# MATRIX REPRESENTATION OF THE GENERALIZED MOYAL ALGEBRA

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It is shown that the isomorphism between the generalized Moyal algebra and the matrix algebra follows in a natural manner from the generalized Weyl quantization rule and from the well known matrix representation of the annihilation and creation operators.

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This short note is motivated by Merkulov's paper "The Moyal product is the matrix product" [1], where the canonical isomorphism between the Moyal algebra and an infinite matrix algebra has been found.

Here we are going to show how the results of previous works [2–4] and the well known in quantum mechanics [5,6] representation of the position $\widehat{x}$ and the momentum $\widehat{p}$ operators lead to isomorphisms between various $*$-algebras and infinite matrix algebra.

First remind the basic theorems [3,4].

Let $P[[x,p,\hbar]]$ be the $\mathbb{C}$ linear space of all formal power series of $x$, $p$ and $\hbar$ where $(x,p)\in\mathbb{R}\times\mathbb{R}$ are the coordinates of the phase space $\Gamma=\mathbb{R}\times\mathbb{R}$ and $\hbar$ is a real parameter (the deformation parameter). The phase space $\Gamma=\mathbb{R}\times\mathbb{R}$ is endowed with usual symplectic form

$$\omega = dp\wedge dq\,. \tag{1}$$

Let also $\widehat{P}[[\widehat{x},\widehat{p},\hbar]]$ be an associative algebra over $\mathbb{C}$ of the formal power series of $\widehat{x}$, $\widehat{p}$, $\hbar\widehat{1}$. The self-adjoint operators $\widehat{x}$ and $\widehat{p}$ act in a Hilbert space $\mathcal{H}$ and satisfy the commutation relation

$$[\widehat{x},\widehat{p}] := \widehat{x}\widehat{p} - \widehat{p}\widehat{x} = i\hbar\widehat{1}\,. \tag{2}$$

As usual, $\widehat{1}$ denotes the unity operator. $\widehat{P}[[\widehat{x},\widehat{p},\hbar]]$ is the enveloping algebra of the Heisenberg–Weyl algebra generated by $\widehat{x}$, $\widehat{p}$, $\hbar\widehat{1}$.

The following theorem holds [3,4]

**Theorem 1** *There exists a vector space isomorphism*

$$W_g : P[[x,p,\hbar]] \longrightarrow \widehat{P}[[\widehat{x},\widehat{p},\hbar]]\,,$$

*such that*

$$\text{(i)} \quad W_g(1) = \widehat{1}\,, \quad W_g(p^m x^n) = \sum_{s=0}^{\min(m,n)} g(m,n,s)\hbar^s \widehat{p}^{m-s}\widehat{x}^{n-s},$$
$$m,n \in N, \quad m+n \neq 0, \quad g(m,n,s) \in \mathbb{C}, \quad g(m,n,0) = 1\,,$$

$$\text{(ii)} \quad i\hbar W_g\left(\{x,A\}_{\mathcal{P}}\right) = [\widehat{x}, W_g(A)]\,, \qquad i\hbar W_g\left(\{p,A\}_{\mathcal{P}}\right) = [\widehat{p}, W_g(A)]\,,$$

*for every* $A \in P[[x,p,\hbar]]$, *with* $\{\cdot,\cdot\}_{\mathcal{P}}$ *denoting the Poisson bracket.*

*Moreover, every isomorphism* $W_g : P[[x,p,\hbar]] \longrightarrow \widehat{P}[[\widehat{x},\widehat{p},\hbar]]$ *satisfies the conditions (i) and (ii) if and only if*

$$g(m,n,s) = \frac{(-1)^s\, m!n!}{s!\,(m-s)!\,(n-s)!}\left.\frac{d^s f(y)}{dy^s}\right|_{y=0}, \tag{3}$$

*where* $f(y) = \sum_{k=0}^{\infty} f_k y^k$, $f_0 = 1$, *is a formal series independent of* $\hbar$. ■

(Of course, one can easily recognize in the conditions *(ii)* of Theorem 1, the modified Dirac quantization rules.)

Then, the second theorem reads [4]

**Theorem 2** *Let* $W_g : P[[x,p,\hbar]] \longrightarrow \widehat{P}[[\widehat{x},\widehat{p},\hbar]]$ *be the vector space isomorphism defined in Theorem 1.*

*Then for any* $A, B \in P[[x,p,\hbar]]$

$$W_g(A)\,W_g(B) = W_g(A *_g B)\,, \tag{4}$$

*where*

$$
\begin{aligned}
A *_g B &= \widehat{\alpha}^{-1}\left[(\widehat{\alpha}A) * (\widehat{\alpha}B)\right] , \\
\widehat{\alpha} &:= \alpha\left(-\hbar\frac{\partial^2}{\partial x\partial p}\right) = f\left(-\hbar\frac{\partial^2}{\partial x\partial p}\right)\exp\left\{\frac{i}{2}\left(-\hbar\frac{\partial^2}{\partial x\partial p}\right)\right\}
\end{aligned}
\quad (5)
$$

*and "∗" stands for the usual Moyal product*

$$
\begin{aligned}
A * B &= A\exp\left\{\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}}\right\}B , \\
A\overleftrightarrow{\mathcal{P}}B &:= \{A,B\}_{\mathcal{P}} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} . \qquad \blacksquare
\end{aligned}
\quad (6)
$$

It can be also shown that $W_g(A)$ is a symmetric operator for every real $A \in P[[x,p,\hbar]]$ if and only if, the formal series $\alpha = \alpha(y) = f(y)\exp\left\{\frac{i}{2}y\right\}$ is real.

In terms of $\alpha$ we have

$$
g(m,n,s) = \left(\frac{i}{2}\right)^s \frac{m!n!}{(m-s)!(n-s)!}\sum_{k=0}^{s}\frac{(2i)^k}{(s-k)!}\alpha_k , \quad (7)
$$

where $\alpha_k$ are defined by

$$
\alpha(y) = \sum_{k=0}^{\infty}\alpha_k y^k, \qquad \alpha_0 = 1 . \quad (8)
$$

Now we introduce the well known in quantum mechanics operators, $\widehat{a}$ ("the annihilation operator") and its Hermitian conjugate $\widehat{a}^\dagger$ ("the creation operator") such that

$$
\begin{aligned}
\widehat{x} = \frac{1}{2}\left(\widehat{a}^\dagger + \widehat{a}\right) , \qquad \widehat{p} = i\hbar\left(\widehat{a}^\dagger - \widehat{a}\right) , \\
\left[\widehat{a}, \widehat{a}^\dagger\right] = 1 .
\end{aligned}
\quad (9)
$$

It is an easy matter to show that

$$
\begin{aligned}
\widehat{x} &= \exp\left\{\frac{1}{2}\left(\widehat{a}^\dagger\right)^2\right\}\exp\left\{\frac{1}{4}\widehat{a}^2\right\}\widehat{a}^\dagger\exp\left\{-\frac{1}{4}\widehat{a}^2\right\}\exp\left\{-\frac{1}{2}\left(\widehat{a}^\dagger\right)^2\right\} , \\
\widehat{p} &= \exp\left\{\frac{1}{2}\left(\widehat{a}^\dagger\right)^2\right\}\exp\left\{\frac{1}{4}\widehat{a}^2\right\}(-i\hbar\widehat{a})\exp\left\{-\frac{1}{4}\widehat{a}^2\right\}\exp\left\{-\frac{1}{2}\left(\widehat{a}^\dagger\right)^2\right\} .
\end{aligned}
\quad (10)
$$

Therefore, one can define an algebra isomorphism

$$L := \widehat{P}[[\widehat{x},\widehat{p},\hbar]] \longrightarrow \widehat{P}[[\widehat{a}^{\dagger},-i\hbar\widehat{a},\hbar]]\,,$$

by

$$L\left(\widehat{x}\right)=\widehat{a}^{\dagger} \quad \text{and} \quad L\left(\widehat{p}\right)=-i\hbar\widehat{a}\,. \tag{11}$$

Consequently, by Theorem 1 and Theorem 2 we obtain the algebra isomorphism

$$L\circ W_g : P[[x,p,\hbar]] \longrightarrow \widehat{P}[[\widehat{a}^{\dagger},-i\hbar\widehat{a},\hbar]]\,,$$

$$L\circ W_g\left(p^m x^n\right)=\sum_{s=0}^{\min(m,n)}\frac{(-\hbar)^s\, m!n!}{s!\,(m-s)!\,(n-s)!}\left.\frac{d^s f\left(y\right)}{dy^s}\right|_{y=0}(-i\hbar\widehat{a})^{m-s}\left(\widehat{a}^{\dagger}\right)^{n-s},$$

$$\begin{aligned}\left(L\circ W_g\left(A\right)\right)\left(L\circ W_g\left(B\right)\right) \;&=\; L\circ W_g\left(A *_g B\right)\,,\\ A,B \;&\in\; P[[x,p,\hbar]]\,.\end{aligned} \tag{12}$$

Now, employing the standard matrix representation of $\widehat{a}$ and $\widehat{a}^{\dagger}$ [5,6]

$$\widehat{a}\longmapsto a=\begin{pmatrix} 0 & 1 & 0 & 0 & \dots\\ 0 & 0 & \sqrt{2} & 0 & \dots\\ 0 & 0 & 0 & \sqrt{3} & \dots\\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$\widehat{a}^{\dagger}\longmapsto a^{\dagger}=\begin{pmatrix} 0 & 0 & 0 & \dots\\ 1 & 0 & 0 & \dots\\ 0 & \sqrt{2} & 0 & \dots\\ 0 & 0 & \sqrt{3} & 0\\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{13}$$

and substituting the matrices $a$ and $a^{\dagger}$ instead of $\widehat{a}$ and $\widehat{a}^{\dagger}$, respectively, into (12) one finds the algebra isomorphism $\widetilde{W_g}$ between the generalized Moyal algebra $(P[[x,p,\hbar]],*_g)$ and the matrix algebra $P[[a^{\dagger},-i\hbar a,\hbar]]$.

Denote $F^{(m,n)} := (-i\hbar a)^m\left(a^{\dagger}\right)^n$. Simple calculations lead to the following non vanishing elements of the matrices $F^{(m,n)}\quad(m+n>0)$

$$\begin{aligned}\left(F^{(m,0)}\right)_{j,j+m} \;&=\; (-i\hbar)^m\sqrt{j\left(j+1\right)\dots\left(j+m-1\right)}\,,\\ \left(F^{(0,n)}\right)_{j+n,j} \;&=\; \sqrt{j\left(j+1\right)\dots\left(j+n-1\right)}\,,\end{aligned}$$

$$\begin{aligned}
\left(F^{(m,n)}\right)_{j,j+m-n} &= (-i\hbar)^m (j+m-n) \\
&\quad ...(j+m-1)\sqrt{j(j+1)...(j+m-n-1)}\,, \\
\text{for} \quad m &> n > 0; \\
\left(F^{(m,m)}\right)_{j,j} &= (-i\hbar)^m j(j+1)...(j+m-1)\ , \\
\left(F^{(m,n)}\right)_{j+n-m,j} &= (-i\hbar)^m (j+n-m) \\
&\quad ...(j+n-1)\sqrt{j(j+1)...(j+n-m-1)}\,, \\
\text{for} \quad n &> m > 0\,.
\end{aligned} \tag{14}$$

Finally, we have

$$\widetilde{W}_g\left(p^m x^n\right) = \sum_{s=0}^{\min(m,n)} \frac{(-\hbar)^s\, m!n!}{s!\,(m-s)!\,(n-s)!} \left.\frac{d^s f(y)}{dy^s}\right|_{y=0} F^{(m-s,n-s)}\,. \tag{15}$$

This formula corresponds to Merkulov's result but in slightly different representation and in our case we deal with generalized Moyal products $*_g$.

**Examples**

(1) *The Moyal $*$-algebra*

It is well known that this algebra is induced by the Weyl ordering of operators [2–4]. In this case the operator $\widehat{\alpha} = 1$. Hence, by (5)

$$f(y) = \exp\left\{-\frac{i}{2}y\right\} \Longrightarrow \left.\frac{d^s f(y)}{dy^s}\right|_{y=0} = \left(-\frac{i}{2}\right)^s$$

and we get now (the index "g" is omitted)

$$\widetilde{W}\left(p^m x^n\right) = \sum_{s=0}^{\min(m,n)} \frac{(i\hbar)^s\, m!n!}{2^s s!\,(m-s)!\,(n-s)!} F^{(m-s,n-s)}\,, \tag{16}$$

(compare with Merkulov's result).

(2) *The $*_{(\mathrm{st})}$-algebra*

This algebra follows from the standard ordering

$$p^m x^n \longmapsto \widehat{x}^n \widehat{p}^m.$$

Here $\alpha(y) = \exp\left\{-\frac{i}{2}y\right\}$ . Hence,

$$f(y) = \exp\{-iy\} \Longrightarrow \left.\frac{d^s f(y)}{dy^s}\right|_{y=0} = (-i)^s \ .$$

Consequently

$$\widetilde{W}_{st}\left(p^m x^n\right) = \sum_{s=0}^{\min(m,n)} \frac{(i\hbar)^s\, m!n!}{s!\,(m-s)!\,(n-s)!} F^{(m-s,n-s)} \, . \tag{17}$$

(3) *The* $*_{(\mathrm{ast})}$*-algebra*

This is the algebra which follows from the anti-standard ordering

$$p^m x^n \longmapsto \widehat{p}^m \widehat{x}^n .$$

Now $\alpha(y) = \exp\left\{\frac{i}{2}y\right\}$ . Hence $f(y) = 1$ and it remains only one term with $s = 0$ in (15).

Hence,

$$\widetilde{W}_{\mathrm{ast}}\left(p^m x^n\right) = F^{(m,n)} \, , \tag{18}$$

(compare with Merkulov's paper [1]).

(4) *The* $*_{(\mathrm{sym})}$*-algebra*

Here we deal with the algebra generated by the symmetric ordering. So one has $\alpha(y) = \cos(y/2)$ . Therefore,

$$f(y) = \frac{1}{2}\Big(1 + \exp\{-iy\}\Big) \Longrightarrow \left.\frac{d^s f(y)}{dy^s}\right|_{y=0} = \frac{1}{2}\left(\delta_{s,0} + (-i)^s\right) \, .$$

Consequently

$$\widetilde{W}_{\mathrm{sym}}\left(p^m x^n\right) = F^{(m,n)} + \sum_{s=1}^{\min(m,n)} \frac{(i\hbar)^s\, m!n!}{2(s!)\,(m-s)!\,(n-s)!} F^{(m-s,n-s)}. \tag{19}$$

Finally we consider

(5) *The* $*_{\mathrm{BJ}}$*-algebra*

This algebra follows from the Born–Jordan ordering.

Now $\alpha(y) = (\sin(y/2))/(y/2)$. Therefore,

$$f(y) = \frac{1}{iy}\Big(1 - \exp\{-iy\}\Big) \Longrightarrow \left.\frac{d^s f(y)}{dy^s}\right|_{y=0} = \frac{(-i)^s}{s+1} \, .$$

Hence,

$$\widetilde{W}_{\rm BJ}\left(p^m x^n\right) = \sum_{s=0}^{\min(m,n)} \frac{(i\hbar)^s\, m!n!}{(s+1)!\,(m-s)!\,(n-s)!}\, F^{(m-s,n-s)}\,.$$

**Final comments**

The results presented here correspond to a generalization of the matrix representation found by Merkulov [1]. However, as was pointed out to us by Zachos, the idea of an isomorphism between Moyal and matrix algebras has a long history which started with the distinguished work by Groenewold [7] (see also [8]). Some new insight into this problem was given by Fairlie and Zachos [9] and by Fairlie, Fletcher and Zachos [10]. But, of course, in such a short note as ours, we are not able to deal with all these problems. Some applications of a matrix representation of the Moyal algebra have been discussed in our previous work [11].

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