

SOLUTIONS OF THE QUANTIZATION CONDITIONS FOR THE ODDERON CHARGE q_3 AND CONFORMAL WEIGHT h

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The quantization conditions which come from the requirement of the singlevaluedness of the odderon wave function are formulated and solved numerically in the 4 dimensional space of the odderon charge q_3 and the conformal weight h . It turns out that these conditions are fulfilled along one dimensional curves parametrized by a discrete set of values of $\text{Re } h$ in 3 dimensional subspace $(\text{Im } h, \text{Im } q_3, \text{Re } q_3)$. The odderon energy calculated along these curves corresponds in all cases to the intercept lower than 1.

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1. Introduction

The leading contribution to the total elastic scattering amplitude of two hadrons (A, B) can be written in Quantum Chromodynamics (QCD) in a so called Regge limit

$$s \rightarrow \infty, \quad t = \text{const.} \quad (1.1)$$

as a power series in a strong coupling constant α_s of the partial amplitudes with a given number n of reggeized gluons (Reggeons) propagating in the t channel:

$$A(s, t) = \sum_{n=2}^{\infty} \alpha_s^{n-2} A_n(s, t), \quad (1.2)$$

$$A_n(s, t) = i \sum_{\{\alpha\}} \beta_A^{n, \alpha_n}(t) \beta_B^{n, \alpha_n}(t) s^{\alpha_n(t)}. \quad (1.3)$$

Here $\varkappa_n$ denotes quantum numbers of the n Reggeon state and the residue functions $\beta_A^{n,\varkappa_n}$ and $\beta_B^{n,\varkappa_n}$ measure the overlap between the hadronic wave functions and the wave function of a compound state of n reggeized gluons. The n -Reggeon's partial amplitudes are proportional to $s^{\alpha_n(t)}$ where $\alpha_n(0)$ is called an intercept.

It is of great importance to calculate the intercepts $\alpha_n(0)$ in QCD, not only because they govern the high energy behavior of the forward elastic amplitudes but also because, *e.g.* for $n = 2$, they are responsible for the small Bjorken x behavior of the deep inelastic structure functions. The lowest non-trivial contribution for $n = 2$ was calculated in the leading logarithmic approximation by Balitsky, Fadin, Kureav and Lipatov [1,2], who derived and solved equation for the Pomeron intercept. The equation for three and more Reggeons was formulated in Refs. [3–5]. It took, however, almost 20 years before the solution for $n = 3$ was obtained in Refs. [6,7].

The real progress started with an observation that the n -Reggeon exchange is equivalent to an eigenvalue problem of a Schrödinger like equation with calculable interaction Hamiltonian $\hat{\mathcal{H}}_n$. Here the eigen-energy is related to $\alpha_n(0) - 1$. This Schrödinger problem is exactly solvable [8–10] which means that there exist $n - 1$ integrals of motion $(\hat{q}_2, \dots, \hat{q}_n)$ which commute with $\hat{\mathcal{H}}_n$ and among themselves. The eigenvalue of $\hat{q}_2$ is equal $-h(h - 1)$ where h is called a conformal weight.

In the present work we shall concentrate the odderon exchange, *i.e.* on the case with $n = 3$. It is easier to conduct the calculations in the impact parameter space, that is in the transverse spatial coordinates of n Reggeons (x_j, y_j) . After introducing the complex coordinates $(z_j = x_j + y_j, z_j^* = x_j - y_j)$ for j -th reggeized gluon, the odderon Hamiltonian becomes holomorphically separable

$$\hat{\mathcal{H}}_3 = \hat{H}_3 + \hat{\bar{H}}_3 = \frac{\alpha_s N_c}{4\pi} \sum_{k=1}^3 \left[\hat{H}(z_k, z_{k+1}) + \hat{H}(z_k^*, z_{k+1}^*) \right], \quad (1.4)$$

where N_c is a number colors and $z_1 = z_{n+1}$. The Hamiltonian $\hat{\mathcal{H}}_3$ is conformally invariant. Its eigenfunction is given as a bilinear form $\Phi = \bar{\Psi} \times \Psi$ where Ψ ($\bar{\Psi}$) is the solution of the Schrödinger equation in the holomorphic (antiholomorphic) sector.

There have been many attempts either to directly find the values of E_3 [11–15] or to find the spectrum of odderon charge $\hat{q}_3$ [6,16]. Finally, in Ref. [7] the singlevaluedness conditions for the wave function Φ were formulated and the spectrum of $\hat{q}_3$ was found. This allowed to calculate the energy [6] (hence also the odderon intercept) for the conformal weight $h = 1 - \bar{h} = 1/2$ which supposedly gives the largest contribution to the elastic amplitude. It is, however, interesting to see explicitly whether $h = 1 - \bar{h} = 1/2$ gives

really the largest intercept and whether there exist other solutions to the singlevaluedness conditions than the ones found in [7]. The first attempt in this direction has been undertaken in Ref. [17] where the spectrum of $\hat{q}_3$ for arbitrary conformal weight but for non-physical quantization conditions $h = \bar{h}$ was found. In Ref. [18] the spectrum of $\hat{q}_3$ for the specific choice of the conformal weight $h = 1 - \bar{h} = 1/2 + i\nu$ in the limit of small ν has been studied. This result of Ref. [18] was confirmed and extended for arbitrary ν in Ref. [17].

The values of the conformal weight h and eigenvalues of $\hat{q}_3$ form a four dimensional space. In the present work we construct an algorithm and numerical code which allows to find the points in the (h, q_3) space which satisfy the physical quantization conditions $h = 1 - \bar{h}$ and singlevaluedness conditions of Refs. [7,17]. It turns out that these points form one dimensional curves in the four dimensional space (h, q_3) . We have found families of curves which are numbered by discrete values of $\text{Re } h = 1/2 + m/2$ with $m \in 3\mathbb{Z}$. Therefore for given $\text{Re } h$ these curves are effectively embedded in 3 dimensional subspace $(\text{Im } h, \text{Re } q_3, \text{Im } q_3)$.

Applying the method from Ref. [6] which allows to calculate the odderon energy for arbitrary h and q_3 , we have calculated the odderon energy along the singlevaluedness lines $q_3(h)$. As expected, the odderon energy has a maximum for $h = 1 - \bar{h} = 1/2$, and is always negative. Our numerical procedures are precise enough to find 17 values of q_3 for $h = 1 - \bar{h} = 1/2$ with 9 digits accuracy.

The paper is organized as follows: in Sect. 2, following Ref. [17], we write the odderon equation in terms of variable ξ suggested in Ref. [19] and find its solutions around $\xi = \pm 1$ and ∞ . Recurrence relations for the for these solutions are collected in Appendix A. Next, in Sect. 3, we recapitulate the method of Ref. [6] and construct a singlevalued odderon wave function Φ relegating the detailed form of the singlevaluedness constraints to Appendix B. The resulting spectrum of q_3 and h is calculated and discussed in Sect. 4. The numerical algorithm used in this Section is described in detail in Appendix C. Finally in Sect. 5 we calculate the odderon energy along the singlevaluedness curves found in Sect. 4. Conclusions are presented in Sect. 6.

2. Solution of the eigenequation for the odderon charge $\hat{q}_3$

2.1. Origin of the equation

As already said in the Introduction it is possible to find a family of commuting operators $\hat{q}_k$ which commute with the holomorphic n -Reggeon Hamiltonian $\hat{H}_n$ [8]:

$$[\hat{H}_n, \hat{q}_k] = 0, \quad k = 2, \dots, n. \quad (2.1)$$

It follows that the Hamiltonian $\hat{H}_n$ and the operators $\hat{q}_k$ have the same set of eigenfunctions. In terms of the holomorphic coordinates $\hat{q}_k$ have the following form:

$$\hat{q}_k = \sum_{n \geq i_1 > i_2 > \dots > i_k \geq 1} i^k z_{i_1 i_2} z_{i_2 i_3} \dots z_{i_k i_1} \partial_{i_1} \partial_{i_2} \dots \partial_{i_k}, \quad (2.2)$$

where $k = 2, \dots, n$, $z_{jk} \equiv z_j - z_k$ and $\partial_j \equiv \partial_{z_j}$.

For the odderon case, $n = 3$, we have only 2 operators

$$\begin{aligned} \hat{q}_2 &= \sum_{n \geq j > k \geq 1} z_{jk}^2 \partial_j \partial_k, \\ \hat{q}_3 &= \sum_{n \geq j > k > l \geq 1} -i z_{jk} z_{kl} z_{lj} \partial_j \partial_k \partial_l. \end{aligned} \quad (2.3)$$

Following Ref. [8] we will use conformally covariant Ansatz for Ψ

$$\Psi(z_1, z_2, z_3) = z^{h/3} \psi(x), \quad (2.4)$$

where

$$z = \frac{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}{(z_1 - z_0)^2(z_2 - z_0)^2(z_3 - z_0)^2}, \quad x = \frac{(z_1 - z_2)(z_3 - z_0)}{(z_1 - z_0)(z_3 - z_2)}, \quad (2.5)$$

h is a conformal weight and z_0 represents an arbitrary reference point. A particular feature of this Ansatz (2.4) is that $\hat{q}_2$ is automatically diagonal

$$\hat{q}_2 \Psi(z_1, z_2, z_3) = -h(h-1) \Psi(z_1, z_2, z_3). \quad (2.6)$$

In representation (2.4) the eigenvalue equation for $\hat{q}_3$ takes the following form

$$\begin{aligned} i \hat{q}_3 \psi(x) &= \left(\frac{h}{3}\right)^2 \left(\frac{h}{3} - 1\right) \frac{(x-2)(x+1)(2x-1)}{x(x-1)} \psi(x) \\ &+ \left[2x(x-1) - \frac{h}{3}(h-1)(x^2 - x + 1) \right] \psi'(x) \\ &+ 2x(x-1)(2x-1) \psi''(x) + x^2(x-1)^2 \psi'''(x) = i q_3 \psi(x). \end{aligned} \quad (2.7)$$

Equation (2.7) has been studied in Ref. [7] where the quantization conditions for $\hat{q}_3$ were found by introducing the singlevaluedness constraints on the whole wave function of the odderon, Φ . The singlevaluedness conditions are much simpler when we rewrite equation (2.7) in terms of a new variable suggested in Ref. [19]

$$\xi = i \frac{1}{3\sqrt{3}} \frac{(x-2)(x+1)(2x-1)}{x(x-1)}. \quad (2.8)$$

Putting (2.8) into (2.7) we have [17]

$$\left[\frac{1}{2}(\xi^2 - 1)^2 \frac{d^3}{d\xi^3} + 2\xi(\xi^2 - 1) \frac{d^2}{d\xi^2} + \left(\frac{4}{9} - \beta_h(\xi^2 - 1) \right) \frac{d}{d\xi} + \rho_h \xi + \tilde{q} \right] \varphi(\xi) = 0, \quad (2.9)$$

where

$$\beta_h = \frac{(h+2)(h-3)}{6}, \quad \rho_h = \frac{h^2(h-3)}{27}, \quad \tilde{q} = \frac{q_3}{3\sqrt{3}},$$

and q_3 is the eigenvalue of the operator $\hat{q}_3$.

As we shall shortly see the odderon equation (2.9) is less singular than Eq. (2.7) and the solutions of the indicial equation around $\xi = \pm 1$ do not depend on h .

2.2. Solution of the odderon equation

The odderon equation (2.9) has three regular singular points at $\xi = -1$, $\xi = 1$ and $\xi = \infty$. We shall solve this equation using the power series method. It is a third order ordinary differential equation therefore it has three linearly independent solutions. We can write them as a vector

$$\vec{u}(\xi; q_3) = \begin{bmatrix} u_1(\xi; q_3) \\ u_2(\xi; q_3) \\ u_3(\xi; q_3) \end{bmatrix}. \quad (2.10)$$

2.2.1. Solution of the equation around $\xi = \pm 1$

Solutions of equation (2.9) around $\xi = \pm 1$ have the following form [17]:

$$u_i^{(\pm 1)}(\xi; q_3) = (1 \mp \xi)^{s_i} \sum_{n=0}^{\infty} u_{i,n}^{(\pm 1)} (\xi \mp 1)^n, \quad (2.11)$$

where s_i are solutions of the indicial equation

$$s_1 = \frac{2}{3}, \quad s_2 = \frac{1}{3}, \quad s_3 = 0 \quad (2.12)$$

and do not depend on h . The coefficients $u_{i,n}^{(\pm 1)}$ are defined in Appendix A.

2.2.2. Solutions around $\xi = \infty$

The solution of equation (2.9) around $\xi = \infty$ has a more complicated form. In this case we perform a substitution $\xi = 1/\eta$ and then solve the problem around $\eta = 0$:

$$-\frac{1}{2}(1-\eta^2)^2\eta^2\frac{d^3u}{d\eta^3} + [2(1-\eta^2)\eta - 3\eta(1-\eta^2)^2]\frac{d^2u}{d\eta^2} + \left[-3(1-\eta^2)^2 - \frac{4}{9}\eta^2 + (4+\beta_h)(1-\eta^2)\right]\frac{du}{d\eta} + \left[\rho_h\frac{1}{\eta} + \tilde{q}\right]u = 0. \quad (2.13)$$

The solutions of the indicial equation r_i depend on the conformal weight h

$$r_1 = \frac{2h}{3}, \quad r_2 = 1 - \frac{h}{3}, \quad r_3 = -\frac{h}{3}. \quad (2.14)$$

and are identical as the solutions of the indicial equation in the case of equation (2.7). Since $r_2 - r_3$ is equal to an integer number, one of the solutions, $u_3^{(\infty)}(\xi)$, contains a logarithm. The other two differences $r_2 - r_1$, $r_1 - r_3$ become integer as well if h is an integer itself. Therefore we have to distinguish several cases.

For $h \notin \mathbb{Z}$ and $q_3 \neq 0$ the solution of (2.9) in vicinity of $\xi = \infty$ reads:

$$\begin{aligned} u_1^{(\infty)}(\xi; q_3) &= \left(\frac{1}{\xi}\right)^{r_1} \sum_{n=0}^{\infty} u_{1,n}^{(\infty)} \left(\frac{1}{\xi}\right)^n, \\ u_2^{(\infty)}(\xi; q_3) &= \left(\frac{1}{\xi}\right)^{r_2} \sum_{n=0}^{\infty} u_{2,n}^{(\infty)} \left(\frac{1}{\xi}\right)^n, \\ u_3^{(\infty)}(\xi; q_3) &= \left(\frac{1}{\xi}\right)^{r_3} \sum_{n=0}^{\infty} u_{3,n}^{(\infty)} \left(\frac{1}{\xi}\right)^n, + u_2^{(\infty)}(\xi; q_3) \text{Log} \left(\frac{1}{\xi}\right), \end{aligned} \quad (2.15)$$

where the logarithm $\text{Log}(z)$ is defined as:

$$\text{Log}(z) = \ln|z| + i\text{Arg}(z), \quad |\text{Arg}(z)| < \pi. \quad (2.16)$$

The coefficients entering (2.15) are collected in Appendix A.

The remaining cases, *i.e.* when $q_3 = 0$ and/or $h \in \mathbb{Z}$ ¹, should be considered separately.

¹ Solutions of the equation (2.13) around $\xi = \infty$ for $h \in \mathbb{Z}$ are not considered in this work.

2.2.3. Solutions around $\xi = \infty$ for $q_3 = 0$

It is easy to observe that in equation (A.4) for $q_3 = 0$ the term $u_{3,0}^{(\infty)} = \frac{1-h}{2\bar{q}}$ tends to infinity. In this case the solution of equation (2.9) should be constructed separately. With $q_3 = 0$ and $h \notin \mathbb{Z}$ the solution is given by

$$u_i^{(\infty)}(\xi; q_3 = 0) = \left(\frac{1}{\xi}\right)^{r_i} \sum_{n=0}^{\infty} u_{i,2n}^{(\infty; q_3=0)} \left(\frac{1}{\xi}\right)^{2n}. \quad (2.17)$$

The coefficients $u_{i,2n}^{(\infty; q_3=0)}$ are defined in Appendix A.

2.3. Antiholomorphic sector

For given h and q_3 we can find the solutions of Eq. (2.9) around all singular points by means of Eqs. (2.11), (2.15) and (2.17). Analogously, we can construct the solutions in the antiholomorphic sector. Here instead of using the conformal weight h and charge q_3 we use their antiholomorphic equivalents: $\bar{h}$ and $\bar{q}_3$ ². Similarly to Eq. (2.10) we write the three linearly independent solutions as a vector:

$$\vec{v}(\xi^*; \bar{q}_3) = \begin{bmatrix} v_1(\xi^*; \bar{q}_3) \\ v_2(\xi^*; \bar{q}_3) \\ v_3(\xi^*; \bar{q}_3) \end{bmatrix}. \quad (2.18)$$

3. Quantization conditions for the odderon charge $\hat{q}_3$

3.1. Transition matrices

Each of the solutions around $\xi = \xi_{1,2,3}$ where $\xi_{1,2,3} = \pm 1, \infty$ has a convergence radius equal to the difference between two singular points: the point around which the solution is defined and the nearest of the remaining singular points. In order to define the global solution which is convergent in the entire complex plane we have to glue the solutions defined around different singular points. This can be done by expanding one set of solutions defined around ξ_i in terms of the solutions defined around ξ_j for ξ belonging to the overlap region of the two solutions considered. Thus, in the overlap region we can define the transition matrices Δ , Γ , Ω , where

$$\begin{aligned} \vec{u}^{(\infty)}(\xi; q_3) &= \Delta(q_3) \vec{u}^{(-1)}(\xi; q_3), \\ \vec{u}^{(-1)}(\xi; q_3) &= \Gamma(q_3) \vec{u}^{(+1)}(\xi; q_3), \\ \vec{u}^{(+1)}(\xi; q_3) &= \Omega(q_3) \vec{u}^{(\infty)}(\xi; q_3). \end{aligned} \quad (3.1)$$

² *bar* does not denote complex conjugation for which we use an *asterisk*.

Matrices Δ , Γ and Ω are constructed in terms of the ratios of certain determinants. For example to calculate the matrix Γ we construct the Wronskian

$$W = \begin{vmatrix} u_1^{(+1)}(\xi; q_3) & u_2^{(+1)}(\xi; q_3) & u_3^{(+1)}(\xi; q_3) \\ u_1'^{(+1)}(\xi; q_3) & u_2'^{(+1)}(\xi; q_3) & u_3'^{(+1)}(\xi; q_3) \\ u_1''^{(+1)}(\xi; q_3) & u_2''^{(+1)}(\xi; q_3) & u_3''^{(+1)}(\xi; q_3) \end{vmatrix}. \quad (3.2)$$

Next, we construct determinants W_{ij} which are obtained from W by replacing j -th column by the i -th solution around $\xi = -1$, *i.e.* for $i = 1$ and $j = 2$ we have

$$W_{12} = \begin{vmatrix} u_1^{(+1)}(\xi; q_3) & u_1^{(-1)}(\xi; q_3) & u_3^{(+1)}(\xi; q_3) \\ u_1'^{(+1)}(\xi; q_3) & u_1'^{(-1)}(\xi; q_3) & u_3'^{(+1)}(\xi; q_3) \\ u_1''^{(+1)}(\xi; q_3) & u_1''^{(-1)}(\xi; q_3) & u_3''^{(+1)}(\xi; q_3) \end{vmatrix}. \quad (3.3)$$

The matrix elements Γ_{ij} are defined as

$$\Gamma_{ij} = \frac{W_{ij}}{W}. \quad (3.4)$$

Matrix Γ does not depend on ξ , but only on q_3 and h . In a similar way we can get matrices Δ and Ω and their antiholomorphic equivalents: $\overline{\Delta}$, $\overline{\Gamma}$, $\overline{\Omega}$.

3.2. Quantization conditions and singlevaluedness of the wave function

The odderon charge q_3 is connected to its antiholomorphic equivalent by

$$\overline{q}_3 = -q_3^*, \quad (3.5)$$

where an *asterisk* denotes complex conjugation. There exist two possible choices for $\overline{q}_3$: the one given by Eq. (3.5) and a similar one with the *plus* sign. This follows from the fact that the eigenvalues of holomorphic and antiholomorphic Hamiltonian, ε_3 and $\overline{\varepsilon}_3$, are symmetric functions of q_3 and $\overline{q}_3$, respectively [20]. Only one choice, namely (3.5), leads to the nonvanishing solution of the quantization conditions³.

The odderon wave function can be written as [17]

$$\Phi_{h\overline{h}q_3\overline{q}_3}(z, z^*) = z^{h/3}(z^*)^{\overline{h}/3}\vec{v}^T(\xi^*; \overline{q}_3)A(\overline{q}_3, q_3)\vec{u}(\xi; q_3), \quad (3.6)$$

³ Note, that because of the factor i in the definition ξ (2.8), our $\overline{q}_3$ has different sign than the one in Ref. [7].

where $\xi = \xi(z)$. The wave function $\Phi_{h\bar{h}q_3\bar{q}_3}(z, z^*)$ contains the solutions of equation (2.9) and its antiholomorphic counterpart, $\vec{u}(\xi; q_3)$ and $\vec{v}(\xi^*; \bar{q}_3)$ respectively, and a 3×3 matrix $A(\bar{q}_3; q_3)$ "sewing" the solutions of the both sectors.

The wave function Φ has to be singlevalued. This means that it should not depend on the choice of the Riemann sheet for the variables z and ξ . In formula (3.6) the term $z^{h/3}(z^*)^{\bar{h}/3}$ is uniquely defined only if $h/3 - \bar{h}/3 \in \mathbb{Z}$. This leads to the quantization condition for the conformal weight h

$$h = \frac{1}{2}(\mu + m) + i\nu \quad \text{and} \quad \bar{h} = \frac{1}{2}(\mu - m) + i\nu, \quad (3.7)$$

where μ and ν are real and $m/3 \in \mathbb{Z}$. The latter condition follows from the invariance under the Lorentz spin transformations. The normalization condition of the wave function requires that $\mu = 1$ for the physical odderon solution.

The fact that the wave function Φ should be singlevalued imposes certain conditions on the form of matrix A . It follows from Eq. (3.6) that for the solutions (2.11) around $\xi = \pm 1$ the matrix element A_{ji} is multiplied by a factor $(1 \mp \xi)^{s_i}(1 \mp \xi^*)^{s_j}$. This expression is singlevalued only if $s_i - s_j \in \mathbb{Z}$. For s_i of Eq. (2.12) this is true only for $i = j$. Therefore the matrices $A^{(\pm 1)}$ have a diagonal form

$$A^{(-1)}(\bar{q}_3, q_3) = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad A^{(+1)}(\bar{q}_3, q_3) = \begin{bmatrix} \alpha' & 0 & 0 \\ 0 & \beta' & 0 \\ 0 & 0 & \gamma' \end{bmatrix}. \quad (3.8)$$

For the solutions around $\xi = \infty$ (2.15), (2.17), and for $h \notin \mathbb{Z}$, the matrix element A_{ji} is multiplied by a factor $(1/\xi)^{r_i}(1/\xi^*)^{\bar{r}_j}$. One should notice that solutions of the indicial equation r_2 , $\bar{r}_2$, r_3 and $\bar{r}_3$ differ by an integer⁴, therefore terms which correspond to the elements A_{23} and A_{32} , do not vanish. Furthermore, terms with a logarithm appear in the solutions $u_3^{(\infty)}$ and $v_3^{(\infty)}$ for $q_3 \notin \mathbb{Z}$. One can see that when $A_{23} = A_{32}$ then in the sum the ambiguous arguments of the logarithms cancel out. Moreover, for $q_3 \notin \mathbb{Z}$ the term which corresponds to the matrix element A_{33} is not singlevalued. It contains a square of the logarithm which does not occur in any other terms. For this reason the element $A_{33}(q_3 \neq 0, \bar{q}_3 \neq 0)$ has to vanish.

Thus the matrices A , defined around $\xi = \infty$, have the following form

$$A^{(\infty)}(\bar{q}_3 \neq 0, q_3 \neq 0) = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \sigma & \tau \\ 0 & \tau & 0 \end{bmatrix}, \quad A^{(\infty)}(\bar{q}_3 = 0, q_3 = 0) = \begin{bmatrix} \rho' & 0 & 0 \\ 0 & \sigma' & v' \\ 0 & \varsigma' & \tau' \end{bmatrix}. \quad (3.9)$$

⁴ $\bar{r}_2$ and $\bar{r}_3$ are solutions of the indicial equation in the antiholomorphic sector.

Substituting equation (3.1) into the wave function (3.6), one finds the following conditions for the matrices $A(\bar{q}_3, q_3)$

$$\bar{\Delta}^T(\bar{q}_3)A^{(\infty)}(\bar{q}_3, q_3)\Delta(q_3) = A^{(-1)}(\bar{q}_3, q_3), \quad (3.10)$$

$$\bar{\Gamma}^T(\bar{q}_3)A^{(-1)}(\bar{q}_3, q_3)\Gamma(q_3) = A^{(+1)}(\bar{q}_3, q_3), \quad (3.11)$$

$$\bar{\Omega}^T(\bar{q}_3)A^{(+1)}(\bar{q}_3, q_3)\Omega(q_3) = A^{(\infty)}(\bar{q}_3, q_3). \quad (3.12)$$

Each of the formulae (3.10)–(3.12) is equivalent to a set of nine equations which can be conveniently written in terms of the following 4 vectors:

$$\vec{a} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} \alpha' \\ \beta' \\ \gamma' \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} \rho \\ \sigma \\ \tau \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} \rho' \\ \sigma' \\ \varsigma' \\ v' \\ \tau' \end{bmatrix},$$

We can now rewrite equations (3.10), (3.11), (3.12) in the following form:

- equation (3.10):

– for $q_3 \neq 0$ as

$$B_{\text{up}}\vec{c} = 0, \quad B_{\text{low}}\vec{c} = 0, \quad B_{\text{diag}}\vec{c} = \vec{a}, \quad (3.13)$$

– for $q_3 = 0$ as

$$B'_{\text{up}}\vec{d} = 0, \quad B'_{\text{low}}\vec{d} = 0, \quad B'_{\text{diag}}\vec{d} = \vec{a}. \quad (3.14)$$

- equation (3.11) as

$$C_{\text{up}}\vec{a} = 0, \quad C_{\text{low}}\vec{a} = 0, \quad C_{\text{diag}}\vec{a} = \vec{b}, \quad (3.15)$$

- equation (3.12):

– for $q_3 \neq 0$ as

$$D_{\text{up}}\vec{b} = 0, \quad D_{\text{low}}\vec{b} = 0, \quad D_{\text{diag}}\vec{b} = \vec{c}, \quad (3.16)$$

– for $q_3 = 0$ as

$$D'_{\text{up}}\vec{b} = 0, \quad D'_{\text{low}}\vec{b} = 0, \quad D'_{\text{diag}}\vec{b} = \vec{d}. \quad (3.17)$$

Definitions of matrices $B, C, D \dots$ are given in Appendix B.

Equations (3.13)–(3.17) have nonvanishing solutions only if the determinants of matrices with subscripts *up* and *low* are equal zero. Moreover there should exist the unique solutions of these equations: $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ which depend only on one free parameter which can be fixed by normalizing the wave function Φ . As we shall show in the next Sections these two requirements fix uniquely the “boundary conditions” for the eigen equation of the operator $\hat{q}_3$ and allow to calculate its spectrum.

4. Spectrum of the odderon charge $\hat{q}_3$

4.1. Eigenvalues of $\hat{q}_3$ for $h = 1/2$

Let us first discuss physical solutions found in Ref. [7] which correspond to the conformal weight $h = 1/2$ and $\text{Re } q_3 = 0$. In order to calculate the eigenvalues q_3 we have solved Eqs. (3.13)–(3.17) requiring vanishing of the *up* and *low* matrix determinants. After that, we have also checked the uniqueness of obtained solutions. The results, also for unphysical values of q_3 with $\text{Im } q_3 = 0$ are displayed in Table I. Entries labeled from 0 to 4, 12 and 13 agree with the ones of Refs. [7,17], while the remaining ones are new.

TABLE I

Eigenvalues of the odderon charge q_3 for $h = 1/2$.

No.	q_3	No.	q_3	No.	q_3
0	0	6	68.600522343 <i>i</i>	12	1.475327424
1	0.205257506 <i>i</i>	7	109.214406900 <i>i</i>	13	12.947047037
2	2.343921063 <i>i</i>	8	163.296192765 <i>i</i>	14	44.413830163
3	8.326345902 <i>i</i>	9	232.769867177 <i>i</i>	15	105.872614615
4	20.080496894 <i>i</i>	10	319.559416811 <i>i</i>	16	207.320706051
5	39.530550304 <i>i</i>	11	425.588828106 <i>i</i>	17	358.755426678

In fact the eigenvalues q_3 for $h = 1/2$ form a discrete set of points symmetrically distributed on the real and complex axis in complex q_3 plane. Therefore in Table I only a half of the spectrum is displayed. It has been shown [6,7] that only the imaginary values of q_3 are relevant for the odderon problem; real eigenvalues correspond to a wave function which is not totally symmetric under the exchange of the neighboring Reggeons. There exists also one eigenvalue $q_3 = 0$ which does not correspond to a normalizable solution [20], see however [21].

4.2. Eigenvalues $\hat{q}_3$ for the arbitrary conformal weights h

As seen from Eq. (3.7) the conformal weight h depends on two parameters: $m/3 \in \mathbb{Z}$ and $\nu \in \mathbb{R}$. In order to find the manifolds on which conditions (3.13)–(3.17) are satisfied we have extended the domain of h and allowed m to be any real number. This defines a four dimensional space of the conformal weight and the odderon charge $(h, q_3) \in \mathbb{R}^4$. In each group of equations (3.13)–(3.17) there are two equations which contain matrices with subscripts ‘up’ and ‘low’. The determinants of these two matrices are complex, so we can define the following fourvalued functions:

$$\begin{aligned}
f_B : (\operatorname{Re} h, \operatorname{Im} h, \operatorname{Re} q_3, \operatorname{Im} q_3) \longrightarrow \\
\begin{cases} (\operatorname{Re}(\det B_{\text{up}}), \operatorname{Im}(\det B_{\text{up}}), \operatorname{Re}(\det B_{\text{low}}), \operatorname{Im}(\det B_{\text{low}})) & \text{for } q_3 \neq 0 \\
(\operatorname{Re}(\det B'_{\text{up}}), \operatorname{Im}(\det B'_{\text{up}}), \operatorname{Re}(\det B'_{\text{low}}), \operatorname{Im}(\det B'_{\text{low}})) & \text{for } q_3 = 0, \end{cases}
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
f_C : (\operatorname{Re} h, \operatorname{Im} h, \operatorname{Re} q_3, \operatorname{Im} q_3) \longrightarrow \\
(\operatorname{Re}(\det C_{\text{up}}), \operatorname{Im}(\det C_{\text{up}}), \operatorname{Re}(\det C_{\text{low}}), \operatorname{Im}(\det C_{\text{low}})),
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
f_D : (\operatorname{Re} h, \operatorname{Im} h, \operatorname{Re} q_3, \operatorname{Im} q_3) \longrightarrow \\
\begin{cases} (\operatorname{Re}(\det D_{\text{up}}), \operatorname{Im}(\det D_{\text{up}}), \operatorname{Re}(\det D_{\text{low}}), \operatorname{Im}(\det D_{\text{low}})) & \text{for } q_3 \neq 0 \\
(\operatorname{Re}(\det D'_{\text{up}}), \operatorname{Im}(\det D'_{\text{up}}), \operatorname{Re}(\det D'_{\text{low}}), \operatorname{Im}(\det D'_{\text{low}})) & \text{for } q_3 = 0. \end{cases}
\end{aligned} \tag{4.3}$$

Thus, in order to calculate the spectrum of the operator $\hat{q}_3$, we should find common zeros of all functions f_B , f_C and f_D :

$$f_B = 0, \quad f_C = 0, \quad f_D = 0. \tag{4.4}$$

Furthermore one should verify the uniqueness of the solutions for (3.13)–(3.17).

In Appendix C we have described the numerical algorithm constructed to find roots of Eqs. (4.4). Our numerical findings can be summarized as follows:

1. Although we have formally allowed m to be a continuous real parameter, the solutions of Eqs. (4.4) exist only for $m/3 \in \mathbb{Z}$ (*e.g.* $\operatorname{Re} h = 1/2$ or $\operatorname{Re} h = 2$).
2. For the above discrete values of m , that is for fixed $\operatorname{Re} h$, the solutions of Eqs. (4.4) form continuous curves in 3 dimensional subspace $(\operatorname{Im} h, \operatorname{Re} q_3, \operatorname{Im} q_3)$.
3. It turned out that each of 3 equations (4.4) yields the same set of curves, provided that the solutions of Eqs. (3.13)–(3.17) are unique.

In the following we discuss two sets of solutions to Eqs. (4.4), namely for $\operatorname{Re} h = 1/2$ and $\operatorname{Re} h = 2$.

4.2.1. Spectrum of $\hat{q}_3$ for $\text{Re}(h) = 1/2$

In figure 1 we plot spectrum of the odderon charge $\hat{q}_3$ as a function of h for $\text{Re } h = 1/2$. One can see two sets of curves: the ones in the plane of $\text{Re } q_3 = 0$ and in the perpendicular plane of $\text{Im } q_3 = 0$ and the line $q_3 = 0$ which belongs to the both classes. For all these curves, except for $q_3 = 0$, the minimum of $|q_3|$ occurs for $h = 1/2$. Going away from this point the absolute value of q_3 increases monotonically. Minimal values of $|q_3|$ correspond to the points listed in Table I. We have not found any curve located outside of $\text{Re } q_3 = 0$ or $\text{Im } q_3 = 0$ planes.

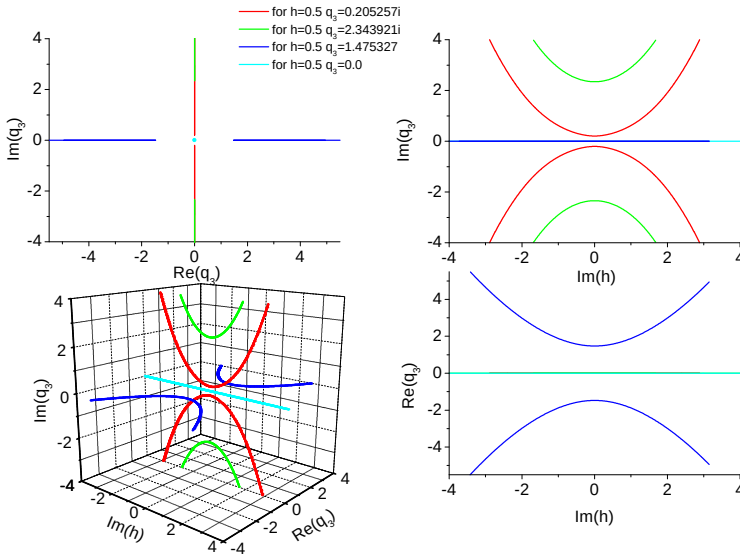


Fig. 1. Odderon charge q_3 as a function of h for $\text{Re } h = 1/2$ Presented curves are described with point with the higher value of $|q_3|$.

The curves in the plane of $\text{Re } q_3 = 0$ have been earlier found in Ref. [22] and also in Refs. [17, 23].

The spectrum of q_3 has the following symmetries:

1. $\text{Re } q_3 \rightarrow -\text{Re } q_3$,
2. $\text{Im } q_3 \rightarrow -\text{Im } q_3$,
3. $h \rightarrow 1 - h$

which follow from the symmetries of the odderon equation (2.9). Indeed, equation (2.9) is invariant under the transformation $\xi \rightarrow -\xi$ and $q_3 \rightarrow -q_3$. The last symmetry follows from the exchange symmetry between the holomorphic and antiholomorphic sectors.

4.2.2. Spectrum of $\hat{q}_3$ for $\text{Re}(h) = 2$

In figure 2 we plot the odderon charge q_3 as a function of the conformal weight for $\text{Re } h = 2$. Except of a line with $q_3 = 0$ which is analogous to the case of $\text{Re } h = 1/2$, the remaining curves have more complicated character. Still, they obey the following symmetries:

1. $q_3 \rightarrow -q_3$,
2. $\text{Im } h \rightarrow -\text{Im } h \quad \text{Re } q_3 \rightarrow -\text{Re } q_3$,
3. $\text{Im } h \rightarrow -\text{Im } h \quad \text{Im } q_3 \rightarrow -\text{Im } q_3$.

These symmetries are similar to the case of $h = 1/2$.

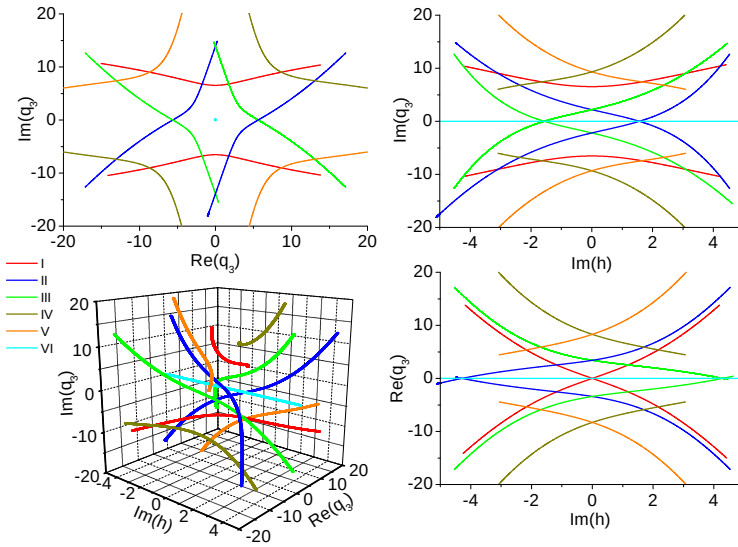


Fig. 2. Odderon charge q_3 as a function of h for $\text{Re } h = 2$

The first one is connected to Bose symmetry ($\xi \leftrightarrow -\xi$) and the others correspond to exchange of holomorphic and antiholomorphic sectors. For $h = 2$, that is, if the imaginary part of h vanishes, equation (2.9) has not been solved. In this case the eigenvalues q_3 have been obtained by an interpolation of the neighboring points which satisfied the quantization conditions.

4.2.3. Case for $q_3 = 0$

During the numerical computations we have noticed that similarly to the other matrices A , the matrix $A^{(\infty)}(q_3 = 0, \bar{q}_3 = 0)$ depends only on three parameters. The remaining ones vanish identically. In the case $\text{Re } h = 1/2$, ς' and v' vanish and for $\text{Re } h = 2$, $\sigma' = \tau' = 0$.

5. Odderon energy

The odderon energy is defined as [20]

$$E_3 = \frac{\alpha_s N_c}{4\pi} [\varepsilon_3(h, q_3) + \bar{\varepsilon}_3(\bar{h}, \bar{q}_3)] = \frac{\alpha_s N_c}{2\pi} \text{Re}(\varepsilon_3(h, q_3)), \quad (5.1)$$

where ε_3 and $\bar{\varepsilon}_3$ are the largest eigenvalues of the holomorphic and antiholomorphic odderon Hamiltonian, respectively. Applying the Bethe Ansatz we have for $n = 3$ [20, 24]

$$\varepsilon_3 = i \left(\frac{\dot{Q}_3(-i)}{Q_3(-i)} - \frac{\dot{Q}_3(i)}{Q_3(i)} \right) - 6, \quad (5.2)$$

where $Q_3(\lambda)$ satisfies the following Baxter equation [25]:

$$(\lambda + i)^3 Q_3(\lambda + i) + (\lambda - i)^3 Q_3(\lambda - i) - (2\lambda^3 + \lambda q_2 + q_3) Q_3(\lambda) = 0. \quad (5.3)$$

Equation (5.3) was solved in Ref. [6] by a substitution

$$\begin{aligned} \lambda^k Q_k(\lambda) &= \int_{C_z} \frac{dz}{2\pi i} K(z, \lambda) \hat{P}^k Q(z) \\ &\quad - i \sum_{m=1}^k \int_{C_z} \frac{dz}{2\pi i} \frac{d}{dz} \left[z(z-1) \hat{L}^{k-m} K(z, \lambda) \hat{P}^{m-1} Q(z) \right], \end{aligned} \quad (5.4)$$

where

$$K(z, \lambda) = z^{-i\lambda-1} (z-1)^{i\lambda-1}, \quad \hat{L} = \left(-i \frac{d}{dz} z(z-1) \right), \quad \hat{P} = \left(iz(z-1) \frac{d}{dz} \right). \quad (5.5)$$

Choosing the proper integration contour C_z and boundary conditions one arrives at a differential equation for $Q(z)$:

$$\left[\left(z(z-1) \frac{d}{dz} \right)^3 - q_2 z^2 (z-1)^2 \frac{d}{dz} - iq_3 z(1-z) \right] Q(z) = 0. \quad (5.6)$$

Similarly to the solutions of equation (2.9), the solutions of Eq. (5.6) depend on the conformal weight h and the odderon charge q_3 . Using the spectrum $q_3(h)$ calculated in the last section, we have calculated the energy of the odderon along the curves from figures 1 and 2.

Analyzing the spectra of energy we can conclude that the odderon energy is always negative. This means that the intercept

$$\alpha(t=0) = E_3 + 1 \quad (5.7)$$

is lower than one, so the odderon partial amplitude $A_3(s, t)$ (1.3) is described by the convergent series in Regge limit (1.1).

5.1. Spectrum of the energy for $\text{Re } h = 1/2$

TABLE II

Maximal values of $\text{Re}(\varepsilon_3)$ for $\text{Re } h = 1/2$.

q_3	$\text{Re}(\varepsilon_3)$
0	-0.73801
$0.20526i$	-0.49434
$2.34392i$	-5.16930
1.47533	-4.23462

In figure 3 we plot a real part of the holomorphic odderon energy $\text{Re}(\varepsilon_3)$ as a function of $\text{Im } h$. The picture is plotted for curves from figure 1, that is for $\text{Re } h = 1/2$. All these curves have a maximum in $h = 1/2$. The maximal values are displayed in Table II. Going away from the maximum, the energy decreases monotonically. Our results agree with the values from Ref. [7]. The energy spectrum has the following symmetry

$$h \longrightarrow 1 - h. \quad (5.8)$$

5.2. Spectrum of the energy for $\text{Re } h = 2$

In figure 4, similarly to Fig. 3, we plot $\text{Re}(\varepsilon_3)$ as a function of $\text{Im } h$ for curves from figure 2, that is for $\text{Re } h = 2$. In this case the curves in (h, q_3) space have much more complicated character and location of the energy maxima occurs not always for $\text{Im } h = 0$.

For plots I, VI, $\text{Re}(\varepsilon_3)$ has a maximum in $h = 2$. In others cases the energy has a maximum in vicinity of $h = 2$, *i.e.* for the II-nd curve the energy has a maximum in $(h = 2.0 + 0.107i, q_3 = -3.508 + 2.050i)$. The maximal values of energy are given in Table III. Similarly to the case of $\text{Re } h = 1/2$ going away from the maximum, energy decreases monotonically.

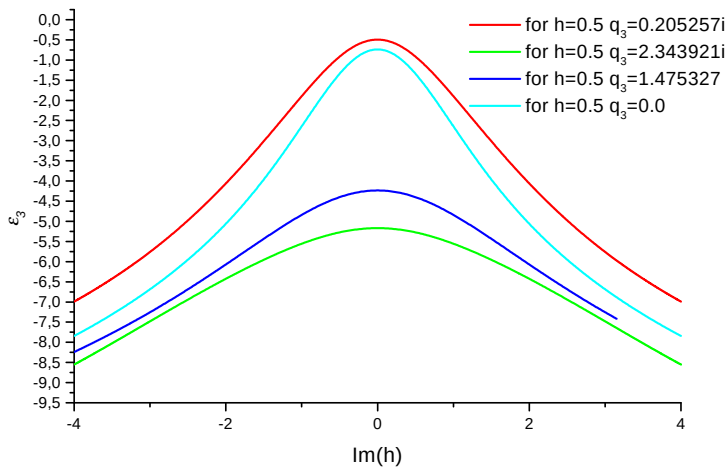


Fig. 3. Real part of holomorphic energy of odderon for $\text{Re } h = 1/2$. Picture is plotted for curves from figure 1.

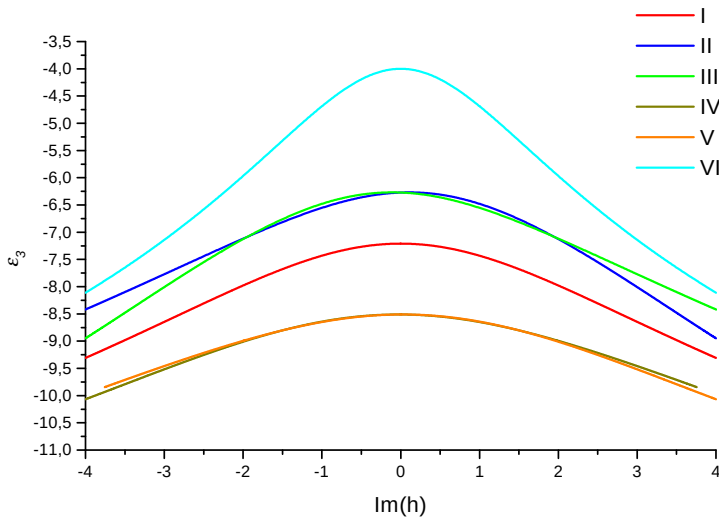


Fig. 4. Real part of the holomorphic odderon energy for $\text{Re } h = 2$. The picture is plotted for the curves from figure 2.

6. Summary and conclusions

The aim of the present paper was to look for the solutions of the odderon equation (2.9) in the entire four dimensional space of the conformal weight h and odderon charge q_3 . So far these solutions have been found only for some

TABLE III

Maximal value $\text{Re}(\varepsilon_3)$ for $\text{Re } h = 2$.

curve symbols	$\text{Re}(\varepsilon_3)$
I	-7.21
II, III	-6.27
IV, V	-8.51
VI	-4.00

specific values of h [6,18,22,23] or for arbitrary h but unphysical quantization condition $h = \bar{h}$ [17]. To this end we have constructed and implemented the algorithm which is in detail described in Appendix C. This algorithm can be easily extended to more dimensional cases [26].

The calculations were performed in the holomorphic variable ξ which respects Bose symmetry and was proposed in Refs. [17,19]. The odderon equation (2.9) is a third order ordinary differential equation with three regular singular points at $\xi = -1, 1$ and ∞ . Solutions around $\xi = \pm 1$ have been already found in Ref. [17]. Here we have also calculated solutions around $\xi = \infty$ (2.13).

The singlevaluedness conditions imposed on the odderon wave function Φ were found to be fulfilled along the discrete sets of continuous one dimensional curves. These sets are numbered by values of $\text{Re } h = 1/2 + m/2$ ($m/3 \in \mathbb{Z}$) and lie effectively in 3 dimensional subspace ($\text{Im } h, \text{Re } q_3, \text{Im } q_3$). In this way we have obtained numerically the known spectrum of the Lorentz spin m .

Although there are in principle 3 different singlevaluedness conditions obtained by gluing solutions around each of 3 singular points, which have to be fulfilled *simultaneously*, it turned out that it was enough to satisfy only one of them to get a complete set of solutions. It was therefore enough to consider two singular points, namely ± 1 for which the solutions of the characteristic equation do not depend on h .

Finally, we have calculated the odderon energy along the singlevaluedness curves $q_3(h)$. For all cases the energy turned out to be negative which means that the odderon intercept is smaller than 1. The maximal value of the odderon energy corresponds, as earlier conjectured, to $h = 1/2$ and $q_3 = 0.205257506 \times i$ which can be seen on Fig. 3.

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Appendix A

Definition of series from solutions of the equation for $\hat{q}_3$

A.A Solutions around $\xi = \pm 1$

The solutions of the equation (2.9) around $\xi = \pm 1$ have a form (2.11), where coefficients are defined by

$$\begin{aligned} u_{i,0}^{(\pm 1)} &= 1, & u_{i,1}^{(\pm 1)} &= a_{i,0} u_{i,0}^{(\pm 1)} / m_{i,1}, \\ u_{i,n}^{(\pm 1)} &= \left(a_{i,n-1} u_{i,n-1}^{(\pm 1)} + b_{i,n-2} u_{i,n-2}^{(\pm 1)} \right) / m_{i,n}, \end{aligned} \quad (\text{A.1})$$

while

$$\begin{aligned} a_{i,n-1} &= \mp 4(n + s_i - 1) [(n + s_i - 2)(n + s_i) - \beta_h] + 2\rho_h \pm 2\tilde{q}, \\ b_{i,n-2} &= -(n + s_i - 2) [(n + s_i - 3)(n + s_i) - 2\beta_h] + 2\rho_h, \\ m_{i,n} &= 4(n + s_i) [(n + s_i)(n + s_i - 1) + 2/9]. \end{aligned} \quad (\text{A.2})$$

The upper sign corresponds to solutions around $\xi = 1$ and the lower one to solutions around $\xi = -1$.

A.B Solutions around $\xi = \infty$

The coefficients of solutions of equation (2.15) for $q_3 \neq 0$ and $h \notin \mathbb{Z}$ we can write as

- for $i = 1, 2$:

$$\begin{aligned} u_{i,0}^{(\infty)} &= 1, & u_{i,1}^{(\infty)} &= a_{i,0} u_{i,0}^{(\infty)} / m_{i,1}, \\ u_{i,2}^{(\infty)} &= \left(a_{i,1} u_{i,1}^{(\infty)} + b_{i,0} u_{i,0}^{(\infty)} \right) / m_{i,2}, \\ u_{i,3}^{(\infty)} &= \left(a_{i,2} u_{i,2}^{(\infty)} + b_{i,1} u_{i,1}^{(\infty)} \right) / m_{i,3}, \\ u_{i,n}^{(\infty)} &= \left(a_{i,n-1} u_{i,n-1}^{(\infty)} + b_{i,n-2} u_{i,n-2}^{(\infty)} + c_{i,n-4} u_{i,n-4}^{(\infty)} \right) / m_{i,n}, \end{aligned} \quad (\text{A.3})$$

- for $i = 3$:

$$\begin{aligned}
 u_{3,0}^{(\infty)} &= \frac{1-h}{2\tilde{q}}, \quad u_{3,1}^{(\infty)} = 0, \\
 u_{3,2}^{(\infty)} &= \left(a_{3,1}u_{3,1}^{(\infty)} + b_{3,0}u_{3,0}^{(\infty)} + d_{3,1}u_{2,1}^{(\infty)} \right) / m_{i,2}, \\
 u_{3,3}^{(\infty)} &= \left(a_{3,2}u_{3,2}^{(\infty)} + b_{3,1}u_{3,1}^{(\infty)} + d_{3,2}u_{2,2}^{(\infty)} + f_{3,0}u_{2,0}^{(\infty)} \right) / m_{3,3}, \\
 u_{3,4}^{(\infty)} &= \left(a_{3,3}u_{3,3}^{(\infty)} + b_{3,2}u_{3,2}^{(\infty)} + c_{3,0}u_{3,0}^{(\infty)} + d_{3,3}u_{2,3}^{(\infty)} + f_{3,1}u_{2,1}^{(\infty)} \right) / m_{3,4}, \\
 u_{3,n}^{(\infty)} &= \left(a_{3,n-1}u_{3,n-1}^{(\infty)} + b_{3,n-2}u_{3,n-2}^{(\infty)} + c_{3,n-4}u_{3,n-4}^{(\infty)} \right. \\
 &\quad \left. + d_{3,n-1}u_{2,n-1}^{(\infty)} + f_{3,n-3}u_{2,n-3}^{(\infty)} + g_{3,n-5}u_{2,n-5}^{(\infty)} \right) / m_{i,n}, \quad (\text{A.4})
 \end{aligned}$$

where

$$\begin{aligned}
 a_{i,n-1} &= 2\tilde{q}, \\
 b_{i,n-2} &= 2(n+r_i-2) [(n+r_i-1)(n+r_i-2) - \beta_h - 4/9], \\
 c_{i,n-4} &= -(n+r_i-4)(n+r_i-3)(n+r_i-2), \\
 d_{i,n-1} &= -(n+r_i) [3(n+r_i-2) + 4] + 2(1+\beta_h), \\
 f_{i,n-3} &= 2[(n+r_i-2)(3(n+r_i-4) + 8) - \beta_h - 4/9], \\
 g_{i,n-5} &= -3(n+r_i-4)(n+r_i-2) - 2, \\
 m_{i,n} &= (n+r_i)^2(n+r_i-1) - 2(n+r_i)(1+\beta_h) - 2\rho_h. \quad (\text{A.5})
 \end{aligned}$$

Similarly the coefficients for the solution (2.17) for $q_3 = 0$ and $h \notin \mathbb{Z}$ have a form

$$\begin{aligned}
 u_{i,0}^{(\infty;q_3=0)} &= 1, \quad u_{i,2}^{(\infty;q_3=0)} = b_{i,0}u_{i,0}^{(\infty)} / m_{i,2}, \\
 u_{i,2n}^{(\infty;q_3=0)} &= \left(b_{i,2(n-1)}u_{i,2(n-1)}^{(\infty;q_3=0)} + c_{i,2(n-2)}u_{i,2(n-2)}^{(\infty;q_3=0)} \right) / m_{i,2n}. \quad (\text{A.6})
 \end{aligned}$$

Appendix B

Definition of matrices connected to singlevaluedness constraints on Φ

In Section 3.2 we defined quantization conditions for operator $\hat{q}_3$. The matrices in formula (3.13) have the following form

$$B_{\text{up}} = \begin{bmatrix} \overline{\Delta}_{11}\Delta_{12} & \overline{\Delta}_{21}\Delta_{22} & \overline{\Delta}_{31}\Delta_{22} + \overline{\Delta}_{21}\Delta_{32} \\ \overline{\Delta}_{11}\Delta_{13} & \overline{\Delta}_{21}\Delta_{23} & \overline{\Delta}_{31}\Delta_{23} + \overline{\Delta}_{21}\Delta_{33} \\ \overline{\Delta}_{12}\Delta_{13} & \overline{\Delta}_{22}\Delta_{23} & \overline{\Delta}_{32}\Delta_{23} + \overline{\Delta}_{22}\Delta_{33} \end{bmatrix}, \quad (\text{B.1})$$

$$B_{\text{low}} = B_{\text{up}}(\Delta \leftrightarrow \bar{\Delta}), \quad (\text{B.2})$$

$$B_{\text{diag}} = \begin{bmatrix} \bar{\Delta}_{11}\Delta_{11} & \bar{\Delta}_{21}\Delta_{21} & \bar{\Delta}_{31}\Delta_{21} + \bar{\Delta}_{21}\Delta_{31} \\ \bar{\Delta}_{12}\Delta_{12} & \bar{\Delta}_{22}\Delta_{22} & \bar{\Delta}_{32}\Delta_{22} + \bar{\Delta}_{22}\Delta_{32} \\ \bar{\Delta}_{13}\Delta_{13} & \bar{\Delta}_{23}\Delta_{23} & \bar{\Delta}_{33}\Delta_{23} + \bar{\Delta}_{23}\Delta_{33} \end{bmatrix}. \quad (\text{B.3})$$

We can write matrices from (3.14) as

$$B'_{\text{up}} = \begin{bmatrix} \bar{\Delta}_{11}\Delta_{12} & \bar{\Delta}_{21}\Delta_{22} & \bar{\Delta}_{31}\Delta_{22} & \bar{\Delta}_{21}\Delta_{32} & \bar{\Delta}_{31}\Delta_{32} \\ \bar{\Delta}_{11}\Delta_{13} & \bar{\Delta}_{21}\Delta_{23} & \bar{\Delta}_{31}\Delta_{23} & \bar{\Delta}_{21}\Delta_{33} & \bar{\Delta}_{31}\Delta_{33} \\ \bar{\Delta}_{12}\Delta_{13} & \bar{\Delta}_{22}\Delta_{23} & \bar{\Delta}_{32}\Delta_{23} & \bar{\Delta}_{22}\Delta_{33} & \bar{\Delta}_{32}\Delta_{33} \\ \bar{\Delta}_{12}\Delta_{11} & \bar{\Delta}_{22}\Delta_{21} & \bar{\Delta}_{32}\Delta_{21} & \bar{\Delta}_{22}\Delta_{31} & \bar{\Delta}_{32}\Delta_{31} \\ \bar{\Delta}_{13}\Delta_{11} & \bar{\Delta}_{23}\Delta_{21} & \bar{\Delta}_{33}\Delta_{21} & \bar{\Delta}_{23}\Delta_{31} & \bar{\Delta}_{33}\Delta_{31} \end{bmatrix}, \quad (\text{B.4})$$

$$B'_{\text{low}} = \begin{bmatrix} \bar{\Delta}_{11}\Delta_{12} & \bar{\Delta}_{21}\Delta_{22} & \bar{\Delta}_{31}\Delta_{22} & \bar{\Delta}_{21}\Delta_{32} & \bar{\Delta}_{31}\Delta_{32} \\ \bar{\Delta}_{11}\Delta_{13} & \bar{\Delta}_{21}\Delta_{23} & \bar{\Delta}_{31}\Delta_{23} & \bar{\Delta}_{21}\Delta_{33} & \bar{\Delta}_{31}\Delta_{33} \\ \bar{\Delta}_{13}\Delta_{12} & \bar{\Delta}_{23}\Delta_{22} & \bar{\Delta}_{33}\Delta_{22} & \bar{\Delta}_{23}\Delta_{32} & \bar{\Delta}_{33}\Delta_{32} \\ \bar{\Delta}_{12}\Delta_{11} & \bar{\Delta}_{22}\Delta_{21} & \bar{\Delta}_{32}\Delta_{21} & \bar{\Delta}_{22}\Delta_{31} & \bar{\Delta}_{32}\Delta_{31} \\ \bar{\Delta}_{13}\Delta_{11} & \bar{\Delta}_{23}\Delta_{21} & \bar{\Delta}_{33}\Delta_{21} & \bar{\Delta}_{23}\Delta_{31} & \bar{\Delta}_{33}\Delta_{31} \end{bmatrix}, \quad (\text{B.5})$$

$$B'_{\text{diag}} = \begin{bmatrix} \bar{\Delta}_{11}\Delta_{11} & \bar{\Delta}_{21}\Delta_{21} & \bar{\Delta}_{31}\Delta_{21} & \bar{\Delta}_{21}\Delta_{31} & \bar{\Delta}_{31}\Delta_{31} \\ \bar{\Delta}_{12}\Delta_{12} & \bar{\Delta}_{22}\Delta_{22} & \bar{\Delta}_{32}\Delta_{22} & \bar{\Delta}_{22}\Delta_{32} & \bar{\Delta}_{32}\Delta_{32} \\ \bar{\Delta}_{13}\Delta_{13} & \bar{\Delta}_{23}\Delta_{23} & \bar{\Delta}_{33}\Delta_{23} & \bar{\Delta}_{23}\Delta_{33} & \bar{\Delta}_{33}\Delta_{33} \end{bmatrix}. \quad (\text{B.6})$$

The matrices occurring in (3.15) have a form

$$C_{\text{up}} = \begin{bmatrix} \bar{\Gamma}_{11}\Gamma_{12} & \bar{\Gamma}_{21}\Gamma_{22} & \bar{\Gamma}_{31}\Gamma_{32} \\ \bar{\Gamma}_{11}\Gamma_{13} & \bar{\Gamma}_{21}\Gamma_{23} & \bar{\Gamma}_{31}\Gamma_{33} \\ \bar{\Gamma}_{12}\Gamma_{13} & \bar{\Gamma}_{22}\Gamma_{23} & \bar{\Gamma}_{32}\Gamma_{33} \end{bmatrix}, \quad (\text{B.7})$$

$$C_{\text{low}} = C_{\text{up}}(\Gamma \leftrightarrow \bar{\Gamma}), \quad (\text{B.8})$$

$$C_{\text{diag}} = \begin{bmatrix} \bar{\Gamma}_{11}\Gamma_{11} & \bar{\Gamma}_{21}\Gamma_{21} & \bar{\Gamma}_{31}\Gamma_{31} \\ \bar{\Gamma}_{12}\Gamma_{12} & \bar{\Gamma}_{22}\Gamma_{22} & \bar{\Gamma}_{32}\Gamma_{32} \\ \bar{\Gamma}_{13}\Gamma_{13} & \bar{\Gamma}_{23}\Gamma_{23} & \bar{\Gamma}_{33}\Gamma_{33} \end{bmatrix}. \quad (\text{B.9})$$

Similarly we can write matrices from (3.16) as

$$D_{\text{up}} = \begin{bmatrix} \bar{\Omega}_{11}\Omega_{12} & \bar{\Omega}_{21}\Omega_{22} & \bar{\Omega}_{31}\Omega_{32} \\ \bar{\Omega}_{11}\Omega_{13} & \bar{\Omega}_{21}\Omega_{23} & \bar{\Omega}_{31}\Omega_{33} \\ \bar{\Omega}_{13}\Omega_{13} & \bar{\Omega}_{23}\Omega_{23} & \bar{\Omega}_{33}\Omega_{33} \end{bmatrix}, \quad (\text{B.10})$$

$$D_{\text{low}} = \begin{bmatrix} \bar{\Omega}_{12}\Omega_{11} & \bar{\Omega}_{22}\Omega_{21} & \bar{\Omega}_{32}\Omega_{31} \\ \bar{\Omega}_{13}\Omega_{11} & \bar{\Omega}_{23}\Omega_{21} & \bar{\Omega}_{33}\Omega_{31} \\ \bar{\Omega}_{13}\Omega_{12} - \bar{\Omega}_{12}\Omega_{13} & \bar{\Omega}_{23}\Omega_{22} - \bar{\Omega}_{22}\Omega_{23} & \bar{\Omega}_{33}\Omega_{32} - \bar{\Omega}_{32}\Omega_{33} \end{bmatrix}, \quad (\text{B.11})$$

$$D_{\text{diag}} = \begin{bmatrix} \frac{\overline{\Omega}_{11}\Omega_{11}}{\overline{\Omega}_{12}\Omega_{12} + \overline{\Omega}_{12}\Omega_{13}} & \frac{\overline{\Omega}_{21}\Omega_{21}}{\overline{\Omega}_{22}\Omega_{22} + \overline{\Omega}_{22}\Omega_{23}} & \frac{\overline{\Omega}_{31}\Omega_{31}}{\overline{\Omega}_{32}\Omega_{32} + \overline{\Omega}_{32}\Omega_{33}} \\ \frac{\overline{\Omega}_{12}\Omega_{12}}{\overline{\Omega}_{13}\Omega_{12} + \overline{\Omega}_{12}\Omega_{13}} & \frac{\overline{\Omega}_{22}\Omega_{22}}{\overline{\Omega}_{23}\Omega_{22} + \overline{\Omega}_{22}\Omega_{23}} & \frac{\overline{\Omega}_{32}\Omega_{32}}{\overline{\Omega}_{33}\Omega_{32} + \overline{\Omega}_{32}\Omega_{33}} \\ \frac{\overline{\Omega}_{13}\Omega_{12} + \overline{\Omega}_{12}\Omega_{13}}{2} & \frac{\overline{\Omega}_{23}\Omega_{22} + \overline{\Omega}_{22}\Omega_{23}}{2} & \frac{\overline{\Omega}_{33}\Omega_{32} + \overline{\Omega}_{32}\Omega_{33}}{2} \end{bmatrix}. \quad (\text{B.12})$$

and the matrices from (3.17) look like

$$D'_{\text{up}} = \begin{bmatrix} \overline{\Omega}_{11}\Omega_{12} & \overline{\Omega}_{21}\Omega_{22} & \overline{\Omega}_{31}\Omega_{32} \\ \overline{\Omega}_{11}\Omega_{13} & \overline{\Omega}_{21}\Omega_{23} & \overline{\Omega}_{31}\Omega_{33} \\ \overline{\Omega}_{12}\Omega_{11} & \overline{\Omega}_{22}\Omega_{21} & \overline{\Omega}_{32}\Omega_{31} \end{bmatrix}, \quad (\text{B.13})$$

$$D'_{\text{low}} = D'_{\text{up}}(\Omega \leftrightarrow \overline{\Omega}), \quad (\text{B.14})$$

$$D'_{\text{diag}} = \begin{bmatrix} \overline{\Omega}_{11}\Omega_{11} & \overline{\Omega}_{21}\Omega_{21} & \overline{\Omega}_{31}\Omega_{31} \\ \overline{\Omega}_{12}\Omega_{12} & \overline{\Omega}_{22}\Omega_{22} & \overline{\Omega}_{32}\Omega_{32} \\ \overline{\Omega}_{13}\Omega_{12} & \overline{\Omega}_{23}\Omega_{22} & \overline{\Omega}_{33}\Omega_{32} \\ \overline{\Omega}_{12}\Omega_{13} & \overline{\Omega}_{22}\Omega_{23} & \overline{\Omega}_{32}\Omega_{33} \\ \overline{\Omega}_{13}\Omega_{13} & \overline{\Omega}_{23}\Omega_{23} & \overline{\Omega}_{33}\Omega_{33} \end{bmatrix}. \quad (\text{B.15})$$

Appendix C

Manifolds determined by set of M equations in N dimensional space

C.A Method of finding zeros of the function $\vec{F}$ in N dimensions

We shall be looking for solutions of a set of equations

$$F_j(x_1, x_2, \dots, x_N) = 0 \quad j = 1, 2, \dots, M. \quad (\text{C.1})$$

Let $\vec{x}$ denote the vector of values x_i , and $\vec{F}$ the vector of functions F_j . Let us expand $F_j(\vec{x})$ in a Taylor series

$$F_j(x_1 + \delta x_1, x_1 + \delta x_1, \dots, x_N + \delta x_N) = F_j(x_1, x_2, \dots, x_N) + \sum_{i=1}^N \frac{\partial F_j}{\partial x_i} \delta x_i + O(\delta x^2). \quad (\text{C.2})$$

The matrix of partial derivatives appearing in equation (C.2) is the rectangular Jacobian matrix $\mathbf{J}$. Thus, in matrix notation equation (C.2) reads:

$$\vec{F}(\vec{x} + \delta \vec{x}) = \vec{F}(\vec{x}) + \mathbf{J} \delta \vec{x} + O(\delta \vec{x}^2). \quad (\text{C.3})$$

We are interested in zeros of $\vec{F}(\vec{x})$, *i.e.* we are looking for such $\delta\vec{x}$ that $\vec{F}(\vec{x} + \delta\vec{x}) = 0$. Neglecting terms of the order $O(\delta\vec{x}^2)$, we obtain a set of linear equations for the corrections $\delta\vec{x}$

$$\mathbf{J}\delta\vec{x} = -\vec{F}. \quad (\text{C.4})$$

Equation (C.4) describes a set of M linear equations with n values of solution $\delta\vec{x}$. Each of these equations defines $(N - 1)$ dimensional plane in N dimensional space. In order to solve (C.4) we should find the intersection of these $N - 1$ -planes.

One should consider three cases:

1. If the number of linearly independent equations is equal to the number of coordinates $M = N$ then the set of equations has only one solution $\delta\vec{x}$;
2. If the number of linearly independent equations is lower than the number of coordinates $M < N$ and the equations are not contradictory, the set (C.4) has an infinite number of solutions which form a $N - M$ dimensional plane. Then one selects the solution from this $N - M$ plane which has the lowest norm.
3. In the case when $\delta\vec{x}$ doesn't exist, which means that the equations are contradictory, we adopt a procedure which tries to find some $\delta\vec{x}'$ which decreases the test function (C.3). If our set of equations is not contradictory then the algorithm reduces itself to two other cases.
 - (a) Let m be a number of linearly independent equations which are not contradictory. Then, from the set of equations (C.4) we can construct k sets of m linearly independent equations. Each of these sets determines m entries of the N dimensional vector $\delta\vec{x}^{(i) '}$. Here i ($= 1, \dots, k$) corresponds to the i^{th} set of equations.
 - (b) The remaining $(N - m)$ entries are set in a such way that $\delta\vec{x}^{(i) '}$ has the lowest norm.
 - (c) Next we choose such j that sum of the angles between a vector $\delta\vec{x}^{(j) '}$ and remaining vectors $\vec{x}^{(i) '}$ is minimal. $\delta\vec{x}^{(j) '}$ has the nearest direction to the average direction of other vectors $\vec{x}^{(i) '}$.

Instead of finding a zero of the M -dimensional function $\vec{F}$ we shall be looking for a global minimum of a function

$$f = \frac{1}{2} \vec{F} \cdot \vec{F}. \quad (\text{C.5})$$

Of course, there can be some local minima of Eq. (C.5) that are not solutions of Eq. (C.4).

Our step $\delta\vec{x}$ is usually in the descent direction of f

$$\vec{\nabla} f \cdot \delta\vec{x} = (\vec{F}\mathbf{J}) \cdot (-\mathbf{J}^{-1}\vec{F}) = -\vec{F} \cdot \vec{F} < 0 \quad (\text{C.6})$$

which is true if only $\mathbf{J}^{-1}$ exists.

It is convenient to define

$$g(\lambda) \equiv f(\vec{x}_{\text{old}} + \lambda\vec{p}), \quad (\text{C.7})$$

where $\vec{p} = \delta\vec{x}$. We always first try the full step *i.e.* $\lambda = 1$. If the proposed step does not reduce f we backtrack along the same direction until we have an acceptable step

$$\vec{x}_{\text{new}} = \vec{x}_{\text{old}} + \lambda\vec{p}, \quad 0 < \lambda \leq 1, \quad (\text{C.8})$$

i.e. we look for λ which sufficiently reduces $g(\lambda)$. This is done by approximating $g(\lambda)$ by polynomials in λ . Initially we know $g(0)$ and $g'(0)$ and also $g(1)$ which is known from the first trial ($\lambda = 1$). Since we can easily calculate the derivative of $g(\lambda)$

$$g'(\lambda) = \vec{\nabla} f \cdot \vec{p}. \quad (\text{C.9})$$

we can approximate $g(\lambda)$ by a quadratic polynomial in λ :

$$g(\lambda) \simeq [g(1) - g(0) - g'(0)] \lambda^2 + g'(0)\lambda + g(0) \quad (\text{C.10})$$

and look for the minimum of (C.10). If this step also fails, we model $g(\lambda)$ as a cubic polynomial in λ and so on until the satisfactory value of λ is found.

It is obvious that because of the linearity of the algorithm

$$\vec{F}(\vec{x} + \delta\vec{x}) = O(\delta\vec{x}^2). \quad (\text{C.11})$$

But, if $f(\vec{x} + \delta\vec{x}) \leq f(\vec{x})$, then our procedure leads towards the solution of (C.1), provided we are not in a vicinity of a false (*i.e.* local) minimum of f . In the latter case we have to change the initial conditions and start the whole procedure again.

C.B Algorithm for finding the curves

We shall describe a curve as a set of points placed along some path where the distance r between all neighboring points, should be constant. By definition all points should be zeros of function $\vec{F}$.

1. The input data of our algorithm are:
 - two points $\vec{y}_1, \vec{y}_2$, which should be placed in the vicinity of the sought curve,
 - distance r between these adjacent points.
2. Making use of the algorithm from Section C.A we find a root $\vec{x}_1$, situated in the vicinity of $\vec{y}_1$.
3. We define the point $\vec{x}_1$ as a center of a hyperspherical coordinate system. Next, we look on the sphere with radius r for the root $\vec{x}_2$ which has similar coordinates as point $\vec{y}_2$.
4. We shift the center of the coordinate system to the point $\vec{x}_2$ and look for a zero of $\vec{F}$ on the hypersphere of radius r in vicinity of point $\vec{y}_k (k = 3)$ extrapolated from previously found roots $\vec{x}_i$ with $i = 1, 2, \dots, k - 1$.
5. Iterating the above procedure we construct a curve of zeros of $\vec{F}$.

Using this algorithm we can find not only curves, but also k dimensional hypersurfaces. This can be done by fixing the values of $k - 1$ coordinates. Then for each choice depending on which coordinates were fixed, we can find curves from which we can, in principle, reconstruct the hypersurface. However, in our case it turned out that the zeros of $\vec{F}$ lie on one dimensional curves.

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