SIGNATURES OF RANDOMNESS IN QUANTUM SPECTRA*

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We investigate toy dynamical models of energy-level repulsion in quantum (quasi) energy eigenvalue sequences. We focus on parametric (with respect to a running coupling or "complexity" parameter) stochastic processes that are capable of relaxing towards a stationary regime (*e.g.* equilibrium, steady state asymptotic measure). In view of ergodic property, that makes them appropriate for the study of short-range fluctuations in any disordered, randomly-looking spectral sequence (as exemplified *e.g.* by empirical nearest-neighbor spacings histograms of various quantum systems). The pertinent Markov diffusion-type processes (with values in the space of spacings) share a general form of forward drifts $b(x) = (N-1)/2x - x$, where $x > 0$ stands for the spacing value. Here $N = 2, 3, 5$ correspond to the familiar (generic) random-matrix theory inspired cases, based on the exploitation of the Wigner surmise (usually regarded as an approximate formula). $N = 4$ corresponds to the (non-generic) non-Hermitian Ginibre ensemble. The result appears to be exact in the context of 2×2 random matrices and indicates a potential validity of other non-generic $N > 5$ level repulsion laws.

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1. Regular *versus* irregular in quantum theory

The vague notion of so-called quantum chaos, normally arising in conjunction with semiclassical quantum mechanics of chaotic dynamical systems [1], currently stands for a key-word capturing continued efforts to give a proper account to what extent quantization destroys, preserves, or qualitatively reproduces major features of classical chaos. There is no general agreement about what actually *is* to be interpreted as "quantum chaos" or

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its definite manifestations. That, in part, derives from an inherent ambiguity of quantization schemes for nonlinear, possibly nonconservative, driven and damped classical problems, and is intrinsically entangled with a delicate reverse problem of a reliable semiclassical limit procedure for once given quantum system. Other origins of this elusiveness seem to be rooted in the diversity of meanings attributed in the literature to the concepts of *regular* and/or *irregular* behavior of a physical system, irrespective of whether it is classical or quantum.

Mathematical definitions of classical chaos emphasize an apparent appearance of *randomness* in deterministic situations [2, 3]. That involves a deep question of when specific features of a physical system appear to be (or can be interpreted as) random. Quite typically, while dealing with an irregular behavior, we need to quantify an interplay between chance and order in terms of suitable measures of randomness (if random, then "how much?") [4, 6, 7].

Disorder, irregularity and randomness are casually perceived as synonyms and are interpreted to stay at variance with notions of order and regularity. Albeit order and randomness may as well coexist as "two faces of the same mysterious coin" [4].

One of basic problems in the quantum chaos theory is to establish whether the classical order–disorder interplay induces any unambiguous imprints ("signatures of chaos" [5]) in quantum systems. In this context, familiar concepts of regular and irregular spectra [8–10] were coined to characterize distinctive differences between semiclassical distributions of energy eigenvalues for *generic* quantum systems. The term "generic" basically means "more or less typical", since one excludes from considerations all systems which do not behave *properly,* although there are many of them. (More stringent definition invokes suitable symmetry properties of the quantum system.)

Nowadays it is clear that an irregularity alone of any particular spectral series (possibly interpreted in terms of an irregular sequence of consecutive energy or quasi-energy levels) is not an adequate criterion for quantum manifestations of chaos. In fact, semiclassical spectra corresponding to many [9] classical systems, be them integrable or chaotic, have an irregular appearance. That was the motivation for attempts to classify such spectra in terms of the "degree of randomness of the sequence of eigenvalues" [11], *cf.* also [6] and [7] for related argumentation.

Let us, however, stress that a primordial question of whether a given energy level sequence can at all be regarded as random has been left untouched. Seldom one may have at disposal a complete analytic information about quantum spectra. Usually some experimentation is needed to extract the data and most of available spectral information relevant for quantum

chaos studies, comes either from a genuine experiments (microwave analogs of quantum billiards, realistic nuclear data) or computer simulations, always with a definite beginning and an end. In each case that produces a finite string of data and it is known that no finite sequence can be interpreted as truly random. Fortunately, if a data sequence generated by a stochastic process of any origin (deterministic algorithms included) is sufficiently long, then it will always satisfy a test for randomness with fine-tuned confidence level, *cf.* [6, 12, 13].

We emphasize an issue of randomness, because various probability laws (and densities of invariant, steady state measures) are omnipresent in the quantum chaos research. In view of that, a stochastic modeling will be our major tool in below.

A rich class of classically integrable (hence regarded as regular) systems displays random-looking, locally uncorrelated sequences of energy eigenvalues [9,14,15]. However, there are well known classically chaotic (hence regarded as irregular) systems whose quantum spectral statistics appears to bear no distinctive imprints of classical chaos and look appropriate for the completely integrable case, [14]. Therefore, a supplementary rule is necessary to typify various classes of spectral irregularities and of the involved types of randomness being random, but possibly "random otherwise"), if those are to be interpreted as consequences of irregular characteristics of the related classical system.

A possible hint might have originated from discriminating between the spatial regularity and irregularity of the corresponding quantum eigenfunctions. It is the spatial pattern of wave functions that appears to have a decisive effect on the spectral pattern of eigenvalues [10,15]. However, a minor obstacle still persists: not all classically ergodic systems (irregular case of Ref. [8]) would semi-classically yield irregular eigenfunctions [10]. Consequently, one usually tries to narrow the class of quantum system that are suspected to show un-doubtful "signatures of chaos" to those which have irregular eigenfunctions, with no specific reference to their classical (chaotic or non-chaotic) behavior. In this class one ultimately attempts to specify those systems which remain in a consistent semiclassical relationship with their chaotic classical partners. Those systems quite justifiably would deserve to be named generic and would more or less naturally fall into various spectral universality classes, in accordance with the random-matrix classification scheme [17–19].

Under rather plausible assumptions [10], quantum systems with spatially irregular wave functions were found to exhibit *level repulsion*, hence to "avoid" degeneracies which is basically an indication of non-integrability, hence not necessarily that of any links with chaos.

We recall that an opposite spectral effect of level clustering, combined with the conjectured absence of correlations between levels, is characteristic for a large class ("almost all" according to [15,16], see, however, [20]) of classically integrable systems. Typically they display so-called Poisson statistics (strictly speaking, there is an exponential law of probability involved [21]) of adjacent level spacings: small spacings are predominant and there is enough room for multiply degenerate levels. One says then [22] that energy levels occur in a completely random way via a Poisson process on the energy axis.

Therefore, *level repulsion,* when regarded as an *emergent* spectral symptom of level correlations (usually interpreted as a certain departure from purely random behavior), may be viewed as a necessary condition to deal either with quantum imprints of classical chaos or, in the least, with a classically non integrable phase-space irregularity.

Interestingly enough, this viewpoint finds some support in the discovery of pseudo integrable systems (variously shaped billiards, sometimes with singular scattering obstacles) which appear to be neither integrable nor chaotic, but give rise to various forms of "wave chaos" while quantized [23,24]. The corresponding distribution of adjacent level spacings is named semi-Poisson and combines various forms (including fractional powers) of level repulsion with Poisson (exponential) statistics, hence purely random behavior in the spectral series. The repulsion phenomenon is here a manifestation of the topologically complicated phase space (an invariant manifold is not topologically equivalent to a torus but to a higher genus manifold), which was conjectured to preclude integrability and thus the standard torus (EBK) semiclassical quantization, see *e.g.* [23–26].

In view of the wide usage of such terms like "universality" in the quantum chaos literature, one should always keep in mind that harmonic oscillators display level repulsion [9], although they seem to be exemplary cases of classical and quantum regularity at their extreme. Another spectacular exception is the hydrogen atom spectrum. Like all higher dimensional harmonic oscillators, or a square billiard [16,27], the Coulomb spectrum belongs to a distinctive group of "pathologically non-generic" spectral problems [5].

Nonetheless, we shall confine our attention to the suggestive, random matrix theory universality classification that is considered to be faithful for local fluctuations in quantum spectra of (generic) systems that display global chaos in their classical phase spaces. Our hunch is to mimic (or rather extract) those features of the level-spacing classification which may bear imprints of pure randomness or in reverse — depart from randomness.

Studying classical manifestations of chaos in terms of probability measures (including their densities or distributions and their dynamics) is a respectable strategy [32]. In quantum theory, in view of Born's statistical interpretation postulate alone, probability measures are ubiquitous. On the other hand, various probability laws and distributions naturally pervade the

familiar random-matrix theory [17, 18]. This statistical theory of spectra, models a symmetry-limited spectral disorder in terms of statistical ensembles of complex quantum systems (*e.g.* heavy nuclei). Apart from an ensemble input, random-matrix theory forms a convenient vehicle to interpret spectroscopic properties of a concrete (single!) quantized version of a complex classical model. (The classical complexity notion refers *e.g.* to the phase-space organization specific to a system and various complication degrees of its dynamics related to ergodicity, mixing and exactness.)

However, one should keep in mind that the universality hypothesis in the context of quantum chaos proper, derives from exploiting a spectral affinity of an *ensemble* of large (with size ultimately growing to infinity) random matrices, with a once given *individual* quantum system (take into consideration the Sinai billiard or periodically kicked pendulum/rotator). Therefore, we may justifiably ask how an individual (Hamiltonian or Floquet-type) quantum eigenvalue problem may capture all conceivable statistical properties of suitable random-matrix ensemble spectra? Told otherwise, how may we justify a comparison of a statistical ensemble of disordered spectral series with *one* only specific energy (or quasi-energy) level sequence of an *a priori* chosen quantum system?

To our knowledge this conceptual obstacle, except for preliminary investigations of Ref. [28], has not received much attention in the quantum chaos literature. A partial answer to that question [5], points towards certain *ergodicity* properties appropriate for models of the parametric level dynamics (Coulomb gas, plasma or else, evolving in "fictitious time"), that provide a reinterpretation of random-matrix theory in terms of an equilibrium statistical mechanics for a fictitious N-particle system (with N allowed to grow indefinitely), *cf.* also [29, 30].

In the framework of random-matrix theory, an ergodic problem for Gaussian ensembles was analyzed long ago in Ref. [31], with a focus on the ergodic behavior for the eigenvalue density and k-point correlation functions of individual random matrices and their statistical ensembles. That involves a local version of the ergodic theorem, where *e.g.* the spectral averaging over a finite energy span of the level density is compared with the matrix ensemble mean of the level density. That suggests analogies with disordered many-body quantum systems where ensemble averaging is a standard analytic tool, while for an individual system, only an energy averaging should be employed [30].

We know that the distribution of spacings of highly excited quantum systems may involve definite laws like *e.g.* the exponential or Wigner-type distributions. Such laws may be related to definite stochastic processes as invariant (steady state) measures, in particular as asymptotic measures to which the process does relax.

We shall focus upon the parametric dynamics (parametric interpolation) scenario for the nearest-neighbor spacing distributions of irregular quantum systems where asymptotic invariant (with respect to the parametric process) probability measures are ultimately involved. Let us mention that a concept of parametric dynamics involves the possibly troublesome "fictitious time" parameter. Its possible interpretation is that of a running coupling constant measuring the strength of the chaotizing perturbation, or more generally that of a "complexity parameter" whose growth to infinity gives account of the complexity increase in the spectral properties of a quantum system.

Disregarding the origins of randomness in diverse settings, we shall take the view that stochastic processes are mathematically appropriate models when the time evolution (parametric "dynamics" being included) of random phenomena is involved. Whenever probability laws are in use, random phenomena and stochastic processes are always at hand [32,33].

The major difference of our strategy, if compared to other approaches, amounts to considering exclusively the parametric evolution (relaxation) towards equilibria of nearest neighbor spacing distributions as the major source of probabilistic information. We arrive here at prototype invariant (equilibrium) measures and stationary (steady state) probability densities of limiting stationary stochastic processes. We do not invoke any explicit eigenvalue (*e.g.* a solution of the spectral problem for the quantum system or the related random-matrix model) nor level dynamics input, since those data prove to be irrelevant for the primary ergodic behavior that is displayed by the adjacent spacing probability densities. At least in the considered approximation regime, where probability densities surmised by Wigner are assumed to be adequate (in reality, they have the status of reliable approximate formulas).

An exploitation of ergodicity (in fact strong mixing and/or exactness [32, 34, 35]) of certain (parametrically evolving) Markovian stochastic processes is here found to provide a supplementary (probabilistic) characterization of quantum signatures of chaos.

2. Poissonian level sequences

2.1. Exponential random variable and semi-Poisson laws

"Poissonian" matrix ensembles with independent random diagonal elements are often used to model spectral properties of integrable Hamiltonian systems (we disregard an issue of various, even quite remarkable, deviations from an exact Poisson-type statistics [16,20,36]). Indeed, many regular (integrable) systems, semi classically exhibit spacings between adjacent energy levels which are distributed according to the exponential probability density $p(s) = \exp(-s)$ on R^+, where we tacitly assume a normalization of the first

moment (mean spacing) of the probability measure (hence the unfolding of the energy spectrum [5, 16]).

A canonical statement in this respect [9], conveys a message that "for generic regular systems" $p(s)$ is "characteristic of a Poisson process with levels distributed at random" and "the levels are not correlated". (A discussion of serious violations of the Berry–Tabor conjecture can be found in Ref. [20].)

Since the regular spectrum is perfectly deterministic and for each set of quantum numbers the corresponding energy level is obtained from an explicit formula (via Einstein–Brillouin–Keller semiclassical argument, or directly by solving the spectral problem for *e.g.* rectangular billiard) it is far from obvious that levels may come as a realization of a random variable. Even though probability distributions are thought to arise in near classical quantum systems, when the number of levels in any range of energy is very large (and indefinitely increases when the classical limit is approached).

Following Refs. [16, 19], let us consider a sequence of numbers (we keep an explicit energy notation, although an unfolded sequence is rescaled to be non-dimensional):

$$E_{i+1} = E_i + x_i = E_0 + \sum_{j=1}^{i} x_j \,, \tag{1}$$

where $E_0 = 0$ and x_j with $j = 1, 2, ...$ are outcomes of independent trials of the exponentially distributed random variable X taking values in R_+.

The resulting sequence $(E_1, E_2, ...)$ of nonnegative numbers is a particular model realization (sample) for what is commonly named a Poisson spectrum. Here, randomly sampled (independent, in accordance with the exponential distribution) increments $x_i = E_{i+1} - E_i$ play the role of adjacent level spacings. Let us emphasize that the Poissonian random-matrix ensemble would comprise all possible sequences of the above form, each obtained as a result of independent sampling procedures.

At this point let us turn to an explicit probabilistic lore (*cf.* [21, 33]) whose absence is painfully conspicuous in major quantum chaos publications.

Let $X_1, X_2, ...$ be independent random variables with common for all exponential probability law $\mu(x) = \alpha \exp(-\alpha x)$, $\alpha > 0$ with mean $1/\alpha$ and variance $1/\alpha^2$.

Furthermore let us denote $S_n = X_1 + X_2 + ... + X_n$, $n = 1, 2, ...$. Then the random variable S_n has a probability density:

$$p_n(x) = \frac{\alpha^n \, x^{n-1}}{(n-1)!} \exp(-\alpha x) \,, \tag{2}$$

coming from an $(n-1)$-fold convolution of exponential probability densities

on R^+. The law (2) is infinitely divisible [21,33]:

$$p_{n+m}(x) = (p_n * p_m)(x) = \int_0^x p_n(x-y)p_m(y)dy\,, \tag{3}$$

where $p_1(x) = \mu(x)$ and $n, m = 1, 2, \ldots$.

In particular, note that $X_i + X_j$ for any $i, j, \in N$ has a probability density

$$p_2(x) = \alpha^2\, x\, \exp(-\alpha x)\,. \tag{4}$$

which upon setting $\alpha = 2$ and $x = s$ stands for an example of a semi-Poisson law $P(s) = 4s\, \exp(-2s)$ [5,26], which has been identified to govern the adjacent level statistics for a subclass of pseudo integrable systems.

It is also obvious that other (plasma-model related [26]) semi-Poisson laws come directly from distributions appropriate for S_n. For example, S_3 has a density $p_3(x)$ which upon substituting $\alpha = 3$ and $x = s$ gives rise to $P(s) = (27/2)s^2\, \exp(-3s)$. Analogously, S_5 yields $p_5(x)$ and upon setting $\alpha = 5$ implies $P(s) = (3125/24)s^4\, \exp(-5s)$, *cf.* Eq. (36) in Ref. [26].

2.2. Gaussian regime

Both in the quantum chaos and random-matrix theory contexts, the regime of $n \gg 1$ is of utmost importance. Since the primary random variable X has an exponential density with mean $\mu = 1/\alpha$ and variance $\sigma^2 = 1/\alpha^2$, we stay within the conditions of the central limit theorem [21]. First of all we know that for every $\varepsilon > 0$:

$$P\left[\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right] \longrightarrow 0 \tag{5}$$

when $n \to \infty$. Hence $(1/n)S_n \to \mu$ with probability 1.
Furthermore, we have:

$$P\left[a < \frac{S_n - n\mu}{\sigma\sqrt{n}} < b\right] \longrightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_a^b \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx\,. \tag{6}$$

To give a pedestrian intuition about the above formal observations, let us ask for a probability that there holds

$$\left|\frac{S_n}{n} - \mu\right| < a\frac{\sigma}{\sqrt{n}} \tag{7}$$

for any $a > 0$. In the regime of large n, an integral

$$\frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \exp(-y^2)\, dy$$

gives a reliable answer. The same integral determines the probability that $|S_n - n\mu| < a\sigma\sqrt{n}$, hence tells us how S_n fluctuates about $n\mu$ (and S_n/n about μ) with the growth of n.

2.3. Whence Poisson process on the energy axis?

The probability density of the random variable S_n allows us to evaluate a probability that the n-th level energy value E_n is actually located in an interval $[E, E + \Delta E]$ about a fixed nonnegative number E. It is easily obtained by redefining the previous $p_n(x)$, *cf.* Eq. (2)

$$\begin{aligned} P[E \le S_n \le E + \Delta E] &= \frac{\alpha^n E^{n-1}}{(n-1)!} \exp(-\alpha E)\, \Delta E \\ &= \frac{S^{n-1}}{(n-1)!} \exp(-S)\, \Delta S \\ &= P_n(S)\, \Delta S\,, \end{aligned} \tag{8}$$

where $x = E$, $S = (E/\langle E \rangle)$ and $1/\alpha = \langle E \rangle$ is the mean adjacent level spacing. The probability density $P_n(S)$, in Ref. [5] is interpreted as "probability density for finding the n-th neighbor of a level in the distance increment $[S, S + dS]$, for a stationary Poisson process", while in Ref. [16], while denoted $E(k, L) \to E(n-1, S)$ where L is replaced by our S, stands for the "probability that inside an interval of length S we find exactly $n-1$ levels". Since $E(k, L)$ has the form of a standard Poisson probability law with mean-value and variance L, one may also follow [16] to tell that "if they are on the average L events, then the probability to actually observe k events is given by $E_{\text{Poisson}}(k, L) = (L^k/k!) \exp(-L)$".

Indeed, if $E > 0$ is a fixed energy value and we ask for a probability that there are exactly n energy levels below E, then probability distributions for S_n and S_{n+1} combine together to yield the Poisson distribution with mean αE:

$$P\left[S_n \le E < S_{n+1}\right] = P[N(E) = n] = \frac{(\alpha E)^n}{n!} \exp(-\alpha E)\,. \tag{9}$$

In this connection, let us recall that a random variable N taking discrete integer values $0, 1, 2, ...$ is said to have Poisson distribution with the mean (and variance) λ if the probability of $N = k$ reads $P[N = k] = \lambda^k/k! \exp(-\lambda)$. Clearly, $\sum_{k=0}^{\infty} P[N=k]=1$ and $E[N] = \sum_{0}^{\infty} kP[N=k]=\lambda$.

Let us, however, stress that *no* explicit Poisson process was involved anywhere in the above, since its precise mathematical definition [21,37] refers to a counting process with a one parameter family of random variables $[N_t = N(t) = n] = [S_n \leq t < S_{n+1}]$ obeying the Poisson probability law for all $t \in R^+$:

$$P[S_n \leq t < S_{n+1}] = P[N_t = n] = \frac{(\alpha t)^n}{n!} \exp(-\alpha t)\,. \tag{10}$$

The Poisson process has stationary independent increments: $N_{t_1}, N_{t_2}–N_{t_1}, ...$ for $0 < t_1 < t_2 < ...$ with the Poisson probability distribution for each increment

$$P[N_t - N_s = n] = \frac{[\alpha(t-s)]^n}{n!} \exp\left[-\alpha(t-s)\right], \tag{11}$$

where $N_0 = 0$ with probability 1. Here, by denoting $P_n(t) = P[N_t = n]$ and $P_n(t-s) = P[N_t - N_s = n]$ we easily check that $\int_0^t P_n(t-s)P_m(s)(\alpha ds) = P_{n+m}(t)$.

The related intensity (parameter, mean) of the Poisson "process" equals $E[N_t] = \alpha t$ and displays the linear growth when t increases. Notice also that $N_t/t \to \alpha$ with probability 1 as $t \to \infty$.

(The Poisson process is a particular example of a Markovian process *in law* [33]. We deal here with a temporally homogeneous process N_t, $t > 0$ associated with an infinitely divisible probability distribution $\mu(k) = \frac{c^k}{k!} \exp(-c)$, The process in law is here recovered by simply setting $\mu^t(k) = \frac{(ct)^k}{k!} \exp(-ct)$ where $\mu^1(k) = \mu(k)$.)

The Poisson process involves time dependent probabilities: $P_0(t) = \exp(-\alpha t)$, $P_1(t) = \alpha t \exp(-\alpha t)$, ..., which should be compared with previous outcomes for the exponential random variable. By recalling Eq. (2) we immediately arrive at a formal identification of probability distributions:

$$p_{k+1}(t) = \alpha\, P_k(t)\,. \tag{12}$$

In the above, the exponential probability density is labeled by time t. Let us stress that $p_{k+1}(t)\,\Delta t$ stands for a probability that the random variable S_{k+1} takes its value in the interval $[t, t + \Delta t]$, while $P_k(t)$ is a probability that $N_t = n$.

Notwithstanding, Eq. (9) is formally identical with Eq. (10) and, therefore, we can in principle vary the parameter E, so setting (9) in a direct equivalence with a parametric (evolving in fictitious time) Poisson process. This formal equivalence underlies a Poisson process lore of the quantum chaos literature.

Instead of paying attention to the exponential probability rule which is responsible for the randomness of the collection of "time" instants on R^+, one is tempted to tell that it is the Poisson process which dictates those rules of the game. The standard way of thinking refers to the observation of the number of signals recorded up to an instant t (actually, number of jumps of N_t or the number of levels that are below E) [21].

2.4. Ergodicity

Sample paths of the Poisson process N_t are nondecreasing functions of t with integer values. If we attempt to draw a sample path, we begin from the value $N_t = 0$ which is maintained up to the time instant $S_1 = t_1$ when the jump occurs to $N_t = 1$. This value stays constant up to the time $S_2 = x_1 + x_2$. Then, a new jump to $N_t = 3$ occurs, and that value survives until $S_3 = x_1 + x_2 + x_3$ is sampled. The sample path construction for the Poisson process strictly parallels a time series construction in terms of points on R^+ at which jumps of N_t occur. Intervals between consecutive time instant form the sequence $(x_1, x_2, x_3...)$ of adjacent level spacings.

On the other hand, it is Eq. (1) which provides us with a concrete sample sequence of levels $(E_1, E_2, ...)$, drawn in accordance with the exponential probability law for adjacent level spacings $x_i, i \geq 1$. Thus, the set of all realizations of the random variable $E = (S_1, S_2, ...)$ comprises a statistical ensemble of sample sequences $\omega : E(\omega) = (E_1, E_2, ...)$. In fact, those sequences exemplify the Poissonian ensemble of spectra.

(If we set $\alpha = 1$, then a connection with the standard Poissonian reasoning in the random-matrix approach to quantum chaos is immediate. A cataloge of various statistical measures for the Poissonian spectra can be found in [19].)

If we would construct a histogram of adjacent level spacings for a *single* sequence $(E_1, E_2, , ,)$ which was compiled in accordance with the exponential distribution, the familiar Poissonian shape would be revealed.

As well, the very same picture would emerge if we would randomly collect and make a statistical analysis of various finite strings of neighboring energy levels, like in case of the so called nuclear data ensemble composition (there *e.g.* one makes a compilation of 1407 data points from 30 sequences of levels experimentally found for 27 different nuclei) [17,19].

All that is connected with a primitive at this stage notion of *ergodicity* of the exponential *process* .

Namely, let us consider a one-parameter family $(X_n,\ n=1,2,...)$ of exponential random variables as a stochastic process with "discrete time". Since X_n are independent random variables with the *same* for all n probability distribution, then for any real function $(f: x \in R^+ \to f(x) \in R)$ such that $\langle f \rangle = E[f(X_1)]$ exists, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) = \langle f \rangle = \int\limits_{R^+} f(x)\, \mu(dx) \tag{13}$$

for all sample sequences $X(\omega) = (x_1, x_2, ...)$. In that case the random sequence $X_n, n \geq 1$ is known to be *ergodic with respect to* f. That is a standard link between the "time average" and "ensemble average", which is here accomplished by means of the exponential probability measure μ. Indeed, as often happens in the context of stationary stochastic processes, ergodicity property allows us to replace an average over the set of *all* realizations of the process at a chosen time instant, by the time average evaluated along *one* sample trajectory.

If we consider $f(X_n) = X_n$ for all $n \geq 1$, then the ergodicity notion refers to limiting properties of S_n/n. Accordingly, in view of the law of large numbers Eq. (5) and (13) hold true.

Presently, there is no wonder in the fact that single eigenvalue series of a suitable integrable quantum system (like *e.g.* the rectangle billiard of Refs. [9, 16, 27, 36]) may be utilized to generate a statistical information in (approximate [20]) affinity with the ensemble statistics. Numerical research involving *e.g.* about 10^6–10^{19} levels for the eigenvalue series $E_{m,n} = m^2 + \gamma n^2$, $\gamma = \pi/3$, *cf.* [16], allows to generate various statistical data. The nearest neighbor spacing histograms show a very close resemblance to the exponential distribution curve, in agreement with the conjecture of Ref. [9]. Effectively, the eigenvalue sequence of the rectangle billiard can be interpreted as (in fact mimics) a sample path $E(\omega) = (E_1, E_2, ...)$ with adjacent spacings x_i distributed according to the exponential law.

A standard (Poissonian) way of thinking in this context, refers to an "observation of the number of signals recorded up to an instant t" (actually, jumps of N_t or number of levels that are below E) [21]. However, the sample path $E(\omega)$ encodes also a complete information about a sample path of the involved exponential process $X = (X_1, X_2, ...)$.

Our ergodic argument is valid with respect to any chosen sample path $X(\omega) = (X_1(\omega) = x_1,\ X_2(\omega) = x_2, ...)$ of X. An ensemble average is provided by $\int\limits_{R^+} x\mu(dx) = 1/\alpha$ and that value is to coincide with

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_n$$

irrespective of the particular choice of a sample path ω of the exponential process $X_n, n \geq 1$.

Ergodicity property normally embodies the weakest form of complications present in the evolution of physical systems, including those modeled by stochastic processes. There is a well established catalog of irregular behaviors that the dynamics of any type may exhibit and there are stronger types of irregularity than those connected with ergodicity. A corresponding hierarchy of irregularities refers to the properties of mixing and exactness [34,35] which will be exploited in below.

3. Gaussian universality classes: generalities

In the random matrix approach we have *a priori* involved random-looking sequences of energy levels [38], which well agrees with the phenomenology of nuclei where inadequacies of fundamental theoretical models are compensated by resorting to statistical matrix ensembles with appropriate symmetries. The roots of randomness presumably can be attributed to random deformations of the "shape of the nuclei" (bag) in the independent-particle model of nuclear dynamics, [39]. An analog of this reasoning can be found in a recent analysis [40] of a chaotic system in a cavity (billiard) with a parametric control of shape deformations. Then a quantum particle is confined within a continuously deformed boundary, whose parametric dynamics can be as well represented by a stochastic process of any kind.

A concrete quantum system (like *e.g.* a spectral problem for concretely shaped billiard) usually induces its own unique spectrum and there is no need, nor room for any statistical ensemble of systems unless we shall indeed consider a family of quantum systems with a suitable selection of random potentials. We must thus cope with obvious discrepancies underlying otherwise attractive affinities (*e.g.* the universality classes idea for spectral statistics). Useful affinities appear to mask quite deep differences between the underlying physical mechanisms.

It is the level repulsion which is routinely interpreted as a quantum manifestation of classical non integrability and ultimately also of chaos, *cf.* [10].

Normally that is quantified by means of polynomial modifications of the Gaussian probability law (in association with the Wigner–Dyson statistics of adjacent level spacings for *e.g.* unitary, orthogonal and symplectic random matrix ensembles). For completeness of the argument, let us list the standard formulas:

$$P_1(s) = s\frac{\pi}{2}\exp\left(-\frac{s^2\pi}{4}\right), \quad P_2(s) = s^2\frac{32}{\pi^2}\exp\left(-\frac{s^2\pi}{4}\right)$$

$$\text{and} \quad P_4(s) = s^4\frac{2^{18}}{3^6\pi^3}\exp\left(-\frac{s^2 64}{9\pi}\right),$$

corresponding, respectively, to the GOE, GUE and GSE random-matrix theory predictions.

Let us point out [5] that for the most practical cases the Wigner distributions (albeit exact in the 2×2 random-matrix case only) are adequate. Typical spacing histograms drawn from experimental or numerically generated (quasi)energy spectra are too rugged to allow subtle distinctions against the $n \to \infty$ random-matrix size related predictions.

We shall consider mostly the Wigner-type cases, even though neither of those probability laws deserves the status of being an *exact* representation of the real state of affairs. Remember that also in the context of random matrix theories the Wigner spacing formulas are approximations which usually improve in the large matrix size regime.

The nearest neighbor spacing distributions, in the random-matrix approach are the secondary notions and can be derived from an explicit formula for the joint probability density to find the (dimensionless) energy eigenvalues in respective infinitesimal intervals $[x_i, x_i + \Delta x_i]$ with $i = 1, 2, \ldots, N$

$$P(x_1, x_2, \ldots, x_N) = C_{N\beta}\left[\prod_{i>j=1}^{N}\left|x_i - x_j\right|^{\beta}\exp\left(-\frac{1}{2}\sum_{i=1}^{N}x_i^2\right)\right], \qquad (14)$$

where $\beta = 1, 2, 4$ and $C_{N\beta}$ is a normalization constant [5, 17]. The level repulsion has been built into the framework from the very beginning and appropriate level spacing distributions (including the adjacent level case) can be directly evaluated on that basis [17, 41, 42].

There were many attempts to provide convincing (and independent from the definite symmetry and Gaussian randomness inputs, proper to random-matrix theory) arguments that would generate level repulsion through well defined dynamical mechanisms (like *e.g.* the parametric level dynamics) and would lead to statistical predictions as well. A suitable level dynamics scenario may as well give rise to the so-called intermediate statistics and possibly a continuous (parametric) interpolation among them.

In the random matrix theory context a radical probabilistic attempt due to Dyson explicitly involves the (parametric) Brownian motion assumption for each energy level separately [17, 42, 43].

More satisfactory results were obtained by resorting to a fictitious gas of interacting particle representatives of individual energy levels. A corresponding many-particle system is then investigated at suitable "thermal equilibrium" conditions. Then, without introducing *a priori* statistical ensembles of random matrices, level distribution functions are derived by means of ordinary statistical mechanics methods. That approach explicitly involves the many-body Hamiltonian (Calogero model)

$$H = -\sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2} + \frac{\beta(\beta-2)}{4} \sum_{i<j} \frac{1}{(x_i - x_j)^2} + \sum_{i=1}^{n} x_i^2 \,, \tag{15}$$

whose squared ground state function (equilibrium measure density) has the form (14) [30, 41, 42].

Apart from that, explicit quantum mechanical investigations for billiard-type systems provide hints about the potential importance of interpolation studies, especially since various intermediate types of statistics were reported to occur, see *e.g.* [1, 40, 50].

There are two basic approaches to an interpolation issue. One refers explicitly to random matrix theories and their "affinity" with quantum chaotic systems [41, 44, 45]. Another refers to the fictitious gas, interacting many-body analogy, [5, 10, 46–49]. Recently, a related short-range plasma model was proposed to analyze an emergence of the "pseudo-Poisson statistics" [26].

4. Parametric dynamics of adjacent level spacings

4.1. Markov processes defined through their steady state measures

Once we have encountered probability densities on the positive half-line in R^1, it is rather natural to investigate a general issue of parametric stochastic processes which would provide a dynamical model of level repulsion in an irregular quantum system and generate at the same time spacing densities as those of asymptotic invariant (equilibrium) probability measures. Such random processes clearly must run with respect to the previously mentioned "fictitious" time-parameter and take values in the set of all level spacings which are appropriate for a complex quantum system or the corresponding random-matrix ensemble.

Effectively, we wish to introduce a Markovian diffusion-type process which might stand for a reliable approximation of a random walk over *level spacing sizes.*

For future reference let us mention that in the regime of equilibrium (when a stationary measure appears in the large "time" asymptotic), a sample path of such random walk would take the form of an ordered sequence of spacings which are sampled (drawn) according to the prescribed invariant

probability distribution. That is precisely *one* explicit example of the ladder of energy levels, understood as a random sample drawn from a suitable ensemble.

An analysis of statistical features of this spectral sequence involves an ergodicity notion to stay in conformity with the ensemble evaluation of various averages (carried out with respect to the invariant density), [32,51].

We shall consider the previously listed GOE, GUE and GSE probability densities on R^+ (up to suitable rescalings!) as, distorted in view of the spacing size normalization, asymptotic invariant densities of certain parametric Markovian stochastic processes whose uniqueness status can be unambiguously settled.

Let as begin from the observation that probability densities on R^+, of the characteristic form $f(x) \sim x\exp(-x^2/4)$, $g(x) \sim x^2\exp(-x^2/2)$ and $h(x) \sim (x^4/4)\exp(-x^2)$ appear notoriously in various quantum mechanical contexts (harmonic oscillator or centrifugal-harmonic eigenvalue problems), *cf.* [52,54–57]. Notwithstanding, as notoriously they can be identified in connection with special classes of stationary Markovian diffusion processes on R^+ [58].

Anticipating further discussion, let us consider a Fokker–Planck equation on the positive half-line in the form:

$$\partial_t\rho = \frac{1}{2}\Delta\rho - \nabla\left[\left(\frac{\beta}{2x} - x\right)\rho\right], \tag{16}$$

which may be set in correspondence with the stochastic differential equation $dX_t = (\beta/(2X_t) - X_t)dt + dW_t$ formally valid for a random variable X_t with values contained in $(0,\infty)$. Here $\beta \geq 0$ and W_t represents the Wiener process.

Accordingly, if $\rho_0(x)$ with $x \in R^+$ is regarded as the density of distribution of X_0 then for each $t > 0$ the function $\rho(x,t)$, solving Eq. (16), is the density of X_t. In view of a singularity of the forward drift at the origin, we refrain from looking for strong solutions of the above stochastic differential equation and confine attention to weak solutions only and the associated tractable parabolic problem (16) with suitable boundary data, *cf.* [58].

In all those cases a mechanism of repulsion is modeled by the $1/x$ term in the forward drift expression. The compensating harmonic attraction which is modeled by the $-x$ term, saturates the long distance effects of repulsion-induced scattering and ultimately yields asymptotic steady (stationary) probability densities.

To interpret a density $\rho(x)$ as an asymptotic steady state (stationary, invariant) density of a well defined Markovian diffusion process we shall utilize the rudiments of the so-called Schrödinger boundary and stochastic interpolation problem, [52,55,59], see also [53] when specialized to invariant measures.

Let us notice that both in case of the standard Ornstein–Uhlenbeck process and its Bessel (radial) variant, we have emphasized the role of a stochastic process with an asymptotic invariant density. To deduce such processes, in principle we can start from an invariant density and address an easier issue of the associated measure preserving stochastic dynamics. Next we can consider whether the obtained process would drive a given initial density towards a prescribed invariant measure (in that case we can tell about an asymptotic state of equilibrium to which the process relaxes). That feature involves the notion of exactness of the related stochastic process, whose straightforward consequences are the properties of mixing and ergodicity of the corresponding random dynamics [32].

There is a general formula [52, 53, 59] relating the forward drift of the sought for stationary process with an explicit functional form of an invariant probability density. We confine our attention to Markov diffusion processes with a constant diffusion coefficient, denoted $D > 0$. Then, the pertinent formula reads:

$$b(x) = 2D\frac{\nabla\rho^{1/2}}{\rho^{1/2}}\,. \tag{17}$$

In particular, for the familiar Ornstein–Uhlenbeck process we have $\rho^{1/2}(x) = (1/\pi)^{1/4}\exp\left(-x^2/2\right)$ and $D = 1/2$, so we clearly arrive at $b(x) = -x$ as should be. Quite analogously, in case of the GUE-type spacing density, we have $D = 1/2$ and $\rho^{1/2}(x) = 2/(\pi^{1/4})x\exp\left(-x^2/2\right)$. Thus, accordingly $b(x) = (1/x - x)$.

The very same strategy allows us to identify a forward drift of the Markovian diffusion process supported by the GOE-type spacing density. By employing

$$\rho^{1/2}(x) = \sqrt{2x}\exp\left(-\frac{x^2}{2}\right)$$

and setting $D = 1/2$ we arrive at the formula: $b(x,t) = (1/(2x) - x)$.

We immediately identify the above forward drifts with the ones appropriate for the time homogeneous radial Ornstein–Uhlenbeck processes, with a corresponding family of ($N > 1$ and otherwise arbitrary integer) transition probability densities, [58]:

$$\begin{aligned} p_t(y,x) &= p(y,0,x,t) \\ &= 2x^{N-1}\exp\left(-x^2\right)\ \frac{1}{1-\exp(-2t)}\exp\left(-\frac{\left(x^2+y^2\right)\exp(-2t)}{1-\exp(-2t)}\right) \\ &\quad\times\ [xy\exp(-t)]^{-\alpha} I_\alpha\left(\frac{2xy\ \exp(-t)}{1-\exp(-2t)}\right)\,, \end{aligned} \tag{18}$$

where $\alpha = (N-2)/2$ and $I_\alpha(z)$ is a modified Bessel function of order α

$$I_\alpha(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+\alpha}}{(k!)\Gamma\,,(k+\alpha+1)}\,, \tag{19}$$

while the Euler gamma function has a standard form

$$\Gamma(x) = \int_0^\infty \exp(-t)t^{x-1}dt\,.$$

We remember that $\Gamma(n+1) = n!$ and $\Gamma(1/2) = \sqrt{\pi}$.

The resultant forward drift has the general form

$$b(x) = \frac{N-1}{2x} - x\,, \tag{20}$$

and corresponds to $\beta = N-1$.

By setting $N = 2$, and then employing the series representation of $I_0(z)$, we easily recover the asymptotic invariant density for the process:

$$\lim_{t\to\infty} p(y,0,x,t) = 2x\exp(-x^2)\,.$$

We can also analyze the large time asymptotic of $p(y,0,x,t)$, in case of $N=3$ which gives rise to an invariant density in the form $4/(\sqrt{\pi})x^2\exp(-x^2)$. That obviously corresponds to the GUE-type case with $b(x) = (1/x - x)$.

When passing to the GSE case, we are interested in the Markovian diffusion process which is supported by an invariant probability density

$$\rho(x) = \frac{2}{\Gamma(3/2)}x^4\exp(-x^2)\,.$$

Let us evaluate the forward drift of the sought for process (we set $D=1/2$): $b(x,t) = 2/x - x$. Clearly, we deal here with a radial Ornstein–Uhlenbeck process corresponding to $N = 5$. The transition probability density of the process displays an expected asymptotic:

$$\lim_{t\to\infty} p(y,0,x,t) = \frac{4}{\sqrt{\pi}}\,x^4\exp(-x^2)\,.$$

Here we have exploited $\Gamma(1/2) = \sqrt{\pi}$ to evaluate $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$.

The above formulas allow us to formulate a hypothesis that further non-generic repulsion laws may be appropriate for quantifying quantum chaos. Straightforwardly, one can verify that our transition probability densities refer to asymptotic invariant densities of the form:

$$\rho(x) = \frac{2}{\Gamma(N/2)} x^{N-1} \exp(-x^2)\,. \tag{21}$$

In particular we get a direct evidence in favor of $N = 4$, *i.e.* $b(x) = (3/(2x) - x)$, universality class which in fact corresponds to the Ginibre ensemble of non-Hermitian random matrices [5], where a cubic level repulsion appears: $\rho(x) = 2x^3 \exp(-x^2)$ (this formula is exact for 2×2 random matrices).

In principle, processes corresponding to any $N > 5$ may be realizable as well, and thus the related higher-power level repulsion might have relevance in the realm of quantum chaos.

In all considered cases, an asymptotic invariance of probability measures (densities) is sufficient to yield ergodic behavior. For each value of $N > 1$ we deal with an independent repulsion mechanism, albeit all of them belong to the radial Ornstein–Uhlenbeck family.

We have thus identified a universal stochastic law (in fact, a family of the like) behind the functional form of basic, Wigner surmise inspired, spacing probability densities appropriate for quantum chaos.

Let us emphasize at this point that one should keep in mind a number of possible reservations coming from the fact that neither of "universal" or "generic" laws can be regarded as a faithful representation of a real state of affairs. Usually exact laws are derived for two by two (hence of the small size!) random matrices, and are known to reappear again (at least in the generic cases) as approximate spacing formulas in the large random-matrix size regime. That in turn allows to achieve a correspondence with semiclassical quantum spectra of complex systems.

There is no obvious explanation of a physical meaning of the integer parameter N in the radial stochastic process scenario. One hypothesis comes from the random-matrix theory, where $\beta = N - 1 = 1, 2, 4$ would correspond to a number of independent components of a typical matrix entry which is decided by the underlying symmetry of the problem (GOE, GUE, GSE). That can be presumably be extended to the case of $N = 4$.

4.2. Link with Calogero Hamiltonian

Previously we have indicated that a common mathematical basis for various level repulsion mechanisms appropriate to quantum chaos is set by the Calogero–Moser Hamiltonian. At the first glance, our stochastic arguments

may leave an impression that something completely divorced from that setting has been obtained in the present paper. However, things look otherwise and our theoretical framework proves to be compatible with standard techniques for spectral analysis of complex quantum systems.

It is peculiar to the general arguments of Refs. [52, 59] that invariant probability densities give rise to measure preserving stochastic processes in a fully controlled way. One of basic ingredients of the formalism is a proper choice of Feynman–Kac kernel functions, which are the building block for the construction of transition probability densities of the pertinent Markov processes. Feynman–Kac semigroup operators (and their kernels) explicitly involve one particle Hamiltonian operators as generators (in less technical terms one may think at this point about rather standard transformation from the Fokker–Planck operator to the associated self-adjoint one [60]).

For stationary processes, a general formula relating forward drifts $b(x)$ of the stochastic process with potentials of the conservative Hamiltonian system reads (we choose a diffusion coefficient to be equal 1/2) [55,59]:

$$V(x) = \frac{1}{2}\left(b^2 + \nabla b\right) . \tag{22}$$

Upon substituting the general expression for $b(x)$ we arrive at:

$$V(x) = \frac{1}{2}\left[\frac{\beta(\beta-2)}{4x^2} - (\beta+1) + x^2\right], \tag{23}$$

where $\beta = N - 1$. This potential function enters a standard definition of the one particle Hamiltonian operator (physical parameters have been scaled away):

$$H = -\frac{1}{2}\Delta + V(x)\,, \tag{24}$$

where $\Delta = d^2/dx^2$. The operator (24) with $V(x)$ defined by (23) is an equivalent form of a two-particle (actually two-level) version of the Calogero–Moser Hamiltonian, *cf.* [54].

Indeed, the classic Calogero-type problem is defined by

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \frac{\beta(\beta-2)}{8x^2} \tag{25}$$

with the well known spectral solution. The eigenvalues read $E_n(\beta) = 2n + 1 + 1/2[1+\beta(\beta-2)]^{1/2}$, where $n \geq 0$ and $\beta > -1$.

By inspection we can check that all previously considered $N = 2, 3, 4, 5$ radial processes correspond to the Calogero operator of the form $H - E_0$ where E_0 is the ground state ($n = 0$) eigenvalue. Its explicit form relies on

the choice of β and by substituting $\beta = 1, 2, 3, 4$ we easily check that an amusing identity generally holds true for natural numbers β

$$E_0(\beta) = 1 + \frac{1}{2}\Big[1 + \beta(\beta - 2)\Big]^{1/2} = \frac{1}{2}(\beta + 1)\,. \tag{26}$$

Accordingly, all considered radial processes arise as the so-called ground state processes associated with the Calogero Hamiltonians (squared modulus of the ground state wave function stands for the pertinent probability density). Let us recall that the classic Ornstein–Uhlenbeck process can be regarded as the ground state process of the harmonic oscillator Hamiltonian operator. That by the way corresponds to choosing $N = 1$ *i.e.* $\beta = 0$ in the above, plus allowing the whole of R^1 to the process, instead of R^+ only. Like in the standard OU process case, radial OU processes share the property of exactness (while driving any initial density towards suitable equilibrium) and hence ergodicity.

5. Discussion

Our motivations were essentially probabilistic and spectral series with spacing densities governed by Wigner-type laws have emerged in the course of a parametric stochastic process that relaxes towards equilibrium (invariant measure). Such series have thus a definite random origin. It is clear that an approximate value of Wigner densities indicates nonrandom input in realistic (random-matrix related) cases.

Let us point out that in the standard matrix-theory framework Dyson's "threefold way" is based on the demonstration that on general (invariance under symmetry) grounds only three basic ensembles (orthogonal, unitary and symplectic) matter. Hence, the non-generic repulsion behavior we have discussed before, goes beyond the standard framework (under assumptions of the Dyson theorem the non-generic laws are not admissible). Many different ensembles have been used in the literature, but their properties were more specific (less general) than the standard GOE, GUE and GSE cases show up.

The spacing distributions we have addressed (Wigner surmise), fail to be correct in general. The true random-matrix universal distributions differ from them, albeit the discrepancy is known to be small when matrix size is going to infinity [5].

In the discussed parametric relaxation process scenario, one may easily implement a transition of any initial density towards a concrete asymptotic stationary (steady state) one with the wealth of intermediate examples (*e.g.* from Poisson to GOE interpolation). In that case, both the initial and terminal distributions refer to random sequences of numbers (possible energy

eigenvalues). Ergodicity of the ultimate stationary process implies that its sample paths arise as random sequences drawn from the Wigner-type distribution. Clearly, we cannot expect that such purely random sample sequences would reveal long range correlations typical of random-matrix models.

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