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FORCED DYNAMICAL SYSTEMS DERIVABLE FROM BOHMIAN MECHANICS

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Using Bohm's quantum mechanics, a wide class of related two-parameter dynamical systems is proposed and their general properties are briefly discussed, in particular, a possibility of chaotic solutions. When the systems are reduced to a one-parameter family of equations then they are all proved to be completely integrable and integrals of the motion are found in an explicite form. The proposed class of dynamical systems can be cast into the form of Hamiltonian equations forced by a time-dependent non-Hamiltonian, periodic in time, disturbance. A systematic way of generating dynamical systems of this kind is also discussed.

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1. Introduction

A dynamical system is understood as a set of ordinary nonlinear differential equations of the first order, $\boldsymbol{v} = \dot{\boldsymbol{r}} = \boldsymbol{F}(\boldsymbol{r}, t, \boldsymbol{c})$, where the variables $\boldsymbol{c}$ are called control parameters. The set of equations represents a conservative system if $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ or a dissipative one if $\boldsymbol{\nabla} \cdot \boldsymbol{v} < 0$. The most widely discussed low-dimensional dynamical systems with chaotic behaviour are the Lorenz [1] and Rössler [2] models and the Duffing [3] and Van der Pol [4] oscillators. The latter pair belongs to the class of so-called forced dynamical systems. All the systems are intrinsically inequivalent since, among other things, they cannot be deformed into each other.

In what follows we shall derive a class of equivalent conservative systems forced by a non-Hamiltonian, periodic in time, perturbation. Their interesting feature is that for the consecutive members of the class separatrices of growing complexity can be observed. Then, even a small perturbation leads to a chaotic behaviour manifested by the increasing in value largest Lyapunov exponents.

The velocity field formula we shall use here is that given by the Bohmian mechanics [5] or equivalently by the hydrodynamical formulation of nonrelativistic quantum mechanics [6]. In one-particle case described by the wave function $\psi(\boldsymbol{r},t) = R(\boldsymbol{r},t)\exp[(i/\hbar)S(\boldsymbol{r},t)]$ it reads as

$$\boldsymbol{v}^{\psi} = \dot{\boldsymbol{r}} = \frac{\boldsymbol{j}}{|\psi|^2} = \frac{i\hbar}{2m}\frac{\psi\nabla\psi^* - \psi^*\nabla\psi}{|\psi|^2} = \frac{1}{m}\boldsymbol{\nabla}S, \tag{1}$$

where ψ is a solution of the time-dependent Schrödinger equation.

Trajectories following from the guidance formula (1) were utilized in numerous applications as important as: a description of the delayed-choice experiment on the basis of two-slit interference [7], the measurement problem in quantum mechanics [8], the geometric phase [9, 10], calculations of tunneling times [11], the quantum cosmology [12], the problem of identical motion in classical and quantum description [13, 14] or correlations in the motion of identical particles [15, 16].

There is also a large group of papers (see *e.g.* [17–21]) devoted to the study of properties of dynamical systems derivable from Eq.(1). One should remember, however, that the trajectories of Eq.(1) form a highly non-classical velocity field, contrary to the purely classical trajectories discussed in the usual dynamical systems, like those mentioned above. We should also emphasize that the Bohmian systems are neither conservative nor dissipative and their generic feature is that the phase space volumes are not conserved by the flow, *i.e.* $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ is not generally equal to zero. Instead, we can expect [22] vanishing of $\lim_{T\to\infty}\int_0^T \boldsymbol{\nabla}\cdot\boldsymbol{v}d\tau$. The systems are additionally much more difficult to deal with than any other usual ones. It follows from the shape of Eq. (1) and from the fact that wave functions usually have nodes and the vortices around them are quantized [23, 24].

Until quite lately, the problem of whether a "simple" system of equations, belonging to a known class of dynamical systems, can be constructed via Eq. (1) and properly chosen wave functions, was long unsolved. Very recently, we were successful [25] in creating a model of conservative system ($\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$), with some disturbance periodic in time, which manifested chaotic behaviour and the properties of the classical forced dynamical systems that we were looking for.

The question underlying the present paper, is how unique is that model and whether our method of deriving it can be generalized, so as to obtain similar ones. We shall show below that the model is the simplest member [26] of a family of forced Bohmian dynamical systems, the only one known so far for causal trajectories. Our previous study [25] suggests using for this purpose the wave function in the following special form:

$$\psi_N(x,y,t) = \mathrm{e}^{-it}\psi_0(x)\psi_0(y) + \mathrm{e}^{-it(N+1)} \sum_{\substack{n,k=0 \\ (n+k=N)}}^{N} a_{nk}\psi_n(x)\psi_k(y), N = 1, 2, \cdots, \tag{2}$$

where ψ_0 and ψ_j stand for the ground and excited states of oscillator in one dimension, and henceforth the dimensionless units are introduced such that $\hbar = 1$, $m = 1$ and $\omega = 1$. The normalization constant of $\psi_N(x,y,t)$ is omitted since it plays no role in further considerations. Two spatial coordinates in equation (2) are enough to get a set of two non-autonomous equations from (1) which in accordance with the celebrated Poincaré–Bendixson theorem is a necessary condition for chaotic solutions to exist. Our wave function (2) is composed of two stationary states, the second one with $N+1$-fold degeneracy, and energies $E_1 = 1$ and $E_2 = N+1$, respectively. Note, the sum in (2) is over n and k such that $n + k = N$.

The plan of our paper is as follows. In Section 2 we derive equations of the motion for the non-trivial and yet as simple as possible wave functions, show their integrability and transform the system to a Hamiltonian autonomous one. In Section 3 we then introduce some perturbations to the integrable equations and show the way of creating systems with growing degree of chaoticity in their solutions. Finally, the conclusions are given in Section 4.

2. Integrable systems

The case we shall study first is a set of two non-autonomous equations which can be formally proved to be integrable. To this end, the 1D oscillator wave functions are used and thus equation (2) can be written as

$$\psi_N(x,y,t) = \mathrm{e}^{-(1/2)(x^2+y^2)}\left[\mathrm{e}^{-it} + \mathrm{e}^{-it(N+1)}\sum_{k=0}^{N} a_{N-k,k}H_{N-k}(x)H_k(y)\right], \tag{3}$$

where $H_j(z)$ is the Hermite polynomial of order j. If the free expansion coefficients $a_{N-k,k}$ are all real then the equations resulting from (1) are obviously integrable. The situation is much more interesting if at least one of the coefficients for a given N, is chosen as an imaginary quantity. Then, one can always adjust $a_{N-k,k}$'s in such a way to have the sum in (3) equal to

$$\sum_{k=0}^{N} a_{N-k,k}H_{N-k}(x)H_k(y) = a^N(x+iy)^N.$$

This can be proved with the help of an integral representation of the Hermite polynomials [27]. For example, for $N = 1$ this is the case for $a_{10} = (1/2)a$ and $a_{01} = ia/2$, and for $N = 2$, we have $a_{20} = a^2/4$, $a_{02} = -a^2/4$, $a_{11} = ia^2/2$, and so on.

Now, we can write the function (3) in the polar form and its phase

$$S_N = -\tan^{-1}\left(\frac{\sin t + a^N r^N \sin\left[(N+1)t - N\varphi\right]}{\cos t + a^N r^N \cos\left[(N+1)t - N\varphi\right]}\right) \tag{4}$$

is then used in Eq. (1). As a result, we get

$$\begin{aligned} \dot{x} &= \frac{-a^N N r^{N-1}\{\sin\left[Nt - (N-1)\varphi\right] + a^N r^N \sin\varphi\}}{[\cos Nt + a^N r^N \cos N\varphi]^2 + [\sin Nt + a^N r^N \sin N\varphi]^2}, \\ \dot{y} &= \frac{a^N N r^{N-1}\{\cos\left[Nt - (N-1)\varphi\right] + a^N r^N \cos\varphi\}}{[\cos Nt + a^N r^N \cos N\varphi]^2 + [\sin Nt + a^N r^N \sin N\varphi]^2}, \end{aligned} \tag{5}$$

where $r^2 = x^2 + y^2$ and $\varphi = \tan^{-1}(y/x)$. The dynamical system (5) has a single control parameter, represented by a, and it can be proved to be completely integrable. The simplest way to do that is using the transformation $x = r\cos D$, $y = r\sin D$ with $D = N(t - \varphi)$. Thus, the non-autonomous equations (5) can be simplified to the autonomous ones and no chaotic solutions are possible in this case.

Equations (5) have two interesting features: a constant of the motion exists

$$C_N = M_N - a^{2N}\ln M_N - 2a^N r^N \cos\left[N(t-\varphi)\right] + a^{2N}(r^2 - r^{2N}) \tag{6}$$

for an arbitrary integer $N \geq 1$, where $M_N = 1 + a^{2N} r^{2N} + 2a^N r^N \cos\left[N(t-\varphi)\right]$, and the phase space volume is conserved, *i.e.*, $d\dot{x}/dx + d\dot{y}/dy = 0$. The latter property is a consequence of the fact that there must be $\Delta S_N = 0$ which in the 2D space has a general solution $S_N(x,y,t) = f_N(x+iy,t) + g_N(x-iy,t)$ for arbitrary functions f_N and g_N of their arguments.

We can also find a function $\tilde{H}(x,y,t)$ such that $\dot{x} = -\partial\tilde{H}/\partial y$ and $\dot{y} = \partial\tilde{H}/\partial x$. It has a simple form of $\tilde{H}(x,y,t) = (1/2)\ln M_N$. Moreover, we can propose a transformation which transforms the conservative non-autonomous system (5) to a Hamiltonian autonomous one. It has the following form

$$X = \rho\cos\alpha, \quad Y = \rho\sin\alpha, \tag{7}$$

with

$$\begin{aligned} \rho^2 &= X^2 + Y^2 = M_N, \\ \alpha &= Nt - \tan^{-1}\left(\frac{\sin Nt + \operatorname{Im} z^N}{\cos Nt + \operatorname{Re} z^N}\right) = \tan^{-1}\left(\frac{Y}{X}\right), \quad z = ax + iby. \end{aligned} \tag{8}$$

In the new variables with $b = a$, equations (5) take the form of a Hamiltonian system

$$\begin{aligned} \dot{X} &= -NY\left(1 - \frac{Na^2}{\rho^2}\right), \\ \dot{Y} &= -N + NX\left(1 - \frac{Na^2}{\rho^2}\right) \end{aligned} \tag{9}$$

derivable from the Hamiltonian

$$H = \frac{N}{2}(X^2 + Y^2) - \frac{1}{2}N^2 a^2 \ln{(X^2 + Y^2)} - NX. \tag{10}$$

Again, of course, the phase space volume is conserved, *i.e.*, now we have $d\dot{X}/dX + d\dot{Y}/dY = 0$. For each N the set (9) has a pair of fixed points

$$\begin{aligned} Y_1 &= 0, \;\; X_1 = \frac{1}{2}(1 + \sqrt{1 + 4Na^2}), \\ Y_2 &= 0, \;\; X_2 = \frac{1}{2}(1 - \sqrt{1 + 4Na^2}), \end{aligned} \tag{11}$$

the first always being the elliptic fixed point and the second a hyperbolic one. The relative separation between the points is growing with the number of the degenerate states used in (3). We can also find separatrices for the particular values of N. Denoting $H(Y = 0, X = X_2) = H_S$ we have the required equation

$$H_S - \frac{N}{2}(X^2 + Y^2) + \frac{1}{2}N^2 a^2 \ln{(X^2 + Y^2)} + NX = 0. \tag{12}$$

3. Non-integrable perturbed systems

The main conclusion we can draw from the previous section is that the one-parameter dynamical systems proposed here are all completely integrable. The simplest generalization of the above approach, leading to non-integrable equations and hence possibly to chaos, is introducing a second control parameter, say b.

To this end, we shall represent the whole sum over k in (3) as

$$\sum_{k=0}^{N} a_{N-k,k} H_{N-k}(x) H_k(y) = G_N(ax + iby)$$

with two real constants a and b. Unfortunately, this time, when $a \neq b$, the coefficients $a_{N-k,k}$ cannot in any way be adjusted to have $G_N(ax + iby)$

equal to $(ax+iby)^N$. Instead, it is possible to obtain:

$$\begin{aligned} G_1 &= ax+iby, \\ G_2 &= (ax+iby)^2+\frac{1}{2}\Delta, \quad \Delta=b^2-a^2, \\ G_3 &= (ax+iby)^3+\frac{3}{2}\Delta(ax+iby), \\ G_4 &= (ax+iby)^4+3\Delta(ax+iby)^2+\frac{3}{4}\Delta^2. \end{aligned} \tag{13}$$

We were not able to find a general formula for G_N. However, the next members of the set (13) can be obtained with a little effort.

Now, the phase S_N of the function (3)

$$S_N(x,y,t)=-\tan^{-1}\left(\frac{\sin t+\operatorname{Re}G_N\sin[t(N+1)]-\operatorname{Im}G_N\cos[t(N+1)]}{\cos t+\operatorname{Re}G_N\cos[t(N+1)]+\operatorname{Im}G_N\sin[t(N+1)]}\right) \tag{14}$$

generates with the help of (1) the following non-integrable dynamical systems:

$$\begin{aligned} \dot{x} &= \frac{-1}{Q_N}\Big[(\sin Nt+\operatorname{Im}G_N)(\operatorname{Re}G_N)_x-(\cos Nt+\operatorname{Re}G_N)(\operatorname{Im}G_N)_x\Big], \\ \dot{y} &= \frac{-1}{Q_N}\Big[(\sin Nt+\operatorname{Im}G_N)(\operatorname{Re}G_N)_y-(\cos Nt+\operatorname{Re}G_N)(\operatorname{Im}G_N)_y\Big], \end{aligned} \tag{15}$$

where the subscripts x and y denote derivatives of the market expressions with respect to these variables, and

$$Q_N=(\sin Nt+\operatorname{Im}G_N)^2+(\cos Nt+\operatorname{Re}G_N)^2. \tag{16}$$

Equations (15) and (16) hold not only for G_N's listed in (13) but generally for the quantities with all integers $N \geq 1$.

One may cast (15) into the form of conservative equations perturbed by some periodic time-dependent contributions. This is a very tedious task for the particular functions $G_N(ax+iby)$. Nevertheless, it can be done with the help of the transformation (7), where now ρ and α should read as:

$$\begin{aligned} \rho^2 &= X^2+Y^2=Q_N, \\ \alpha &= Nt-\tan^{-1}\left(\frac{\sin Nt+\operatorname{Im}G_N}{\cos Nt+\operatorname{Re}G_N}\right)=\tan^{-1}\left(\frac{Y}{X}\right). \end{aligned} \tag{17}$$

Equations (17) and (8) are equivalent to each other in two cases: for any integer $N \geq 1$ when $b = a$, and for $b \neq a$ when $N = 1$. The transformation (7) with both (8) or (17) is a canonical transformation only for $N = 1$ (precisely when $\rho \to -\rho/ab$ in (7)) and as such it does not change the properties of the studied dynamical system. This point has been discussed in detail in [25]. Hence, for $N > 1$ and $b \neq a$ there is no need to derive equations in the large X and Y coordinates. Therefore, we shall only write down here the equations in the simplest case of $N = 1$. Then, using (17) or (8) for $N = 1$, we get from (7) and (15)

$$\begin{aligned} \dot{X} &= -Y\left(1 - \frac{a^2+b^2}{2\rho^2}\right) + \frac{b^2-a^2}{2\rho^2}\left(X\sin 2t - Y\cos 2t\right), \\ \dot{Y} &= -1 + X\left(1 - \frac{a^2+b^2}{2\rho^2}\right) - \frac{b^2-a^2}{2\rho^2}\left(X\cos 2t + Y\sin 2t\right). \end{aligned} \quad (18)$$

For the particular functions G_N, $N \geq 1$, equations for $\dot{X}$ and $\dot{Y}$ will all have the similar structure with, however, much more complicated perturbation terms. Their common feature is that all the equations will have for $b = a$ the form of (9) and that can be proved with the help of equations (17).

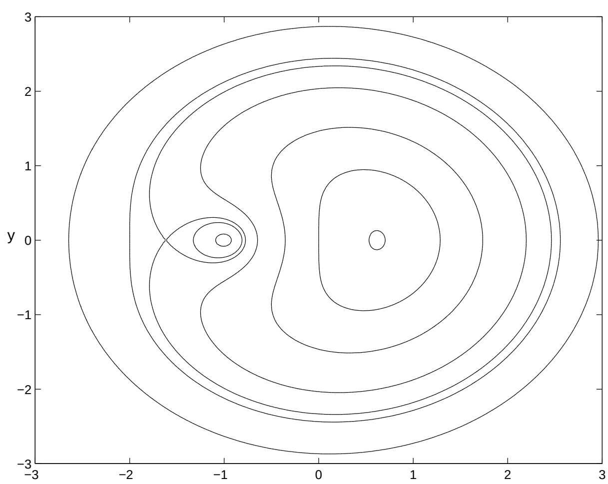

Fig. 1. Contour map of the period $t = 2\pi$ of the orbits from (6) or (15) with $a = b = 1$ and $N = 1$.

In order to show that solutions of (15) may exhibit deterministic chaos three maps of the period $Nt = 2\pi$ have been prepared for $N = 1, 2, 3$ and $b = a$. In the simplest case of $N = 1$, we can clearly observe in figure 1 the homoclinic orbit crossing itself in a hyperbolic point. The position of an elliptic fixed point can easily be traced out in the middle of the large loop of the orbit. With the growing values of N increasing complexity of (15) results in an appearance of additional hyperbolic fixed points and the separatrices get more sophisticated in shape. This can be observed in figure 2 for $N = 2$ and in figure 3 for $N = 3$.

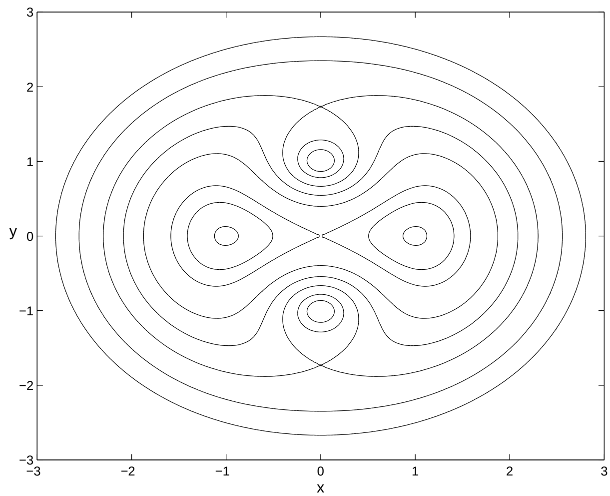

Fig. 2. As in figure 1 but for $N = 2$ and $2t = 2\pi$. One hetero- and one homoclinic orbits are clearly visible.

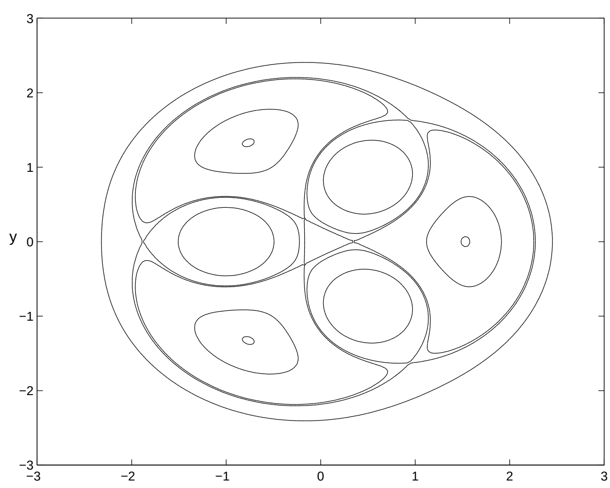

Fig. 3. As in figure 1 but for $N = 3$ and $3t = 2\pi$. One can find here two different separatrices encircling elliptic fixed points.

Now, one may generate here a chaotic behavior even if the parameters a and b differ slightly. To prove that we have calculated the largest Lyapunov exponents λ_N for three values of N for the orbits corresponding to the perturbed homoclinic connections presented in the above figures. In each case $a = 1$, $b = 1.1$ and the initial conditions are: $x(0) = 2.46$, $y(0) = 0$ for $N = 1$; $x(0) = 2.3$, $y(0) = 0$ for $N = 2$ and $x(0) = 0.35$, $y(0) = 0$ for $N = 3$. We have obtained $\lambda_1 = 0.05$, $\lambda_2 = 0.07$ and $\lambda_3 = 0.13$ and this is what we expected. Thus, the formation and break-up of a homo- and heteroclinic trajectories seems to be the cause of chaos for the proposed here class of dynamical systems. A more detailed description of the onset of chaos and the whole route to it will possibly be determined after a systematic study of a few members of the class is conducted.

4. Conclusion

A procedure has been developed that shows a possible way of generating a wide class of dynamical systems. It is based on using the formula (1) and then a suitably chosen wave function. In our study it is composed of two stationary states, one of which has to be at least double degenerate and some of the expansion coefficients in (2) have to be imaginary quantities. Thanks to that, we can create a set of integrable systems and a chaotic behavior is expected when the systems are perturbed by periodic time-dependent terms resulting from introducing the second control parameter. We thus have at our disposal a class of similar systems, the degree of chaoticity of which is to some extent controlable by the number of used degenerate states. The preliminary numerical results we have presented here seem to be interesting enough to study the proposed systems in more detail and systematically in future.

The class of dynamical systems proposed here, is the only one derived so far within the Bohmian mechanics for which known properties of classical forced systems can also be observed for the quantum Bohmian trajectories. Deriving of any similar model seems to be very difficult, if possible at all, since the choice of ψ in Eq. (1) was a rather special one, making our class of systems to some extent unique. The model we have studied in detail recently [25] is the simplest (for $N = 1$) member of the large family of models considered here.

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