

QUASILINEAR DIFFUSION FOR THE CHAOTIC MOTION OF A PARTICLE IN A SET OF LONGITUDINAL WAVES*

D.F. ESCANDE AND Y. ELSKENS

Equipe Turbulence Plasma, unité 6633 CNRS, Université de Provence
Centre de Saint-Jérôme, 13397 Marseille, Cedex 20, France

e-mail: escande@newsup.univ-mrs.fr

e-mail: elskens@newsup.univ-mrs.fr

(Received October 29, 2001)

The rigorous analytical calculation of the diffusion coefficient is performed for the chaotic motion of a particle in a set of longitudinal waves with random phases and large amplitudes ($\sim A$). A first step proves the existence of a quasilinear diffusion on a time scale $\sim A^{-2/3} \ln A$. A second step uses this property to extend the result to asymptotic times by introducing the conditional probability distribution of position and velocity of an orbit at a given time when they are known at a previous time.

PACS numbers: 52.35.-g, 05.45.-a, 05.60.-k, 52.20.-j

1. Introduction

Many chaotic Hamiltonian systems encountered in physics display a chaotic diffusion and in many cases the corresponding diffusion coefficient is given by a so-called quasilinear estimate [1–4]. The proof that this estimate is correct exists for the standard map with large control parameter [5], but is lacking for other systems with a spatially smooth force. We provide this proof for the one-dimensional chaotic motion of a particle in a general set of waves.

This result strengthens the link between the microscopic deterministic (chaotic) dynamics and the macroscopic stochastic motion. Its extension to the self-consistent many-body problem is a central problem to non-equilibrium statistical physics.

* Presented at the XIV Marian Smoluchowski Symposium on Statistical Physics, Zakopane, Poland, September 9–14, 2001.

This paper is organized as follows. First we introduce our model dynamics and stress the core of our argument. Then we recall the traditional argument deriving the quasilinear diffusion over a time short with respect to a characteristic time $\tau_{\text{spread}} \sim A^{-2/3}$ (A being a typical value of the wave amplitudes) and introduce the explicit form of the quasilinear diffusion coefficient. We re-derive this result within our new approach and take advantage of a better understanding of the particle motion to extend the validity of quasilinear diffusion to a time scale $\sim A^{-2/3} \ln A$, which is longer than the traditional scale τ_{spread} for A large. Finally, we introduce the conditional probability distribution of position and velocity of a chaotic orbit at a given time when they are known at a previous time and, thanks to the non-confinement of the velocity of the chaotic orbit, we further extend the quasilinear estimate to asymptotic time scales.

2. Dynamical model and assumptions

We consider the dynamics of a particle in a set of longitudinal waves (*e.g.* Langmuir waves) with random phases and large amplitude, as defined by the Hamiltonian

$$H(p, q, t) = \frac{p^2}{2} + \sum_{m=1}^M A_m \cos(k_m q - \omega_m t + \varphi_m), \quad (1)$$

where the φ_m 's are random variables and the (A_m, k_m, ω_m) 's are prescribed triplets of positive parameters. Such a dynamical system has already been studied in the literature, and for large A_m 's the diffusion coefficient has been found numerically to take on the quasilinear value [4, 6–9] defined below¹. The average over $M \gg A^{2/3} \ln A \gg 1$ random phases is central to our proof, in agreement with the occurrence of uncontrolled phases in many experiments and with the fact that the transport in (1) is much less diffusion-like if one averages only over initial conditions (p_0, q_0) [10]. The large A limit (dynamically speaking, the limit of strong resonance overlap parameter) corresponds to the limit of continuous spectrum often encountered in physics.

In agreement with most of the literature on quasilinear transport, the analysis is performed here in terms of quadratic means, and not in terms of the probability distribution functions, but we indicate at the end of this paper, how our technique could be used to prove the gaussianity of such functions.

¹ $D_{\text{QL}} = \pi A^2/2$ in the case of Ref. [8] (where $M = 2M' + 1$, $A_m = A$, $k_m = 1$ and $\omega_m = m - M' - 1$ for all m 's) and for the standard map, which is a special case of Ref. [8] in the limit $M \rightarrow \infty$ with all phases $\varphi_m = 0$.

The equations of motion are

$$\dot{q} = p, \quad (2)$$

$$\dot{p} = \sum_{m=1}^M A_m k_m \sin(k_m q - \omega_m t + \varphi_m). \quad (3)$$

We consider first the time to be short enough for the orbit to stay close to the unperturbed orbit $q^{(0)}(t) = q_0 + p_0 t$, and let $\Delta q(t) = q(t) - q^{(0)}(t)$, $\Delta p(t) = p(t) - p_0$. We compute their statistical properties when averaging over all φ_m 's.

For completeness, we first evaluate $\Delta p(t)$ by the traditional technique [1, 2] using first order perturbation in the amplitudes:

$$\Delta p(t) = \sum_{m=1}^M \frac{A_m k_m}{\Omega_m} \left[\cos(k_m q_0 + \varphi_m) - \cos(\Omega_m t + k_m q_0 + \varphi_m) \right], \quad (4)$$

with $\Omega_m = k_m p_0 - \omega_m$; if $\Omega_m = 0$ for some m , the corresponding term in the sum is the limit as $\Omega_m \rightarrow 0$. At this order, $\langle \Delta p(t) \rangle = 0$ and

$$\langle \Delta p^2(t) \rangle = \sum_{m=1}^M \left(\frac{A_m k_m}{\Omega_m} \right)^2 [1 - \cos(\Omega_m t)].$$

Let $v_m = \omega_m/k_m$. We assume that $\Delta \Omega_m = \Omega_{m+1} - \Omega_m$ and $\Delta v_m = v_{m+1} - v_m$ have a sign independent of m , which is natural for Langmuir waves and for the dynamics of Ref. [8]. Let

$$D_m \equiv \frac{\pi A_m^2 k_m}{2 |\Delta v_m|} = \lim_{p_0 \rightarrow v_m} \frac{\pi (A_m k_m)^2}{2 |\Delta \Omega_m|}. \quad (5)$$

D_m may fluctuate with m , but we assume (for simplicity only) that for some $L \geq 0$

$$\sum_{j=-L}^L \frac{D_{m+j} |\Delta v_{m+j}|}{|v_{m+L+1} - v_{m-L}|}$$

is a constant D_{QL} , called the quasilinear diffusion coefficient. Let $\Delta \Omega_{LM} = \max |\Omega_{m+L+1} - \Omega_{m-L}|$, $\tau_{\text{discr}} = \Delta \Omega_{LM}^{-1}$ and $\tau_c = (\Omega_{\text{max}} - \Omega_{\text{min}})^{-1}$; τ_{discr} and τ_c are, respectively, the discretization time and the correlation time of the wave spectrum as seen by the particle.

3. Non-chaotic initial quasilinear transport

Assuming $\tau_c \ll t \ll \tau_{\text{discr}}$, one obtains

$$\langle \Delta p(t)^2 \rangle = \frac{2D_{\text{QL}}}{\pi} \int_{-\infty}^{\infty} \Omega^{-2} [1 - \cos(\Omega t)] d\Omega = 2D_{\text{QL}} t,$$

where the discrete sum has been turned into an integral.

As a result, the diffusion coefficient takes on the quasilinear value D_{QL} . A similar calculation for q yields $\langle \Delta q(t) \rangle = 0$ and $\langle \Delta q(t)^2 \rangle = 2D_{\text{QL}} t^3/3$. For $t \ll \tau_c$, Δp grows linearly with time, and $\langle \Delta p^2 \rangle$ grows quadratically, as all modes act with a constant force on the orbit. For $\tau_c \ll t \ll \tau_{\text{discr}}$, the range of m contributing to the diffusion (modes acting with a nearly constant force) narrows like $1/t$. The range of t is further restricted by the condition for the orbit to remain close to the unperturbed one. This is traditionally obtained by requiring $\langle k_{\text{max}}^2 \Delta q^2(t) \rangle \ll 4\pi^2$, namely $t \ll \tau_{\text{spread}}$ with

$$\tau_{\text{spread}} = \left(6\pi^2 k_{\text{max}}^{-2} D_{\text{QL}}^{-1} \right)^{1/3} = 4\gamma_D^{-1}, \quad (6)$$

where we introduce the resonance broadening frequency $\gamma_{Dn} \equiv (k_n^2 D_{\text{QL}})^{1/3}$ and take $\gamma_D \equiv \max_n \gamma_{Dn}$.

In our approach, we evaluate $\Delta p(t)$ as in Ref. [8] by integrating formally the equation of motion for p . This yields $\langle \Delta p(t) \rangle = 0$ over the range $0 \leq t \ll \tau_{\text{QL}}$ defined below, and $\langle \Delta p^2(t) \rangle = \Delta_0 + \Delta_+ + \Delta_-$, with

$$\Delta_j = -\eta_j \int_0^t \int_0^t \sum_{m_1=1}^M \sum_{m_2=1}^M \frac{A_{m_1} k_{m_1} A_{m_2} k_{m_2}}{2} \left\langle \cos [\Phi_{m_1}(t_1) + \eta_j \Phi_{m_2}(t_2)] \right\rangle dt_1 dt_2, \quad (7)$$

where $\Phi_m(t) = k_m \Delta q(t) + \Omega_m t + \varphi_m$, with $\eta_{\pm} = \pm 1$ and $\eta_0 = -1$, and under condition $m_1 \neq m_2$ for $j = -$, and condition $m_1 = m_2$ for $j = 0$. Let $t_- = t_1 - t_2$ and $t_+ = (t_1 + t_2)/2$. For $t_- \ll \tau_{\text{spread}}$

$$\left\langle \exp \left[i k_m \left(\Delta q \left(t_+ + \frac{t_-}{2} \right) - \Delta q \left(t_+ - \frac{t_-}{2} \right) \right) \right] \right\rangle$$

may be considered as equal to 1. Therefore the support in t_- of the integrand in Δ_0 is of the order of τ_c . We assume $\tau_c \ll \tau_{\text{spread}}$. Hence the integration domain in t_- may be restricted to $|t_-| \leq \nu \tau_c$ where ν is a few units. In the limit where $\nu \tau_c \ll t \ll \tau_{\text{discr}}$, we obtain

$$\begin{aligned}
 \Delta_0 &= \sum_{m=1}^M \int_0^t \frac{2D_m}{\pi} \int_0^{\nu\tau_c} \langle \cos[\Omega_m t_-] \rangle \Delta \Omega_m dt_- dt_+ \\
 &= 2D_{\text{QL}} \sum_{m=1}^M (\pi \Omega_m)^{-1} \langle \sin[\Omega_m \nu \tau_c] \rangle \Delta \Omega_m t \\
 &= 2D_{\text{QL}} t,
 \end{aligned}$$

with the discrete sum over m approximated by an integral.

For $t \ll \tau_{\text{spread}}$ we approximate $q(t)$ by its unperturbed value $q^{(0)}(t)$. As this orbit does not depend on the phases, the averaged cosines in (7) are zero for $j = \pm$, and so are the $\Delta_{\pm}$'s. Then our second approach shows again that the diffusion coefficient takes on the quasilinear value. $\langle \Delta q^2(t) \rangle$ too may be computed by integrating the equation of motion². This involves calculating $\langle \Delta p(t_1) \Delta p(t_2) \rangle$, in the same way as $\langle \Delta p^2(t) \rangle$, and one recovers the traditional estimate for $\langle \Delta q^2(t) \rangle$. This provides a way for introducing the condition $t \ll \tau_{\text{spread}}$ without resorting to the traditional perturbative approach, and shows that the usual quasilinear diffusion coefficient may be recovered independently by our second approach.

4. Chaotic trajectory spreading

In fact our second approach is much more powerful. As was pointed out in Ref. [8], $\Delta_{\pm}$ vanishes provided that the dependence of Δq over any $N_{\varphi} = 2$ phases with all other phases fixed is weak, a condition far less stringent than the previous condition $N_{\varphi} = M$ which led to $t \ll \tau_{\text{spread}}$. Ref. [8] estimated the upper bound in time of the initial quasilinear diffusion through numerical calculations for moderate values of the waves amplitude. Here we derive such a bound analytically for large enough amplitudes.

We measure these amplitudes by the parameter $E_n = [2D_{\text{QL}} k_n |\Delta v_n| / \pi]^{1/2}$ which corresponds to the typical electric field of a wave. A related dimensionless quantity characterizes our scaling, namely the Chirikov resonance overlap parameter

$$s(v_n) = \frac{2 \left(A_n^{1/2} + A_{n+1}^{1/2} \right)}{|\Delta v_n|} \quad (8)$$

or equivalently the ratio

$$\mathcal{B}(v_n) \equiv \frac{k_n |\Delta v_n|}{\gamma_{Dn}} \simeq 5s^{-4/3} \quad (9)$$

² The process $\Delta q(t)$ is found to be Gaussian, and its moment generating function reads $\langle e^{i u (\Delta q(t_2) - \Delta q(t_1))} \rangle = e^{-u^2 D_{\text{QL}} [|t_2 - t_1|^3/3 + |t_2 - t_1|^2 \min(t_1, t_2)]}$.

of the frequency mismatch between neighboring waves (in the frame of either wave) to their resonance broadening frequency. As these quantities depend on n , they characterize the dynamics locally. In the following, we are interested in the dense spectrum, or strong overlap, or large amplitude limit. To ensure a genuine scaling, we consider families of dynamics (1) where $E_n = E a_n$ and the reference amplitudes a_n are constant while $E \rightarrow \infty$, or $\mathcal{B}(v_n) = \mathcal{B} b_n$ and the coefficients b_n are constant while $\mathcal{B} \rightarrow 0$.

Apart from the small dimensionless parameter $\mathcal{B}$, we also introduce the Kubo number $\mathcal{K}_c \equiv \tau_c / \tau_{\text{spread}}$. The wide velocity spectrum of the waves ensures that $\mathcal{K}_c \ll 1$.

The limit of interest is the **joint limit** $\mathcal{K}_c \rightarrow 0$ and $\mathcal{B} \rightarrow 0$ (or $\mathcal{K}_c \rightarrow 0$ and $s \rightarrow \infty$).

4.1. Spreading due to a single random phase

In order to avoid too heavy formulas, we give the explicit derivation for the spreading due to one phase, and extend the result to two phases afterwards. To estimate this spreading we study how the orbit which is at (q_0, p_0) at $t = 0$ is modified when phase φ_n changes from 0 to a finite value. Let $(q_{\mathcal{H}}(t), p_{\mathcal{H}}(t))$ be the orbit for $\varphi_n = 0$, let $\delta q_n(t) = q(t) - q_{\mathcal{H}}(t)$ and $\delta p_n(t) = \delta \dot{q}_n(t) = p(t) - p_{\mathcal{H}}(t)$. We assume t to be small enough so that $k_{\max} |\delta q_n(t)| \ll \pi$. As $\delta q_n(t)$ is small, we may linearize the motion

(10)

where $F(t) = \sum_{m=1}^M k_m^2 A_m \cos \Psi_m(t)$, with $\Psi_m = k_m q_{\mathcal{H}}(t) - \omega_m t + \varphi_m$ and $\Psi_{n0} = k_n q_{\mathcal{H}}(t) - \omega_n t$. Then (10) and initial conditions $(\delta q_n(0), \delta p_n(0)) = (0, 0)$ imply

$$\delta q_n(t) = \int_0^t (t - t'') F(t'') \delta q_n(t'') dt'' + \delta q_{n0}(t), \quad (11)$$

where

$$\delta q_{n0}(t) = A_n k_n \int_0^t \int_0^{t'} \left(\sin \Psi_n(t'') - \sin \Psi_{n0}(t'') \right) dt'' dt'.$$

In the short-time limit, the dominant term in this expression (11) for δq_n will be δq_{n0} , but over longer times the first term may self-amplify and overtake the second one.

We estimate only $\langle \delta q_n(t)^2 \rangle$, but $\langle \delta q_n(t) \rangle$ can be computed by the same technique and it turns out to be negligible over the time interval of interest. In a first stage, consider the contribution of δq_{n0} to the variance

$$C_0(t) \simeq \langle \delta q_{n0}(t)^2 \rangle = \frac{k_n^2 A_n^2}{2} \int_0^t \int_0^{t'_1} \int_0^{t'_2} \langle \cos(\Psi_n(t'_1) - \Psi_n(t'_2)) \rangle dt''_2 dt'_2 dt'_1 dt'_1.$$

To estimate this expression, note that

$$\Psi_n(t_1'') - \Psi_n(t_2'') - \Omega_n(t_1'' - t_2'') = k_n(q_{\mathcal{H}}(t_1'') - q_{\mathcal{H}}(t_2'')) - k_n p_0(t_1'' - t_2'')$$

and, for the range of time of interest, $q_{\mathcal{H}}(t'') - q^{(0)}(t'')$ is essentially the sum of $M - 1$ terms in which a random phase φ_m ($m \neq n$) is added to a term which has a weak dependence on φ_m . Therefore, this sum is almost Gaussian, and for $M \gg 1$ we may approximate $\dot{q}_{\mathcal{H}}(t'')$ by a Brownian motion. Furthermore, as $M \gg 1$, we approximate $q_{\mathcal{H}}(t'')$ by $q(t'')$ in the averages. Using the distribution of $\Delta q(t_2) - \Delta q(t_1)$, we find³ the estimate

$$C_0(t) \leq C_{0M}(t) \equiv 0.28 k_n |\Delta v_n| \gamma_{Dn}^2 t^3 = 0.28 \mathcal{B}(\gamma_{Dn} t)^3. \quad (12)$$

For the second stage, we take into account the first term in the right hand side of (10). As δq_n is small, we may treat $F(t)$ as a Gaussian process with moments $\langle F(t) \rangle = 0$ and $\langle F(t_1)F(t_2) \rangle = 2\gamma_{Dn}^3 \delta(t_1 - t_2)$ where $\delta(t)$ is the Dirac distribution. Indeed $q_{\mathcal{H}}(t)$ has a weak dependence on any phase φ_m , which makes $\langle F(t_1)F(t_2) \rangle$ a Bragg-like function with the small width τ_c in $t_1 - t_2$. Higher moments of F are assumed to factorize, *i.e.* F is treated as a white noise, which is consistent with approximating $\dot{q}_{\mathcal{H}}(t)$ by a Brownian motion.

We estimate the spreading of $\delta q_n(t)$ by computing

$$\begin{aligned} C(t) &\equiv \langle \delta q_n(t)^2 \rangle \\ &\simeq \int_0^t \int_0^{t_1'} \int_0^t \int_0^{t_2'} \langle F(t_1'')F(t_2'') \rangle \langle \delta q_n(t_1'')\delta q_n(t_2'') \rangle dt_2'' dt_2' dt_1' dt_1'' + C_0(t) \\ &= \frac{E^2}{2} \int_0^t \int_0^{t_1'} \int_0^{\min(t_1', t_2')} C(t'') dt'' dt_2' dt_1' + C_0(t). \end{aligned} \quad (13)$$

It follows from (10) and our assumptions on F that $C(t) = C_0(t) + LC(t)$ with

$$Lf(t) = \frac{E^2}{2} \int_0^t \int_0^t \int_0^{\min(t_1', t_2')} f(t'') dt'' dt_2' dt_1'.$$

³ By previous footnote, one finds $|\langle \cos(\Psi_n(t_1'') - \Psi_n(t_2'')) \rangle| \lesssim |\cos \Omega_n(t_2'' - t_1'')| \times \exp[-\frac{1}{3}k_n^2 D_{QL}|t_2'' - t_1''|^3] \leq \exp[-\frac{1}{3}k_n^2 D_{QL}|t_2'' - t_1''|^3]$. Integration yields $C_0(t) \leq (k_n A_n)^2 \Gamma(4/3) (k_n^2 D_{QL}/3)^{-1/3} \int_0^t \int_0^t \min(t_1', t_2') dt_2' dt_1'$, where $\Gamma(x)$ is the Euler gamma function.

As $(1-L)^{-1}$ preserves positivity⁴, $C = (1-L)^{-1}C_0 \leq (1-L)^{-1}C_{0M} \equiv C_M$. Applying the Laplace transform to both sides of equation $C_M = C_{0M} + LC_M$, we compute C_M and find

$$C(t) \leq C_M(t) = 0.14 \mathcal{B} k_n^{-2} \left(e^{t'} - 1 + 2g(t') \right) \quad (14)$$

with $t' \equiv 4^{1/3} \gamma_{Dn} t$ and $g(t') = e^{-t'/2} \cos(t' \sqrt{3}/2) - 1$. This estimate for the variance of $\delta q_n(t)$ starts from zero at $t = 0$ and diverges exponentially for $t \rightarrow \infty$. Its exponentiation time scale $\tau_{\text{Liap}} \sim \gamma_{Dn}^{-1} \sim \tau_{\text{spread}}$ is the reciprocal of the Liapunov characteristic instability rate (this is reminiscent of Ref. [4]). However, as the coefficient in front of the exponential goes to zero as $E \rightarrow \infty$, the time needed by our upper estimate on $k_n^2 C(t)$ to reach unity is of the order of

$$\tau_{\text{QL}} = \gamma_D^{-1} |\ln \mathcal{B}|. \quad (15)$$

Though this time goes to zero as $E \rightarrow \infty$, it is $O(\ln \mathcal{B}^{-1})$ times larger than the time τ_{spread} over which the initial quasilinear approximation is traditionally justified.

4.2. Spreading due to two random phases

The result of this discussion is that $q(t)$ depends little on any given phase over a time τ_{QL} . For $M \gg 1$, the argument is easily strengthened into $q(t)$ depends little on any two given phases over a time τ_{QL} . To this end $(q_{\mathcal{H}_1, \mathcal{H}_2}(t), p_{\mathcal{H}_1, \mathcal{H}_2}(t))$ and $(\delta q(t), \delta p(t))$ are defined starting from $\varphi_{m_1} = \varphi_{m_2} = 0$, and a third term similar to the second one adds in the right hand side of (10). The first stage of our iteration procedure now estimates the contribution of both phases φ_{m_1} and φ_{m_2} by a term again of the order of $\mathcal{B}^{-2} \gamma_D^3 t^3$, while the second stage does not change.

As a result, for $t \ll \tau_{\text{QL}}$, the non-quasilinear terms $\Delta_{\pm}$ are negligible since q has a small dependence on any given pair of phases in this time range. Furthermore these terms may be estimated by making explicit in the argument of the cosine of (7) the main dependence over φ_{m_1} and φ_{m_2} through estimates $\delta \Phi_{m_1}$ and $\delta \Phi_{m_2}$ of the type $k_m \delta q_{n0}$ for both phases, and by expanding to second order in these $\delta \Phi$'s. Such estimates hold for $t \ll \beta \tau_{\text{QL}}$ with $0 < \beta < 1$ for E large enough, and yield $\Delta_+ \sim E^4 \tau_c^2 t^2$ and $\Delta_- \sim E^4 t^5$ which are negligible with respect to Δ_0 in the time interval of interest, and do not grow with M although there are $2M^2 - M$ “off-diagonal” terms.

⁴ Indeed, L preserves positivity and is a contraction operator for functions on $[0, \infty]$ with the norm $\|f\|_{\lambda} = \lambda \int_0^{\infty} e^{-\lambda t} |f(t)| dt$, for any $\lambda > 4^{1/3} \gamma_D$.

5. Quasilinear transport over large times

Finally, we show that the quasilinear estimate holds for asymptotic times. Let $p_{\min} = \min(v_m)$ and $p_{\max} = \max(v_m)$. We assume that in the velocity domain $[p_{\min}, p_{\max}]$ the dynamics is chaotic enough for a typical orbit to be unconfined in p within this domain, but that the time of interest is also smaller than the time for the orbit to reach the boundaries of the chaotic domain. Therefore we set the condition

$$\min [(p_0 - p_{\min})^2, (p_0 - p_{\max})^2] \gg D_{\text{QL}} \tau_{\text{QL}} \sim k_n^{-2} \gamma_{Dn}^2 \ln(\mathcal{B}^{-1}),$$

to compute now the diffusion coefficient due to the chaotic motion when M and E are large. We define $\delta q(\tau|p, q, t) = q(t + \tau) - q - p\tau$, where $q(t')$ is the position at time t' of an orbit which is at (p, q) at time t : $\delta q(\tau|p, q, t)$ tells the departure of this orbit from the free motion during the time interval τ .

Integrating formally the equation of motion for p yields

$$\langle \Delta p^2(t) \rangle = - \sum_{m,n=1}^M \sum_{\varepsilon=\pm 1} \varepsilon \frac{A_m k_m A_n k_n}{2} \int_0^t \int_0^t \langle \cos \Phi \rangle dt' dt'', \quad (16)$$

where

$$\begin{aligned} \Phi = & (k_m + \varepsilon k_n) q(t'') + k_m \delta q[t' - t''|p(t''), q(t''), t''] \\ & + k_m p(t'')(t' - t'') - \omega_m t' - \varepsilon \omega_n t'' + \varphi_m + \varepsilon \varphi_n. \end{aligned}$$

We introduce the probability distribution $P(\delta p, t|p_0)$ of $\delta p = p(t) - p_0$ for an orbit started at $p = p_0$ at $t_0 = 0$; it is independent of q_0 .

$\langle \cos[k_m \delta q(t' - t''|p(t''), q(t''), t'')] \rangle$ is independent of $q(t'')$, and its contribution for diagonal ($m = n$, $\varepsilon = -1$) terms to (16) is

$$\begin{aligned} B & \equiv \lim_{t \rightarrow \infty} \sum_{m=1}^M \frac{(A_m k_m)^2}{4t} \int_0^t \int_0^t P(\delta p, t''|p_0) \langle \cos[k_m \delta q(t' - t''|p_0, q(t''), t'')] \rangle_* \\ & \quad \times \cos[k_m(p_0 + \delta p)(t' - t'') - \omega_m(t' - t'')] d\delta p dt' dt'' \\ & = \lim_{t \rightarrow \infty} \sum_{m=1}^M \frac{(A_m k_m)^2}{4t} \Re \int_0^t \int_{-t''}^{t-t''} \tilde{P}(k_m \tau, t''|p_0) \exp[i \Omega_m \tau] \\ & \quad \times \langle \exp[i k_m \delta q(\tau|p_0, q(t''), t'')] \rangle_* d\tau dt'', \quad (17) \end{aligned}$$

where the starred average means the average done with the constraint $p(t'') = p_0 + \delta p$, and where the Fourier transform

$$\tilde{P}(\alpha, t''|p_0) = \int_{-\infty}^{\infty} P(\delta p, t''|p_0) \exp(i \alpha \delta p) d\delta p \quad (18)$$

was used. As δq is computed with the knowledge of p at time t'' which sets only one condition on a set of many phases, an average with the constraint $p(t'') = p_0 + \delta p$ may be computed by using the initial quasilinear estimate at time $|t' - t''| \leq \tau_{\text{QL}}$. Hence the function $\langle \exp[i k_m \delta q(t' - t''|p, q(t''), t'')] \rangle_*$ is correctly computed by the previous quasilinear estimate over its whole support in $t' - t''$ as $\tau_{\text{QL}} \gg \tau_{\text{spread}}$. This estimate is independent of p , and we could set $p = p_0$ in the average cosine. Up to $t = \tau_{\text{QL}}$, the width of P is growing, since we proved $\langle \Delta p^2(t) \rangle$ grows linearly over this time interval. Later on this width cannot decrease because of the locality of chaotic motion [8, 11]. We assume $t \gg \tau_{\text{spread}}$. Then the width w of $\tilde{P}$ is narrow enough for the spread of δq to be negligible over a time $\tau \sim w/k_m$. Therefore $\langle \exp[i k_m \delta q(\tau|p_0, q(t''), t'')] \rangle_* \simeq 1$ in the part of the integration domain over τ where $\tilde{P}$ takes appreciable values in (17), and

$$\begin{aligned} B &= \lim_{t \rightarrow \infty} \sum_{m=1}^M \frac{\pi A_m^2 k_m}{2t} \int_0^t P(v_m - p_0, t''|p_0) dt'' \\ &= \int_0^t \sum_{m=1}^M \frac{D_m \Delta v_m}{2t} P(v_m - p_0, t''|p_0) dt'', \end{aligned} \quad (19)$$

where the inverse Fourier transform was provided by the integral over τ .

Now, if t is large enough for P to be almost constant over the range $[v_{m-L}, v_{m+L}]$ for all m 's, we approximate

$$\sum_{j=-L}^L \frac{D_{m+j} |\Delta v_{m+j}|}{|v_{m+L+1} - v_{m-L}|} \simeq D_{\text{QL}},$$

and substitute the sum over v_m by an integral:

$$B = 2 \int_0^t \int D_{\text{QL}} P(p - p_0, t''|p_0) dp dt'' = 2 D_{\text{QL}} t.$$

The general term of (16) can be estimated by a similar calculation. A sequence of two Fourier transforms is again recovered. After the first one, averages of the kind $\langle \exp i [k_m \delta q(\tau|p_0, q(t''), t'') + \varphi_m + \varepsilon \varphi_n] \rangle_*$ are found. They vanish as the constraint $p(t'') = p_0 + \delta p$ leaves almost free the average on any two phases, and since δq is negligible for τ small. Therefore only B contributes to $\langle \Delta p^2(t) \rangle$ which thus grows in a quasilinear way. This ends our proof of the quasilinear estimate for asymptotic times.

Note that the conditional probability P permits to use the knowledge of initial quasilinear diffusion for proving it over asymptotic times only because we proved before that $\tau_{\text{QL}} \gg \tau_{\text{spread}}$. In contrast with the initial non-chaotic quasilinear regime, the number of modes acting on the particle increases with t . This agrees with the fact that the orbit visits an increasing number of resonances when time increases.

6. Conclusion

Thus we prove the quasilinear character of the diffusion for the motion of a particle in a spectrum of large amplitude longitudinal waves. Our technique can be adapted to systems with a slow dependence of the quasilinear diffusion coefficient on p . As many Hamiltonian systems may be locally reduced to case (1) (see [12]), this further extends its range of applicability and shows that the universality class of quasilinear diffusion is broad. It also provides insight for the case where particles and waves are self-consistently coupled [13].

Higher order moments of Δp could be computed using a similar technique. Indeed, preliminary calculations indicate that the use of conditional probabilities should enable one to retain after Fourier transforms the same terms for the moment of order κ as in the case where $q(t)$ is weakly dependent on any phase provided that $\kappa \ll \mathcal{B}^{-1}$, which yields a Gaussian estimate. Proving the Gaussianity of f would also lead to a Fokker–Planck–Smoluchowski evolution equation for f .

The value of $\mathcal{B}$ (which depends only on local aspects of the spectrum: $A, k, \delta v$) determines the time scale over which the quasilinear approximation holds. Given $\mathcal{B} \ll 1$, this time scale is $t \gg \tau_{\text{QL}}$. On the other hand, we require that the motion remains away from the boundaries p_{min} and p_{max} of the wave spectrum. Given the scaling $\langle \Delta p^2 \rangle \sim 2Dt$, the boundary is reached for $t_{\text{bound}} \sim D^{-1}M^2\Delta v^2 \sim M^2\mathcal{B}\tau_{\text{QL}}$. As M is independent of $\mathcal{B}$, one may let $M \rightarrow \infty$ to ensure t_{bound} to be as large as desirable.

Comments by D. Bénisti and A. Henriët on this work are gratefully acknowledged. YE thanks the organizers and participants of the M. Smoluchowski symposium for discussions.

REFERENCES

- [1] A.A. Vedenov, E.D. Velikhov, R.Z. Sagdeev, *Nucl. Fusion Suppl.* **2**, 465 (1962).
- [2] W.E. Drummond, D. Pines, *Nucl. Fusion Suppl.* **3**, 1049 (1962).
- [3] A.J. Lichtenberg, M.A. Lieberman, *Regular and Stochastic Motion*, Springer, New York 1983.
- [4] A.B. Rechester, M.N. Rosenbluth, R.B. White, *Phys. Rev. Lett.* **42**, 1247 (1979).
- [5] A.B. Rechester, R.B. White, *Phys. Rev. Lett.* **44**, 1586 (1980).
- [6] J.R. Cary, D.F. Escande, A.D. Verga, *Phys. Rev. Lett.* **65**, 3132 (1990).
- [7] O. Ishihara, H. Xia, S. Watanabe, *Phys. Fluids* **B5**, 2786 (1993).
- [8] D. Bénisti, D.F. Escande, *Phys. Plasmas* **4**, 1576 (1997).
- [9] B.R. Ragot, *J. Plasma Phys.* **60**, 299 (1998).
- [10] D. Bénisti, D.F. Escande, *Phys. Rev. Lett.* **80**, 4871 (1998).
- [11] D. Bénisti, D.F. Escande, *J. Stat. Phys.* **92**, 909 (1998).
- [12] D.F. Escande, *Phys. Rep.* **121**, 165 (1985).
- [13] I. Doxas, J.R. Cary, *Phys. Plasmas* **4**, 2508 (1997) and references therein.