NULL AND TENSILE STRINGS IN PERES SPACETIME

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We present the general equations of motion for null and tensile strings in the Peres spacetime and give the general solutions in quadratures in the case of null strings. As for the tensile strings we integrate equations of motion and constraints completely in a closed form for circular and straight string Ansäts.

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1. Introduction

The study of the string equations of motion and constraints in a curved background spacetime has been a topic of an active research over the last few years. Since the equations are nonlinear, it is often quite difficult to obtain exact solutions in a variety of curved backgrounds [1–6]. In papers [7, 20] it was shown that the string equations in the Schwarzschild spacetime are actually non-integrable and exhibit chaotic behavior. It means that it is only possible to find the exact evolution for some special configurations or perform some numerical calculations [3, 7, 8].

Among the various peculiarities of the behavior of strings at the Hagedorn temperature there is the fact that the effective tension vanishes. Strings with vanishing tension were introduced by Schild [9], who considered a generalization of the Nambu–Goto Lagrangian to a more generic action principle. Schild gave them the name "null strings" because their world surface is a null sub-manifold. As it is known, the string tension appears in the Nambu–Goto Lagrangian (as well as in the generalization that is associated with the name of Polyakov) as an overall multiplicative constant and, therefore, cannot be set to zero without the loss of the whole physical content of the theory.

The classical evolution of a null string in a curved background is described by null geodesic equations of general relativity appended by an additional "stringy" constraint. The exact null string configurations were studied in flat and curved backgrounds [10–13, 19]. A very interesting exact solution which describes a null string moving vertically up and down around the photon sphere in the Schwarzschild spacetime was presented by Dabrowski and Larsen [14]. Another reason why it could be interesting to study tensile and null strings is that tensile and null strings can be viewed as classical sources of gravity [15].

In this paper we present general equations of motion for tensile and null strings in the Peres spacetime and we give general solutions of motion completely in a closed form for circular strings.

2. Null strings in the Peres spacetime

The Nambu–Goto string action in a curved spacetime can be presented in the form [16, 17]

$$S = S_0 + S_1 = \int d\tau d\sigma \left[\frac{\det\left(\partial_\mu x^M G_{MN}(x) \partial_\nu x^N\right)}{E(\tau,\sigma)} - \frac{1}{(\alpha')^2} E(\tau,\sigma) \right] , \quad (1)$$

where $E(\tau,\sigma)$ is an auxiliary world-sheet density, $M, N, ... = 0, 1, ..., D-1$; $\mu, \nu = 0, 1$ and $\partial_0 \equiv \partial/\partial\tau$, $\partial_1 \equiv \partial/\partial\sigma$. The equation of motion for $E(\tau,\sigma)$ produced by Eq. (1) is

$$E = \alpha' \sqrt{-\det g_{\mu\nu}} \,, \quad (2)$$

$$g_{\mu\nu} = \partial_\mu x^M G_{MN}(x) \partial_\nu x^N \,. \quad (3)$$

The substitution of $E(\tau,\sigma)$ from Eq. (2) into functional (1) transforms the latter into the Nambu–Goto representation

$$S = -\frac{2}{\alpha'} \int d\tau d\sigma \sqrt{-\det g_{\mu\nu}} \,. \quad (4)$$

Thus, the representations (1) and (4) for the string action (1) are classically equivalent. Unlike the representation (4), the representation (1) includes the string tension parameter $1/\alpha'$ as a constant at an additive world–sheet "cosmological" term playing the role of the potential energy. This term may be considered as a perturbative addition for the case of a weak tension.

We are to consider a dimensional parameter γ or some combination of the parameters defining the metric of the curved space, where the string moves. Without loss of generality one can put that γ has the dimension of L^2 ($\hbar = c = 1$). Then the value of the dimensionless combination

$$\varepsilon = \frac{\gamma}{\alpha'} \tag{5}$$

can be considered as a parameter characterizing the power of string tension. In the gauge

$$E(\tau, \sigma) = -\gamma \left(x^M_{,\sigma} G_{MN}(x) x^N_{,\sigma} \right) \tag{6}$$

accompanied by the ortho-normality condition

$$x^M_{,\tau} G_{MN}(x) x^N_{,\sigma} = 0\,, \tag{7}$$

where $x^M_{,\tau} \equiv \partial x^M / \partial \tau$, $x^M_{,\sigma} \equiv \partial x^M / \partial \sigma$, the variational Euler–Lagrange equations of motion generated by Eq. (1) acquire the form

$$x^M_{,\tau\tau} - \varepsilon^2 x^M_{,\sigma\sigma} + \Gamma^M_{PQ}(G) \left[x^P_{,\tau} x^Q_{,\tau} - \varepsilon^2 x^P_{,\sigma} x^Q_{,\sigma} \right] = 0\,, \tag{8}$$

and they contain the dimensionless parameter ε. This parameter appears in another string constraint

$$x^M_{,\tau} G_{MN}(x) x^N_{,\tau} + \varepsilon^2 x^M_{,\sigma} G_{MN}(x) x^N_{,\sigma} = 0\,, \tag{9}$$

which is additional to constraint (7). For $\varepsilon = 0$ we have the null string. From the above we can see that for the null strings we have the null geodesic equations supplemented by the constraint (7), which ensures that each point of a tensile string (null string) propagates in the direction perpendicular to the string. Thus, knowing the null geodesics in a background spacetime would naturally lead to null string configurations provided all the constraints are satisfied.

Let us now discuss the null string propagation in the Peres spacetime [21, 22]. The metric for such spacetime is represented as

$$dS^2 = 2dtdx - dy^2 - dz^2 + S(t, y, z)dt^2\,. \tag{10}$$

From Einstein–Hilbert equations we have two different solutions:

$$dS^2 = 2dtdx - dy^2 - dz^2 + b(y^2 + z^2)dt^2\,, \tag{11}$$

$$dS^2 = 2dtdx - dy^2 - dz^2 + b(y^2 - z^2)dt^2\,, \tag{12}$$

where b is a constant [18]. Eq. (11) describes radiation field with an isotropic energy-momentum tensor. In the second case (see Eq. (12)) we have spacetime which describes strong gravitational waves.

The null string equations of motion and constraints in the spacetime (11) are given by

$$\begin{aligned} t_{,\tau\tau} &= 0\,, \\ x_{,\tau\tau} + 2b(yy_{,\tau} + zz_{,\tau})t_{,\tau} &= 0\,, \qquad (13) \\ y_{,\tau\tau} + byt^2_{,\tau} &= 0\,, \\ z_{,\tau\tau} + bzt^2_{,\tau} &= 0\,, \qquad (14) \\ b(y^2+z^2)t^2_{,\tau} - y^2_{,\tau} - z^2_{,\tau} + 2t_{,\tau}x_{,\tau} &= 0\,, \qquad (15) \\ b(y^2+z^2)t_{,\tau}t_{,\sigma} - y_{,\tau}y_{,\sigma} - z_{,\tau}z_{,\sigma} + t_{,\tau}x_{,\sigma} + t_{,\sigma}x_{,\tau} &= 0\,. \qquad (16) \end{aligned}$$

Eqs. (13)–(16) easily integrate

$$t(\tau,\sigma) = p_t(\sigma)\tau + t(\sigma)\,, \qquad (17)$$

$$x(\tau,\sigma) = x(\sigma) + g(\sigma)\sin\left(a(\sigma)\tau\right) + w(\sigma)\cos\left(a(\sigma)\tau\right)\,, \qquad (18)$$

$$x^i(\tau,\sigma) = x^i(\sigma)\cos\left(\frac{a(\sigma)\tau}{2}\right) + \frac{2p^i(\sigma)}{a(\sigma)}\sin\left(\frac{a(\sigma)\tau}{2}\right)\,, \qquad (19)$$

where $i = 2,3$ $(x^2 = y,\ x^3 = z,\ p^2 = p_y,\ p^3 = p_z)$,

$$g(\sigma) = \frac{\sqrt{b}}{a^2(\sigma)}\Big\{p_y^2(\sigma) + p_z^2(\sigma) - cp_t^2(\sigma)\left[y^2(\sigma) + z^2(\sigma)\right]\Big\}\,, \qquad (20)$$

$$w(\sigma) = \frac{1}{2p_t(\sigma)}\Big\{y(\sigma)p_y(\sigma) + z(\sigma)p_z(\sigma)\Big\}\,,$$

$$a(\sigma) = 2\sqrt{b}\,p_t(\sigma)\,, \qquad (21)$$

and $t(\sigma)$, $x(\sigma)$, $x^i(\sigma)$, $p_t(\sigma)$, $p_x(\sigma)$, $p^i(\sigma)$ are any functions of σ. The explicit form of the solutions (17)–(19) allows to transform the constraints (13), (14) into those for the Cauchy initial data:

$$2p_t(\sigma)p_x(\sigma) - p_y^2(\sigma) - p_z^2(\sigma) + bp_t^2(\sigma)\left[y^2(\sigma) + z^2(\sigma)\right] = 0\,, \qquad (22)$$

$$\begin{aligned} &2p_x(\sigma)p_t(\sigma)t'(\sigma) + 2p_t^2(\sigma)x'(\sigma) - p_t(\sigma)\left[y'(\sigma)p_y(\sigma) + z'(\sigma)p_z(\sigma)\right] \\ &+p_t(\sigma)\left[y(\sigma)p_y'(\sigma) + z(\sigma)\ p_z'(\sigma)\right] - p_t'(\sigma)\left[y(\sigma)\ p_y(\sigma) + z(\sigma)p_z(\sigma)\right] = 0\,, \qquad (23) \end{aligned}$$

where the primes denote differentiation with respect to σ.

For example, we can consider

$$t(\sigma) = 0\,, \qquad x(\sigma) = 0\,, \qquad y(\sigma) = R\sin\frac{\sigma}{R}\,, \qquad z(\sigma) = R\cos\frac{\sigma}{R}\,,$$

$$p_t(\sigma) = 1\,, \qquad p_x(\sigma) = -\frac{bR^2}{2}\,, \qquad p_y(\sigma) = p_y(\sigma) = 0\,. \qquad (24)$$

This configuration describes a circular null string. Combining Eq. (24) with Eqs. (17)–(19) we obtain

$$x = -\frac{\sqrt{b}\,R^2}{4}\sin\left(2\sqrt{b}\,t\right), \qquad y^2+z^2 = R^2\cos^2\left(\sqrt{b}\,t\right). \tag{25}$$

From Eqs. (25) one can see that radiation field with isotropic energy-momentum tensor play the role of a trap. Fig. 1 shows a world sheet of the null string Eqs. (25).

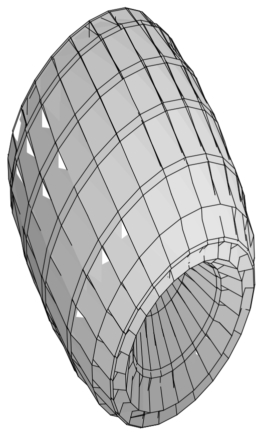

Fig. 1. World-sheet of the null string in the radiation field with an isotropic energy-momentum tensor.

The null string equations of motion for metric Eq. (12) are analogous to Eqs. (13)–(16) and one can easily solve them to get

$$t(\tau,\sigma) = p_t(\sigma)\tau + t(\sigma)\,, \tag{26}$$

$$\begin{aligned} x(\tau,\sigma) = {} & x(\sigma) + \frac{1}{4}\left[\sqrt{b}\,x^2(\sigma) + \frac{p_z^2(\sigma)}{\sqrt{b}\,p_t^2(\sigma)}\right]\sinh\left(2\sqrt{b}\,p_t(\sigma)\tau\right) \\ & -\frac{1}{4}\left[\sqrt{b}\,y^2(\sigma) + \frac{p_y^2(\sigma)}{\sqrt{b}\,p_t^2(\sigma)}\right]\sin\left(2\sqrt{b}\,p_t(\sigma)\tau\right) + \frac{1}{2p_t(\sigma)} \\ & \times\left[z(\sigma)p_z(\sigma)\cosh\left(2\sqrt{b}\,p_t(\sigma)\tau\right) + y(\sigma)p_y(\sigma)\cos\left(2\sqrt{b}\,p_t(\sigma)\tau\right)\right], \end{aligned} \tag{27}$$

$$y(\tau,\sigma) = y(\sigma)\cos\left(\sqrt{b}\,p_t(\sigma)\tau\right) + \frac{p_y(\sigma)}{\sqrt{b}\,p_t(\sigma)}\sin\left(\sqrt{b}\,p_t(\sigma)\tau\right), \tag{28}$$

$$z(\tau,\sigma) = z(\sigma)\cosh\left(\sqrt{b}\,p_t(\sigma)\tau\right) + \frac{p_z(\sigma)}{\sqrt{b}\,p_t(\sigma)}\sinh\left(\sqrt{b}\,p_t(\sigma)\tau\right). \tag{29}$$

Using the explicit form of the solutions Eqs. (26)–(29), we obtain the constraints for the Cauchy initial data:

$$2p_t(\sigma)p_x(\sigma) - p_y^2(\sigma) - p_z^2(\sigma) + bp_t^2(\sigma)\Big[y^2(\sigma) - z^2(\sigma)\Big] = 0\,, \tag{30}$$

$$\begin{aligned} &2p_x(\sigma)p_t(\sigma)t'(\sigma) + 2p_t^2(\sigma)x'(\sigma) + p_t(\sigma)\Big[y(\sigma)p_y'(\sigma) + z(\sigma)p_z'(\sigma)\Big] \\ &-2p_y(\sigma)\left(y(\sigma)p_t(\sigma)\right)' - 2p_z(\sigma)\left(z(\sigma)p_t(\sigma)\right)' = 0\,. \end{aligned} \tag{31}$$

For example, we can consider

$$t(\sigma) = x(\sigma) = y(\sigma) = z(\sigma) = 0\,, \tag{32}$$

$$p_t(\sigma) = 1\,, \qquad p_x(\sigma) = \frac{1}{2}\,, \qquad p_y(\sigma) = \sin\sigma\,, \qquad p_z(\sigma) = \cos\sigma\,.$$

Finally, substituting Eq. (32) into Eqs. (26)–(29), one obtains

$$x = \frac{1}{4\sqrt{b}}\left[\cos^2\sigma\sinh\left(2\sqrt{b}\,t\right) + \sin^2\sigma\sin\left(2\sqrt{b}\,t\right)\right], \tag{33}$$

$$y = \frac{1}{\sqrt{b}}\sin\sigma\sin\left(\sqrt{b}\,t\right), \qquad z = \frac{1}{\sqrt{b}}\cos\sigma\sinh\left(\sqrt{b}\,t\right)\,. \tag{34}$$

Fig. 2 shows world-sheet of the null string — Eqs. (33), (34).

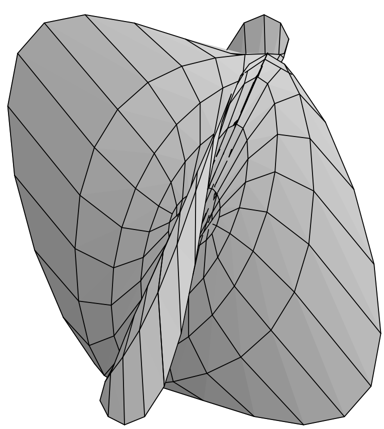

Fig. 2. World-sheet of the null string in the field of strong gravitational waves.

3. Tensile strings in Peres background

In this section we briefly consider the case of tensile strings ($\varepsilon = 1$) and start with the circular Ansätz

$$t = \tau\,, \qquad x = x(\tau)\,, \qquad y = f(\tau)\cos\sigma\,, \qquad z = f(\tau)\sin\sigma\,. \tag{35}$$

We then find from Eqs. (7)–(9) and (11) the following set of ordinary differential equations for $x(\tau)$ and $f(\tau)$:

$$f_{,\tau\tau} + (b+1)f = 0\,, \tag{36}$$

$$x_{,\tau} = \frac{1}{2}\left[(f_{,\tau})^2 + (1-b)f^2\right]. \tag{37}$$

The set of ordinary differential equations (36), (37) can be easily integrated and we have

$$t = \tau, \qquad y = f(\tau)\cos\sigma, \qquad z = f(\tau)\sin\sigma, \tag{38}$$

$$x = \frac{1}{2}\Big(A^2 + B^2\Big)\tau + \frac{b\left(A^2 - B^2\right)}{4\sqrt{b+1}}\sin\left(2\sqrt{b+1}\tau\right) + \frac{ABb}{2\sqrt{b+1}}\cos\left(2\sqrt{b+1}\tau\right), \tag{39}$$

$$f(\tau) = A\cos\left(\sqrt{b+1}\tau\right) + B\sin\left(\sqrt{b+1}\tau\right), \tag{40}$$

where A and B are arbitrary constants. Eqs. (38)–(40) describe a circular tensile string in the y, z plane, centered at the origin and with an oscillating radius $f(\tau)$. In addition, the string moves in the x-direction.

From the solutions (38)–(40) one can see that a tensile string that contracts to a smaller size than its Schwarzschild radius will collapse and form a back hole.

Another simple, but instructive solution in Peres spacetime is a rotating straight tensile string given by

$$t = \tau, \qquad x = (1-b)\left(A^2 + B^2\right)\tau\,, \tag{41}$$

$$y = \left[A\cos\left(\sqrt{1-b}\,\sigma\right) + B\sin\left(\sqrt{1-b}\,\sigma\right)\right]\cos\tau\,, \tag{42}$$

$$z = \left[A\cos\left(\sqrt{1-b}\,\sigma\right) + B\sin\left(\sqrt{1-b}\,\sigma\right)\right]\sin\tau\,, \tag{43}$$

where $0 < b < 1$. Eqs. (41)–(43) identically fulfill the string equations and constraints.

Finally, we notice that the invariant string size (the length of the string) $S(\tau)$ is given by

$$S(\tau) = \int_0^{2\pi} S(\tau,\sigma) d\sigma \,, \tag{44}$$

where

$$S(\tau,\sigma) \equiv \sqrt{-x^M_{,\tau} G_{MN}(x) x^N_{,\tau}} = \left[-b\left(y^2+z^2\right)t'^2 - 2t'x' + y'^2 + z'^2\right]^{1/2} \tag{45}$$

for the null string [14], and

$$S(\tau,\sigma) \equiv \sqrt{x^M_{,\tau} G_{MN}(x)\, x^N_{,\tau}} \tag{46}$$

for the tensile string. By using Eqs. (38)–(40) and (46), we have

$$S(\tau,\sigma) \equiv f(\tau) = A\cos\left(\sqrt{b+1}\tau\right) + B\sin\left(\sqrt{b+1}\tau\right), \tag{47}$$

$$S(\tau) = 2\pi f(\tau)\,. \tag{48}$$

From equations (41)–(43), we have

$$S(\tau,\sigma) = (1-b)\left[\frac{3\left(A^2+B^2\right)}{2} - \frac{A^2-B^2}{2}\cos\left(2\sqrt{1-b}\sigma\right) - AB\sin\left(2\sqrt{1-b}\sigma\right)\right], \tag{49}$$

$$S(\tau) = (1-b)\left[3\pi\left(A^2+B^2\right) - AB - \frac{A^2-B^2}{2}\sin\left(4\pi\sqrt{1-b}\right) + AB\cos\left(4\pi\sqrt{1-b}\right)\right]. \tag{50}$$

It would be interesting to consider other null and tensile strings configurations and apply them to study null and tensile strings dynamics in shock and other strong gravitational waves.

4. Summary

In this article, we have considered a motion of null and tensile strings in Peres spacetime, which traditionally describes radiation field with an isotropic energy-momentum tensor and strong gravitational waves.

The exact, general solutions of equations of motion for a closed null string are obtained. The examples of motion and also the world-sheet for null string in these space are presented. It is shown, that radiation fields with an isotropic energy-momentum tensor play the role of a trap. As for the tensile strings we have integrated equations of motion and constraints completely in a closed form for circular and straight string Ansätz.

Hereinafter, the exact solutions obtained in this article, can be used as the test solutions indispensable for a numerical modeling of motion null and tensile string in the pseudo-Riemannian spaces.

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