# EVOLUTION OF QUANTUM CORRELATIONS FOR JUMP-TYPE QUANTUM STOCHASTIC DYNAMICS*

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Two models of quantum stochastic jump type processes are analyzed with special emphasis on the time evolution of quantum correlations. It is shown that the generalized conditional expectation defining the time evolution of $XXZ$ model contains the proper (*i.e.* genuine quantum) interactions between subsystem and its environment while this is not the case for the stochastic counterpart of the Ising model.

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## 1. Introduction

In the classical theory of particle systems one of the objectives is to produce, describe and analyze dynamical systems with evolution originated from stochastic processes in such a way that their equilibrium states are Gibbs states (*cf.* [2]). A well known illustration is a number of papers describing the so called Glauber dynamics [1]. To perform a detailed analysis of dynamical system of that type, it is convenient to use the theory of Markov processes in the context of $L_p$-spaces. Recently, this program was carried out

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in the setting of quantum mechanics [4–7]. In particular, guided by the classical theory and applying generalized conditional expectations (in the sense of Accardi–Cechnini), it was possible to define the corresponding Markov generators of the underlying quantum Markov–Feller dynamics. Furthermore, such an analysis led to a general scheme for constructing quantum jump processes on a lattice (*cf.* [4]). We emphasize that interpretation of such quantum processes is the same as in the classical case. Namely, having a transition rate (dependent on the state $\rho$), one can describe a Markov semigroup corresponding to quantum Markov process (defined by transition rates). Clearly, while describing a physical process in such a way, we do not know (explicitly) the interactions which are responsible for the underlying transition rates. Therefore, it is natural to pose the following question: Are the interactions proper (*i.e.* genuine quantum ones) or not? To answer this question we proceed with a detailed analysis of properly chosen correlation functions. To be more specific let us consider two-point correlation function $\langle T_t(A)B\rangle_\rho \equiv \mathrm{Tr}(\rho T_t(A)B)$, where $\rho$ is a separable state, $T_t$ is a Markov semigroup, $A$ and $B$ are observables. Obviously, the following function

$$C_{T,\rho}(A,B) \equiv \langle T_t(A)B\rangle_\rho - \langle T_t(A)\rangle_\rho \langle B\rangle_\rho$$

can be taken as a measure of correlation between $T_t(A)$ and $B$. However, only quantum correlation can be considered as an indication of existence of proper interactions in the studied evolution. In other words, we should take away the classical correlations from $C_{T,\rho}(A,B)$. We recall that separable states display classical correlation only. We write

$$\begin{aligned} C_{T,\rho}(A,B) &= \langle T_t(A)B\rangle_\rho - \langle T_t(A)\rangle_\rho\langle B\rangle_\rho = \langle T_t(A)B\rangle_\rho - \langle A\rangle_\rho\langle B\rangle_\rho \\ &= (\langle T_t(A)B\rangle_\rho - \langle AB\rangle_\rho) + (\langle AB\rangle_\rho - \langle A\rangle_\rho\langle B\rangle_\rho)\,. \end{aligned}$$

As the second component measures classical correlations, one can consider the first one as a measure of quantum correlations. Consequently, in order to find an indication of proper interaction in the time evolution $T_t$, we will study

$$C^Q_{T,\rho}(A,B) = \langle T_t(A)B\rangle_\rho - \langle AB\rangle_\rho \tag{1}$$

for a separable state. In our recent paper [8], a detailed analysis of $C^Q_{T,\rho}$ for spin flip type dynamics has been done. Here, we will study $C^Q_{T,\rho}$ for quantum stochastic dynamics, determined by the Gibbs state of the Ising and $XXZ$ model. Throughout the paper we shall use the normalized trace.

## 2. Jump-type dynamics — construction sketch

Consider a composite system $I+II$ associated with a region $\Lambda = \Lambda_I \cup \Lambda_{II}$, where $\Lambda_I, \Lambda_{II} \subset \boldsymbol{Z}^d$. The system $\Lambda$ is described by $\mathcal{H}_1 \otimes \mathcal{H}_2$, $S_1 \otimes S_2$ and $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \cong \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, where $\mathcal{H}_1$ ($\mathcal{H}_2$) are the finite dimensional Hilbert spaces associated with $\Lambda_I$ ($\Lambda_{II}$), $S_1$ ($S_2$), respective sets of density matrices, are the spaces of mixed states, $\mathcal{B}(\mathcal{H}_1)$ ($\mathcal{B}(\mathcal{H}_2)$), the sets of all bounded linear operators, are the algebras of observables. Systems with interactions are described by interaction potentials associated with region $\Lambda$ ($\Lambda_I$, $\Lambda_{II}$, respectively). This leads to the corresponding Hamiltonians $H_\Lambda$ ($H_{\Lambda_I}$, $H_{\Lambda_{II}}$) and to Gibbs state

$$\rho_\Lambda = \frac{\mathrm{e}^{-\beta H_\Lambda}}{\mathrm{Tr}\,(\mathrm{e}^{-\beta H_\Lambda})} \equiv \rho\,,$$

($\rho$ is an invertible operator, *i.e.* $\rho^{-1}$ exists). In this work we study a concrete kind of the jump process, *i.e.* exchange type dynamics. This kind of dynamics is induced by a local symmetry. Consider a symmetry transformation (local automorphism) $\psi$ on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that

$$\psi(A) = A \qquad \text{for} \qquad A \in \mathcal{B}(\mathcal{H}_1), \quad \psi^2 = \mathbf{1}\,.$$

Note that if $\dim(\mathcal{H}_1), \dim(\mathcal{H}_2) < \infty$, the above properties imply $\mathrm{Tr}(\psi(\cdot)) = \mathrm{Tr}(\cdot)$. We shall consider a particular type of symmetries, which are implemented by exchanges of observables between sites of the spin chain. Using transformation $\psi$ one can define a projection $\tau$ on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ as follows

$$\tau(\cdot) \equiv \frac{1}{2}(\mathbf{1} + \psi)(\cdot)\,.$$

We observe that $\tau$ is not a morphism. According to the general theory of semigroups [3] the dynamic $T_t$ induced by the local transformation $\psi$ is of the form

$$T_t(\cdot) \;=\; \exp(t\mathcal{L}(\cdot))\,, \tag{2}$$

with

$$\mathcal{L} \;=\; E - \mathbf{1}\,,$$

where $E\colon \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a generalized conditional expectation in the Accardi–Cechini sense. Performing calculations similar as in the appendix of [7] one can show that for the considered dynamics operator $E$ takes the following form

$$E(A) = \tau(\gamma^* A \gamma)\,, \tag{3}$$

where

$$\gamma = \rho^{\frac{1}{2}} (\tau \rho)^{-\frac{1}{2}}\,. \tag{4}$$

The presented construction has a straightforward generalization to the infinite dimensional case, thus, thermodynamic limit can be performed [4,6]. Again, as a result we get uniformly continuous semigroup $T_t$.

Using (2) and a Taylor expansion of $T_t$ we can write $C^Q_{T,\rho}(A,B)$ as

$$\begin{aligned} C^Q_{T,\rho}(A,B) &= \langle T_t(A)B\rangle_\rho - \langle AB\rangle_\rho \\ &= (\langle AB\rangle_\rho + t\,\langle \mathcal{L}(A)B\rangle_\rho + \ldots) - \langle AB\rangle_\rho \\ &= t\,(\langle E(A)B\rangle_\rho - \langle AB\rangle_\rho) + \ldots\,, \end{aligned}$$

in which the remaining terms are of higher order in time. Thus,

$$\tilde{C}^Q_{T,\rho}(A,B) = \langle E(A)B\rangle_\rho - \langle AB\rangle_\rho \tag{5}$$

describes dominating changes of the chosen dynamics for short times. In this work we will examine function $\tilde{C}^Q_{T,\rho}$ rather than $C^Q_{T,\rho}$, since it is much more computationally tractable and also enables us to answer the main question posed in the introduction.

In order to study concrete physical models we have to deal with concrete Gibbs states. In this work we analyze two models that stem from one dimensional quantum Ising model and one dimensional quantum $XXZ$ model. In particular we will consider a one-dimensional finite $^1/_2$-spin chain with $N+1$ sites indexed from 0 to $N$ and the corresponding algebra of observables generated by

$$\sigma^{i_0} \otimes \sigma^{i_1} \otimes \ldots \otimes \sigma^{i_N},$$

where $i_k \in \{0,1,2,3\}$, $k = 0,\ldots N$, and $\sigma^j$, $j = 0,1,2,3$ are Pauli matrices.

## 3. Ising model

Consider the Ising model as described in the previous section. The Hamiltonian of the system has the form:

$$\begin{aligned} H &= \sum_{n=1}^{N} \lambda_n \sigma^3_{n-1} \sigma^3_n \\ &= \sum_{n=1}^{N} \lambda_n \cdot \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \sigma^3 \otimes \sigma^3 \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}\,, \end{aligned} \tag{6}$$

where $\sigma^3$ denotes 3rd Pauli matrix (*i.e.* $\sigma^3_{11}=1$, $\sigma^3_{22}=-1$, $\sigma^3_{12}=\sigma^3_{21}=0$) and $\sigma^3$ are localized at sites $n-1$ and $n$. We also assume that the model is translationally invariant, which implies that $\lambda_n=\lambda$, $n=1,\ldots,N$. The corresponding Gibbs state $\omega$ is represented by the density matrix

$$\rho = Z^{-1}\exp(-\beta H)\,, \tag{7}$$

where $Z=\mathrm{Tr}(\mathrm{e}^{-\beta H})$ and $\beta$ is the inverse temperature. Here, $Z=\cosh^N(-\beta\lambda)$. Thus, we have

$$\omega(A) = Z^{-1}\,\mathrm{Tr}(\mathrm{e}^{-\beta H}A)\,.$$

We will need the explicit form of the matrix $\rho$. Note that

$$\mathrm{e}^{\xi\sigma^3} = \cosh(\xi) + \sigma^3\cdot\sinh(\xi)\,.$$

Using the above and the fact that all constituents in (6) commute, we obtain

$$\rho = \rho_1\,\rho_2\,, \tag{8}$$

where

$$\rho_1 = (\mathbf{1}\otimes\mathbf{1}+t\cdot\sigma^3_0\otimes\sigma^3_1)\otimes\ldots\otimes(\mathbf{1}\otimes\mathbf{1}+t\cdot\sigma^3_{N-1}\otimes\sigma^3_N)\,, \tag{9}$$

$$\rho_2 = \mathbf{1}\otimes(\mathbf{1}\otimes\mathbf{1}+t\cdot\sigma^3_1\otimes\sigma^3_2)\otimes\ldots\otimes(\mathbf{1}\otimes\mathbf{1}+t\cdot\sigma^3_{N-2}\otimes\sigma^3_{N-1})\otimes\mathbf{1}\,, \tag{10}$$

with $t=\tanh(-\beta\lambda)$, if $N$ is an odd number, and

$$\rho_1 = (\mathbf{1}\otimes\mathbf{1}+t\cdot\sigma^3_0\otimes\sigma^3_1)\otimes\ldots\otimes(\mathbf{1}\otimes\mathbf{1}+t\cdot\sigma^3_{N-2}\otimes\sigma^3_{N-1})\otimes\mathbf{1}\,, \tag{11}$$

$$\rho_2 = \mathbf{1}\otimes(\mathbf{1}\otimes\mathbf{1}+t\cdot\sigma^3_1\otimes\sigma^3_2)\otimes\ldots\otimes(\mathbf{1}\otimes\mathbf{1}+t\cdot\sigma^3_{N-1}\otimes\sigma^3_N)\,, \tag{12}$$

if $N$ is an even number. The subscripts index sites for which Pauli matrices $\sigma^3$ are assigned to.

Having specified regions $\Lambda_I$ and $\Lambda_{II}$ and a local symmetry operator $\psi$ as discussed in Section 2, one can calculate an explicit form of generator $E$ of the dynamics. As an example, let us consider a local transformation $\psi_{kl}$ defined as follows

$$\begin{aligned}&\psi_{kl}(A_1\otimes\ldots\otimes A_k\otimes\ldots\otimes A_l\otimes\ldots\otimes A_N)\\&\quad= A_1\otimes\ldots\otimes A_l\otimes\ldots\otimes A_k\otimes\ldots\otimes A_N\,,\end{aligned} \tag{13}$$

which describes the exchange between the sites. Respective projection $\tau_{kl}$ is defined by $\tau_{kl}(\cdot)=\frac{1}{2}(\mathbf{1}+\psi_{kl})(\cdot)$. In particular, one can choose $l=k+1$ which is related to a possible description of transport properties in the considered model. According to the notation introduced in Section 2, we have that $\Lambda = \{0, 1, \ldots, N\}$, $\Lambda_I = \Lambda \backslash \{k, l\}$, $\Lambda_{II} = \{k, l\}$, while $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ are $2^{N-1}$- and $2^2$-dimensional Hilbert spaces, respectively. First, we calculate an explicit form of the operator $\gamma$. Without loss of generality we can assume that $k$ is an even number. Then, we note that $\rho_1$ is invariant under $\psi_{kl}$. Using this and (4) we get

$$\gamma = (2\rho_2)^{\frac{1}{2}} (\rho_2 + \psi_{kl}\rho_2)^{-\frac{1}{2}}. \tag{14}$$

It is an easy observation that $\gamma$ is nontrivial only in sites $\{k-1, k, k+1, k+2\}$, *i.e.* we have

$$\gamma = \mathbf{1} \otimes \ldots \mathbf{1} \otimes (2\tilde{\rho}_2)^{\frac{1}{2}} (\tilde{\rho}_2 + \psi_{kl}\tilde{\rho}_2)^{-\frac{1}{2}} \otimes \mathbf{1} \ldots \otimes \mathbf{1}. \tag{15}$$

where $\tilde{\rho_2}$ is the restriction of $\rho_2$ to the sub-algebra generated by observables localized in sites $\{k-1, k, k+1, k+2\}$. Obviously for $k = 0$ we would have $\gamma = (2\tilde{\rho}_2)^{\frac{1}{2}} (\tilde{\rho}_2 + \psi_{kl}\tilde{\rho}_2)^{-\frac{1}{2}} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}$, while for $k + 1 = N$ we would have $\gamma = \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes (2\tilde{\rho}_2)^{\frac{1}{2}} (\tilde{\rho}_2 + \psi_{kl}\tilde{\rho}_2)^{-\frac{1}{2}}$. In the sequel we shall assume that $0 < k < N - 1$. If $k = 0$ or $k = N - 1$, all the subsequent considerations can be repeated up to the changes pointed out while discussing the explicit form of $\gamma$. Moreover, we shall use the symbol $\tilde{\gamma}$ denoting the restriction of $\gamma$ to $\{k-1, \ldots, k+2\}$, *i.e.* we have $\gamma = \mathbf{1} \otimes \tilde{\gamma} \otimes \mathbf{1}$. Using (14) and (10) (respectively (12)), one can obtain an explicit form of $\tilde{\gamma}$, which is the following

$$\tilde{\gamma} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + P_0 \otimes \gamma_1 \otimes P_1 + P_1 \otimes \gamma_2 \otimes P_0 \,, \tag{16}$$

where

$$\begin{aligned} \gamma_1 &= \xi_+ \cdot P_0 \otimes P_1 + \xi_- \cdot P_1 \otimes P_0 \,, \\ \gamma_2 &= \xi_- \cdot P_0 \otimes P_1 + \xi_+ \cdot P_1 \otimes P_0 \,, \end{aligned}$$

and

$$\xi_+ = \frac{1+t}{\sqrt{1+t^2}} - 1 \,, \qquad \xi_- = \frac{1-t}{\sqrt{1+t^2}} - 1 \,.$$

$P_0$ and $P_1$ are the spectral projectors of the Pauli matrix $\sigma^3$, *i.e.*

$$(P_0)_{11} = 1 \,,\ (P_0)_{12} = (P_0)_{21} = (P_0)_{22} = 0 \,,$$

$$(P_1)_{22} = 1 \,,\ (P_1)_{11} = (P_1)_{12} = (P_1)_{21} = 0 \,.$$

Now, we are ready to calculate the explicit form of the operator $E$. Observe that $\tilde{\gamma}^* = \tilde{\gamma}$, $\psi_{kl}(\gamma_1) = \gamma_2$ and $\psi_{kl}(\gamma_2) = \gamma_1$. It follows that

$$\psi_{kl}\tilde{\gamma} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + P_0 \otimes \gamma_2 \otimes P_1 + P_1 \otimes \gamma_1 \otimes P_0 \,, \tag{17}$$

Hence, for any $A \in \mathcal{B}(\mathcal{H}_2)$,

$$
\begin{aligned}
&E(A) = \tau_{kl}\left(\tilde{\gamma} A \tilde{\gamma}\right) \\
&= \frac{1}{2}\,\tilde{\gamma} A \tilde{\gamma} + \frac{1}{2}\psi_{kl}(\tilde{\gamma})\psi_{kl}(A)\psi_{kl}(\tilde{\gamma}) = \frac{1}{2}\Big(\mathbf{1} \otimes (A + \psi_{kl}A) \otimes \mathbf{1}\Big) \\
&+ \frac{1}{2}\,P_0 \otimes \Big(\gamma_1 A + A\gamma_1 + \gamma_1 A \gamma_1 + \gamma_2 \psi_{kl}(A) + \psi_{kl}(A)\gamma_2 + \gamma_2 \psi_{kl}(A)\gamma_2\Big) \otimes P_1 \\
&+ \frac{1}{2} P_1 \otimes \Big(\gamma_2 A + A\gamma_2 + \gamma_2 A \gamma_2 + \gamma_1 \psi_{kl}(A) + \psi_{kl}(A)\gamma_1 + \gamma_1 \psi_{kl}(A)\gamma_1\Big) \otimes P_0 . \quad (18)
\end{aligned}
$$

One can show using (18) that the quantum correlations coefficient $\tilde{C}^{Q}_{T,\rho}(A,B)$ (with $A \in \mathcal{B}(\mathcal{H}_2)$, $B \in \mathcal{B}(\mathcal{H}_1)$ arbitrary) equals zero for our particular example of exchange transformation.

Now, we turn to the general case. We are interested in examining $\tilde{C}^{Q}_{T,\rho}(A,B)$ for $A \in \mathcal{B}(\mathcal{H}_2)$ and $B \in \mathcal{B}(\mathcal{H}_1)$. We recall that its nonzero value is an indication of proper interaction in the system as discussed in the introduction. In particular, $\tilde{C}^{Q}_{T,\rho}(A,B) \neq 0$ means that quantum correlations occur between the two parts of the system corresponding to regions $\Lambda_I$ and $\Lambda_{II}$. We shall identify operators on $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ with their embeddings into $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. We will need the following

**Fact 1** *Suppose that $A, B \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Let $\psi$ be a symmetry transformation on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ (i.e. a morphism such that $\psi(G) = G$ for $G \in \mathcal{B}(\mathcal{H}_1)$ and $\psi^2 = \mathbf{1}$). Suppose that $\psi(A) = -A$ and $\psi(B) = B$. Then, we have* $\mathrm{Tr}(AB) = 0$.

*Proof.* Indeed, since $\psi$ is a morphism and $\mathrm{Tr}(\psi(\cdot)) = \mathrm{Tr}(\cdot)$, one has

$$
\mathrm{Tr}(AB) = \mathrm{Tr}(\psi(AB)) = \mathrm{Tr}(\psi(A)\psi(B)) = \mathrm{Tr}(-AB) = -\,\mathrm{Tr}(AB)\,.
$$

Hence, $\mathrm{Tr}(AB) = 0$.

Note that any operator $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ can be represented as a sum $A = A_+ + A_-$, where $\psi(A_+) = A_+$ and $\psi(A_-) = -A_-$ (we have $A_+ = \frac{1}{2}(A + \psi(A))$ and $A_- = \frac{1}{2}(A - \psi(A))$). Obviously, any operator $B \in \mathcal{B}(\mathcal{H}_1)$ is invariant under $\psi$, so $B_+ = B$, $B_- = 0$. We will also use another decomposition of operators from $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. For any $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ we have $A = A_c + A_n$, where $A_c$ commutes with $\rho$, while $A_n$ does not commute with $\rho$. This is because each $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ can be written as a linear combination of simple tensors of the form $\sigma^{i_0} \otimes \ldots \otimes \sigma^{i_N}$, with $\sigma^{i_j}$ being Pauli matrices, $j = 0, 1, \ldots, N$. Then, $A_c = \sum_i \xi_i \sigma_i^{i_0} \otimes \ldots \otimes \sigma_i^{i_N}$ such that $i_j \in \{0, 3\}$, $j = 0, 1, \ldots, N$, and $A_n = \sum_p \xi_p \sigma_p^{p_0} \otimes \ldots \otimes \sigma_p^{p_N}$ such that $p_j \in \{1, 2\}$ for at least one $j \in \{0, 1, \ldots, N\}$.

Now, we are ready to calculate $\tilde{C}^{Q}_{T,\rho}(A,B)$ for the Ising model for any $A \in \mathcal{B}(\mathcal{H}_2)$ and $B \in \mathcal{B}(\mathcal{H}_1)$. Note that in case of Ising model any exchange transformation $\psi$ such as described in Section 2 satisfies $[\psi\rho,\rho]=0$, which also implies that $\gamma^* = \gamma$. In particular, operator $\psi_{kl}$ considered above satisfies this condition. Let $A = A_c + A_n$ be a decomposition of $A$ such that $A_c$ commutes with $\rho$ and $A_n$ does not commute with $\rho$. Each simple tensor in $A_n$ contains at least one Pauli matrix $\sigma^1$ or $\sigma^2$ localized at site belonging to $\Lambda_{II}$. This implies that $\langle A_n B\rangle_\rho = \mathrm{Tr}(\rho A_n B) = 0$. Moreover, all simple tensors in $\gamma A_n \gamma$ must contain at least one matrix with trace zero, localized at site belonging to $\Lambda_{II}$. It follows that

$$\begin{aligned}\langle E(A_n)B\rangle_\rho &= \mathrm{Tr}(\rho\tau(\gamma A_n\gamma)B)\\ &= \frac{1}{2}\,\mathrm{Tr}(\rho(\gamma A_n\gamma)B) + \frac{1}{2}\,\mathrm{Tr}(\rho(\psi(\gamma A_n\gamma))B)\\ &= \frac{1}{2}\,\mathrm{Tr}(\rho(\gamma A_n\gamma)B) + \frac{1}{2}\,\mathrm{Tr}(\psi(\rho)(\gamma A_n\gamma)B) = 0\,.\end{aligned}$$

This proves that $A_n$ does not contribute to $\tilde{C}^{Q}_{T,\rho}(A,B)$. By similar reasoning one can show that also $B_n$ does not contribute to $\tilde{C}^{Q}_{T,\rho}(A,B)$. We have arrived at the following assertion

**Proposition 1** *For any $A \in \mathcal{B}(\mathcal{H}_2)$ and $B \in \mathcal{B}(\mathcal{H}_1)$ the quantum correlations coefficient for Ising model is given by the following formula*

$$\tilde{C}^{Q}_{T,\rho}(A,B) = \tilde{C}^{Q}_{T,\rho}(A_c,B_c)\,.$$

*where $A_c$ ($B_c$) is the part of $A$ ($B$) that commutes with $\rho$.*

By virtue of Proposition 1 we can assume that $[A,\rho]=0$. Then we have $\gamma A\gamma = \gamma^2 A$. Since $\gamma = \rho^{\frac{1}{2}}(\tau\rho)^{-\frac{1}{2}} = \rho^{\frac{1}{2}}(\frac{1}{2}(\rho+\psi\rho))^{-\frac{1}{2}}$, we have $\gamma A\gamma = \rho(\frac{1}{2}(\rho+\psi\rho))^{-1}A$. This means that

$$\begin{aligned}E(A) &= \tau(\gamma A\gamma)\\ &= \frac{1}{2}\left(\rho\left(\frac{1}{2}(\rho+\psi\rho)\right)^{-1}A + (\psi\rho)\left(\frac{1}{2}(\rho+\psi\rho)\right)^{-1}(\psi A)\right).\end{aligned}\tag{19}$$

Consider a decomposition of $A$ into symmetric and antisymmetric part $A = A_+ + A_-$. For $A_+$ we have (from (19))

$$E(A_+) = \frac{1}{2}(\rho+\psi\rho)\left(\frac{1}{2}(\rho+\psi\rho)\right)^{-1}A_+ = A_+\,.$$

It follows that

$$\tilde{C}^{Q}_{T,\rho}(A_+,B) = \langle E(A_+)B\rangle_\rho - \langle A_+B\rangle_\rho = 0\,.\tag{20}$$

For $A_-$ we have

$$E(A_-) = \frac{1}{2}(\rho - \psi\rho)\left(\frac{1}{2}(\rho + \psi\rho)\right)^{-1} A_- = \rho_-\rho_+^{-1}A_- \,.$$

where $\rho_+ \equiv \frac{1}{2}(\rho + \psi\rho)$ and $\rho_- \equiv \frac{1}{2}(\rho - \psi\rho)$. It follows that

$$\begin{aligned}
&\tilde{C}^Q_{T,\rho}(A_-, B) = \langle E(A_-)B\rangle_\rho - \langle A_-B\rangle_\rho = \langle (E(A_-) - A)B\rangle_\rho \\
&= \mathrm{Tr}\Big(\rho(\rho_-\rho_+^{-1} - \mathbf{1})A_-B\Big) = \mathrm{Tr}\Big((\rho_+ + \rho_-)(\rho_-\rho_+^{-1} - \mathbf{1})A_-B\Big) \\
&= \mathrm{Tr}\Big(\rho_- - \rho_+ + \rho_-^2\rho_+^{-1} - \rho_-)A_-B\Big) = \mathrm{Tr}\Big(A_-B(\rho_-^2\rho_+^{-1} - \rho_+)\Big) \,.
\end{aligned}$$

Now, since $B$ and $\rho_-^2\rho_+^{-1} - \rho_+$ are invariant under $\psi$ we have from Fact 1 that

$$\tilde{C}^Q_{T,\rho}(A_-, B) = \langle E(A_-)B\rangle_\rho - \langle A_-B\rangle_\rho = 0 \,. \tag{21}$$

**Theorem 1** *For any $A \in \mathcal{B}(\mathcal{H}_2)$ and $B \in \mathcal{B}(\mathcal{H}_1)$, the quantum correlations coefficient $\tilde{C}^Q_{T,\rho}(A, B)$ for Ising model equals* 0.

*Proof.* See (20) and (21).

The above result shows the classical character of evolution in the Ising model.

## 4. *XXZ* model

Consider the $XXZ$ model as described in Section 2. The Hamiltonian of the system has the form:

$$\begin{aligned}
H &= -\sum_{n=1}^{N}(\sigma^1_{n-1}\sigma^1_n + \sigma^2_{n-1}\sigma^2_n + \Delta\sigma^3_{n-1}\sigma^3_n) \qquad (22) \\
&= -\sum_{n=1}^{N} \mathbf{1}\otimes\ldots\otimes\mathbf{1}\otimes(\sigma^1\otimes\sigma^1 + \sigma^2\otimes\sigma^2 + \Delta\sigma^3\otimes\sigma^3)\otimes\mathbf{1}\otimes\ldots\otimes\mathbf{1}\,,
\end{aligned}$$

where $\sigma^j$, $j = 1, 2, 3$ are Pauli matrices. We recall that $\Delta \neq 1$ is responsible for anisotropy of the model. The corresponding Gibbs state is represented by the density matrix

$$\rho = Z^{-1}\exp(-\beta H), \tag{23}$$

where $Z = \mathrm{Tr}(\mathrm{e}^{-\beta H})$ and $\beta$ is the inverse temperature. Since this model is much more complicated than the Ising one, we shall use a high-temperature expansion, *i.e.* we will use the approximation $\exp(-\beta H) \approx \mathbf{1} - \beta H$, which is valid for small $\beta$. Performing necessary calculation, one can obtain the explicit form of operator $\gamma$ for $XXZ$ model:

$$\gamma = \mathbf{1} + \frac{\beta}{4}(\psi H - H) \equiv \mathbf{1} + \tilde{H}.$$

Observe that $\gamma^* = \gamma$ and $\psi \tilde{H} = -\tilde{H}$. This allows us to get the explicit form of the generator $E$ of our dynamic. For $A \in \mathcal{B}(\mathcal{H}_2)$ we have

$$E(A) = \frac{1}{2}(\mathbf{1} + \tilde{H})A(\mathbf{1} + \tilde{H}) + \frac{1}{2}(\mathbf{1} - \tilde{H})(\psi A)(\mathbf{1} - \tilde{H}), \tag{24}$$

where $A$ in the above formula is identified with its embedding into the algebra $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Leaving out the factors of second order in $\beta$, we can express $E(A)$ as follows

$$E(A) = \frac{1}{2}\Big[(A + \psi(A)) + \tilde{H}(A - \psi(A)) + (A - \psi(A))\tilde{H}\Big]. \tag{25}$$

Assume that $\psi A = A$. Then, we have $E(A) = A$, which implies $\tilde{C}^{Q}_{T,\rho}(A, B) = 0$ for any $B \in \mathcal{B}(\mathcal{H}_1)$. Now suppose that $\psi A = -A$. This implies $E(A) = \tilde{H}A + A\tilde{H}$. Now, we are in position to calculate $\tilde{C}^{Q}_{T,\rho}(A, B)$ for $A \in \mathcal{B}(\mathcal{H}_2)$, $\psi A = -A$ and $B \in \mathcal{B}(\mathcal{H}_1)$. We have

$$\begin{aligned}
\tilde{C}^{Q}_{T,\rho}(A, B) &= \langle E(A)B\rangle_\rho - \langle AB\rangle_\rho = \mathrm{Tr}(\rho(E(A) - A)B) \\
&= \mathrm{Tr}((\mathbf{1} - \beta H)(\tilde{H}A + A\tilde{H} - A)B) \\
&= \mathrm{Tr}((\tilde{H}A + A\tilde{H} + \beta HA)B) - \mathrm{Tr}(AB) - \mathrm{Tr}(\beta H(\tilde{H}A + A\tilde{H})B).
\end{aligned}$$

The second term in the above equality equals zero by Fact 1. The first term can be rewritten in the following form

$$\begin{aligned}
&\mathrm{Tr}((\tilde{H}A + A\tilde{H} + \beta HA)B) \\
&= \frac{\beta}{4}\mathrm{Tr}(((\psi H - H)A + A(\psi H - H) + 4HA)B) \\
&= \frac{\beta}{4}\mathrm{Tr}((\psi H + H)AB) + \frac{\beta}{4}\mathrm{Tr}(A(\psi H + H)B) + \frac{\beta}{2}\mathrm{Tr}((HA - AH)B).
\end{aligned}$$

All these terms equal zero; the first and second one by Fact 1 (since $\psi H + H$ is invariant under $\psi$) and the last one due to the fact that $A$ commutes with $B$. Thus we have obtained

$$\tilde{C}^{Q}_{T,\rho}(A, B) = -\,\mathrm{Tr}(\beta H(\tilde{H}A + A\tilde{H})B),$$

which can be rewritten (using the properties of operator $\psi$ and $\psi A = -A$) as

$$\tilde{C}^{Q}_{T,\rho}(A,B) = \frac{\beta^2}{4}\,\mathrm{Tr}((\psi H + H)(HA + AH)B)\,.$$

Using the decomposition of $A$ into symmetric and antisymmetric part $A = A_+ + A_-$, with $\psi A_+ = A_+$ and $\psi A_- = -A_-$ we can summarize our considerations as follows

**Proposition 2** *For any $A \in \mathcal{B}(\mathcal{H}_2)$ and $B \in \mathcal{B}(\mathcal{H}_1)$ the quantum correlations coefficient for $XXZ$ model is given by the following formula*

$$\tilde{C}^{Q}_{T,\rho}(A,B) = \frac{\beta^2}{4}\,\mathrm{Tr}((\psi H + H)(HA_- + A_- H)B)\,, \tag{26}$$

*where $A_-$ is the antisymmetric part of $A$.*

We observe that $\tilde{C}^{Q}_{T,\rho}(A,B)$ is proportional to $\beta^2$. It is easy to give examples of observables for which nonzero quantum correlations occur in $XXZ$ model. Consider the exchange transformation $\psi_{kl}$ given by (13) (*cf.* Section 3) with $l = k+1$ (recall that we have $\Lambda = \{0,1,\ldots,N\}$, $\Lambda_I = \Lambda\backslash\{k,l\}$, $\Lambda_{II} = \{k,l\}$ in this case). Suppose that $A \in \mathcal{B}(\mathcal{H}_2)$ is such that $A = P_1 \otimes P_0$ and $B \in \mathcal{B}(\mathcal{H}_1)$, $B = \bigotimes_{i=0}^{N} O_n$ with $O_{k-1} = P_0$ and $O_j = \mathbf{1}$ for $j \neq k-1$. This pair of observables has a very straightforward physical interpretation as exchanging of spin directed downwards at $k^{\mathrm{th}}$ site with the spin directed upwards at $(k+1)^{\mathrm{th}}$ site. Using Proposition 2 one can calculate $\tilde{C}^{Q}_{T,\rho}(A,B)$ for our specified $A$ and $B$. Of course, $A_- = \frac{1}{2}(P_1 \otimes P_0 - P_0 \otimes P_1)$. Inserting $A_-$ into (26) and using the explicit form of $H$ we get

$$\tilde{C}^{Q}_{T,\rho}(A,B) = \frac{\beta^2 \Delta^2}{8}\,,$$

which means that nonzero quantum correlations occur for any finite temperature.

## 5. Conclusions

We studied two particular models of quantum stochastic dynamics, *i.e.* the dynamics which can be considered as examples of quantum generalizations of Glauber dynamics. The first model is exchange type dynamics which is originated from one dimensional Ising model with nearest neighbor interactions only while the second model is also exchange type dynamics but originated from $XXZ$ type Hamiltonian. For both cases we took the initial state to be a separable one. In other words, both models have only classical correlations for time $t = 0$. In the first model (Section 3) the Hamiltonian development has a multi-periodic nature for non fixed points (*cf.* [3]). In that sense, such the Hamiltonian model exhibits a behavior typical for classical interactions. Our analysis of stochastic quantum Ising dynamics clearly shows that the transition from that Hamiltonian model to quantum stochastic Ising model preserves the above mentioned property. On the other hand, it is well known that quantum Hamiltonian $XXZ$ model has much more interesting propagation than the Ising model. This feature is also reflected in our analysis of its quantum stochastic generalization (*cf.* [3]). This can be taken as a clear indication that generalized conditional expectations, generating time evolution for that model, contain non-trivial interactions. In other words, the transition rates defining the Markov evolution and determined by the corresponding Hamiltonians are correctly designed for $XXZ$ model in the sense that they contain the proper interaction between the subsystem and its environment. It is worth pointing out that similar results were obtained for another generalization of Glauber dynamics which was considered in [8]. Our results gain interest if we realize that the quantum correlations are closely related to entangled states. In other words, we demonstrated a non-trivial evolution of entanglement for the considered models. As the analysis of relations between quantum correlations and entanglement is a mathematical question and therefore exceeds the scope of this paper, it will be discussed in the forthcoming publication.

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## REFERENCES

[1] R.J. Glauber, *J. Mat. Phys.* **4**, 294 (1963).

[2] T.M. Ligget, *Interacting Particle Systems*, Springer Verlag, 1985.

[3] O. Bratteli, D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Springer Verlag, Vol. I (1979), Vol. II (1981).

[4] A.W. Majewski, B. Zegarlinski, *Math. Phys. Electronic J.* **1**, Paper 2 (1995).

[5] A.W. Majewski, B. Zegarlinski, *Lett. Math. Phys.* **36**, 337 (1996).

[6] A.W. Majewski, B. Zegarlinski, *Rev. Math. Phys.* **8**, 689 (1996).

[7] A.W. Majewski, B. Zegarlinski, *Markov Proc. Rel. Fields* **2**, 87 (1996).

[8] S. Kozieł, A.W. Majewski, LANL preprint `quant-ph/0101033`, (2001).