \mathrm{rot}\boldsymbol{v}_c$ becomes freezing-in field [26]. Using the analogy with superfluid hydrodynamics we believe that in BEC $\mathrm{rot}\boldsymbol{v}_c = 0$ everywhere except the axes of vortices. So, we come to the following conclusion: if there were vortices in the BEC initially and the invariant $\boldsymbol{u}_c\delta\boldsymbol{l}$ is conserved, then there will be the given conserved vorticity in any frame moving with the flux lines in future.

The possible existence of vortices in BEC has been under extensive discussion for a rather long time (see *e.g.* [28–34]). And finally, they were recently obtained in the experiments [35–39].

In the next section we consider the consequences of this conclusion for possible experimental realization of an atom laser.

4. The stability of atom laser beam

A number of atom laser schemes have been proposed during the last few years. Evident progress is already achieved in the realization of pulsed lasers using a matter-wave splitter based on radio frequency (rf) transitions [15,40] and optical Raman transitions [41,42]. Such schemes, however, have several shortcomings the main of which is the difficulties in achieving a continuous refilling. Another schemes, which allow to create the continuous wave atom laser can be clearly divided into two distinct classes: optical cooling [43–45] and evaporative cooling [46–49]. Both models are based, in principle, on the same idea: a source supplies atoms to an upper-lying mode of an atom trap. This source mode is coupled to the ground state mode (condensate) via a particular cooling mechanism. It is hoped that the macroscopic population in this ground state mode, or laser mode, can be built up and coupled to outside world to produce the laser output. Independently, on particular model, cooling process or, in other words, process of increase of the atom population in the ground state mode must satisfy the main criterion: the uncontrolled perturbation of ground state atoms should be minimal.

I shall not limit myself to the frame of the particular experiments (even successful, such as [41, 42] or [15, 16]). Below both the possible situations are considered.

4.1. Continuous models

In the previous section we found that the BEC can have Lagrange invariants. Let us assume that initially all atoms in the trap are in condensate. So, equation (9a) contains the quantity $\Gamma(\boldsymbol{r},t) = \nu_p(\boldsymbol{r},t)$, which is just the rate of pumping, *i.e.* the rate of increase of the population in ground state due to the cooling of atoms from upper-lying mode. Let us find the conditions for $\nu_p(\boldsymbol{r},t)$, under which the pumping of the ground state mode does not break the flux lines (*i.e.* the all local invariants remain to be conserved). As it follows from equation (15), $\boldsymbol{u}_c\delta\boldsymbol{l}$ is always a Lagrange invariant (see equation (16)) if the gauge condition (14) is satisfied. However, if the flux lines are broken so that a vorticity changes, then the condition (14) is necessarily broken as well as $\boldsymbol{u}_c\delta\boldsymbol{l}$ becomes non-invariant. So, we come to the simple conclusion: *to conserve a given vorticity we must keep the regime of pumping to be such one to do the condition (14) being always valid.* Obviously, the most simple requirement is $\theta = \alpha$, or as it follows from equation (14)

$$\mu(\boldsymbol{r},t) = \tfrac{1}{2}v_c^2(\boldsymbol{r},t)\,. \tag{17}$$

Furthermore, this requirement automatically means that $\boldsymbol{u}_c = \boldsymbol{v}_c$ and the invariant $\boldsymbol{v}_v\delta\boldsymbol{l}$ contains the velocity itself.

For simplicity consider the situation, when the velocity of laser mode changes only in given direction. In this case, using equations (9a), (11), and (17), we obtain

$$\frac{\partial n_c}{\partial t}+\nu_p(\boldsymbol{r},t)+\nabla\left\{\sqrt{2}n_c\boldsymbol{e}_v\sqrt{U^{(\mathrm{trap})}(\boldsymbol{r})+g(n_c+2\tilde{n})-\frac{\hbar^2}{2m}\frac{\Delta\sqrt{n_c}}{\sqrt{n_c}}}\right\}=0\,, \tag{18}$$

where $\boldsymbol{e}_v=\boldsymbol{v}_c/|\boldsymbol{v}_c|$. Result (18) gives the connection between experimentally controllable quantities n_c, $\tilde{n}$, $\nu_p(\boldsymbol{r},t)$, and $U^{(\mathrm{trap})}(\boldsymbol{r})$. Of course, this equation should further be solved numerically for particular experimental configurations. Note that, of course, the relation (18) does not solve all problems of atom laser beam stability, but it gives very useful tool for making a choice of the parameters of experimental setup. If the condition (18) is satisfied, then at least the problem of angular momentum transfer is solved in the frame of the made approximations.

4.2. Pulsed models

Here we consider the models of pulsed (not continuously refilled) atom laser on the example of rf-transition scheme [15, 16, 40]. In this scheme the output coupler includes resonant monochromatic radio frequency field transferring atoms in some hyperfine state F from the trapped into untrapped magnetic sublevels. In the case of ^{23}Na atoms $F=1$, so that $s=-1$ corresponds to the trapped state, $s=0$ and $s=1$ corresponds to the untrapped and the repelled sublevels, respectively, (here F is the total angular momentum, s is the magnetic quantum number). Equation (5) now becomes [50]

$$i\hbar\frac{d\tilde{\Phi}_s(\boldsymbol{r},t)}{dt}=\left\{-\frac{\hbar^2}{2m}\Delta+\hbar s\omega_{\mathrm{rf}}U_s^{(\mathrm{trap})}(\boldsymbol{r})+gn_s(\boldsymbol{r},t)\right\}\tilde{\Phi}_s(\boldsymbol{r},t)$$
$$+\hbar\Omega\sum_{s'}(\delta_{s,s'+1}+\delta_{s,s'-1})\tilde{\Phi}_{s'}(\boldsymbol{r},t)\,, \tag{19}$$

where in rotating wave approximation

$$\tilde{\Phi}_s(\boldsymbol{r},t)=\mathrm{e}^{-is\omega_{\mathrm{rf}}t}\langle\hat{\psi}_s(\boldsymbol{r},t)\rangle,\qquad s,s'\in\{-1,0,+1\}\,, \tag{20}$$

$\hat{\psi}_s(\boldsymbol{r},t)$ is the quantum field operator for atoms belong the sublevel with given s, ω_{rf} is the frequency of applied resonant rf field, $n_s=|\tilde{\Phi}_s|^2$. The coupling constant

$$\hbar\,\Omega=g\mu_{\mathrm{Bohr}}\frac{|B|}{\sqrt{2}} \tag{21}$$

refers to the Rabi frequency due to the rf field. For a small coupling strength the process of atoms leaking out of the resonance points is faster than the Rabi oscillations. So, we further neglect the coupling into state $s=+1$ since it is proportional to Ω^4. Using such approximation and writing the relation (7) for each sublevel s we get for the density of atoms in a laser beam n_0 the following relation, analogous to the formula (9a):

$$\frac{\partial n_0}{\partial t} + \nabla(n_0 \boldsymbol{v}_0) = 2\Omega\sqrt{n_c n_0}\sin(\theta_c - \theta_0), \tag{22}$$

where $n_c \equiv n_{-1}$. Equation (22) has a clear physical sense: variation of the atom beam density oscillates due to the differences of the condensate and beam phases.

Equation similar to (9b) looks

$$\left\{\frac{\partial}{\partial t} + (\boldsymbol{v}_0\nabla)\right\}\boldsymbol{v_0} = -\frac{\nabla\tilde{\mu}_0}{m}\,, \tag{23}$$

where

$$\tilde{\mu}_0 = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{n_0}}{\sqrt{n_0}} + U_0^{(\mathrm{trap})}(\boldsymbol{r}) + gn_0(\boldsymbol{r},t) + \hbar\Omega\sqrt{\frac{n_c}{n_0}}\cos(\theta_c - \theta_0) \tag{24}$$

is the new chemical potential. From equations (17), (22), (23), and (24) we easily obtain the condition similar to (18):

$$\frac{\partial n_0}{\partial t} + \nabla\left\{\sqrt{2}n_0\boldsymbol{e}_v\sqrt{U_0^{(\mathrm{trap})}(\boldsymbol{r}) + gn_0(\boldsymbol{r},t) - \frac{\hbar^2}{2m}\frac{\nabla\sqrt{n_0}}{\sqrt{n_0}} + \hbar\Omega\sqrt{\frac{n_c}{n_0}}\cos(\theta_c - \theta_0)}\right\}$$
$$= 2\Omega\sqrt{n_0 n_c}\sin(\theta_c - \theta_0)\,. \tag{25}$$

Note, as it can be seen from the condition (25) the temporal change of the beam atoms population depends on the phase difference $(\theta_c - \theta_0)$, which is determined by the frequency ω_{rf}.

Direct comparison of the formulae (18) and (25) shows that the outcome (*i.e.* laser beam itself) in the continuous case can be stabilized easier than in the pulsed regime.

In conclusion, using the quasi-hydrodynamic approximations we have found the local invariants of Bose–Einstein condensate in trapped alkali gases. Particularly we obtained the Lagrange invariant, which ensures the vorticity to be a freezing-in field. The obtained results can be directly applied to the creation of highly coherent atomic beam generators (atom lasers) with well controlled angular momentum. Both the pulsed laser and laser with continuous refilling are considered. The optimal conditions for the pumping modes have been found (again for the both schemes).

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