DYNAMICS OF AUTONOMOUS SYSTEMS WITH EXTERNAL FORCES

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We consider a geometric framework for analytical mechanics with external forces. Four versions of this framework are considered. A variational principle with boundary terms and external forces.The second and the third versions are the Lagrangian and Hamiltonian formulations, respectively. The last one is the Poisson formulation. An extensive introductory section presents some well known and some little known geometric constructions to put our formulation in the appropriate setting to make the comparison of the different formulations more easy.

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1. Introduction

We distinguish three types of situations to which methods of analytical mechanics can be applied. A mechanical system can be studied as an isolated system not subject to external interference. The study of planetary motion is an example. The motion of a mechanical system in a finite time interval can be studied. The external interaction with the system is limited to setting up or observing the initial and final conditions without interfering with the system during its motion. Such situations are studied in old-time ballistics. The motion of system in a finite time interval can be considered with both the boundary conditions and the motion itself during the time interval are subject to control. Such situations are studied in modern ballistics of guided missiles. The flight of an airplane or the motion of a car are also examples of such situations. Early formulations of analytical mechanics were applicable to all three types of situations. Recent geometrical formulations left out the possibility of analyzing the external interaction with a mechanical system during its motion. The external interaction can take different forms. One possibility is to influence the motion of a mechanical object by subjecting the object to constraints varying in time. Driving a car is an example of this type of control. A geometric study of this type of control was initiated by Marle [1–3]. Another possibility is to control a mechanical system by applying external forces. This happens when trajectory of a space vehicle or the orbit of a satellite are corrected by remotely activating jet engines mounted on the vehicle.

A geometric framework for analytical mechanics with external forces is the subject of the present paper. Four versions of this framework are considered. The first version is a variational formulation. Variational principles found in current literature are almost exclusively versions of the Hamilton principle. A recent paper by Gràcia, Marìn-Solano and Muñoz-Lecanda [4] gives a clear geometric description of the Hamilton principle in presence of constraints. Hamilton's principle does not take into account momenta at the boundary or external forces. A variational principle with boundary momenta was used by Schwinger [5]. The variational principle with boundary terms appearing in [6] is not related to Schwinger's principle. We use a variational principle with boundary terms and external forces. The second version of the framework is the Lagrangian formulation of mechanics and the third is the Hamiltonian formulation. The Lagrangian formulation usually presented is completely equivalent to the Hamilton principle and does not include momenta or external forces. Momenta but not external forces are present in the usual Hamiltonian formulation. In our interpretation a Lagrangian system and the corresponding Hamiltonian system are the same object described in terms of two different generating functions — the Lagrangian and the Hamiltonian. The last version is the Poisson formulation different from the Hamiltonian formulation only in the use of the Poisson structure of the phase space in place of the equivalent symplectic structure.

Our formulations are based on some well known and some little known geometric constructions. These are presented in an extensive introductory section. Work related to our formulations can be found in the references [7–11].

2. Geometric constructions

2.1. Tangent functors

Let M be a differential manifold. A local chart $(x^\kappa) : M \to \mathbb{R}^m$ of M with coordinates $x^\kappa : M \to \mathbb{R}$ will be treated as defined on all of M. Simple modifications have to be applied to constructions involving charts if truly local charts are used.

In the set $C^\infty(M|\mathbb{R})$ of differentiable mappings from $\mathbb{R}$ to M we introduce an equivalence relation. Two mappings γ and γ' are equivalent if $\gamma'(0) = \gamma(0)$ and $\mathrm{D}(f \circ \gamma')(0) = \mathrm{D}(f \circ \gamma)(0)$ for each differentiable function $f : M \to \mathbb{R}$.

The set of equivalence classes will be denoted by $\mathrm{T}M$. The equivalence class of a curve $\gamma : \mathbb{R} \to M$ will be denoted by $\mathbb{t}\gamma(0)$. The set $\mathrm{T}M$ is the set of *tangent vectors* in M or the *tangent bundle* of M.

We have the surjective mapping

$$\tau_M : \mathrm{T}M \to M : \mathbb{t}\gamma(0) \mapsto \gamma(0). \tag{1}$$

From a function $f : M \to \mathbb{R}$ we construct functions

$$f_{1;0} : \mathrm{T}M \to \mathbb{R} : \mathbb{t}\gamma(0) \mapsto (f \circ \gamma)(0) \tag{2}$$

and

$$f_{1;1} : \mathrm{T}M \to \mathbb{R} : \mathbb{t}\gamma(0) \mapsto \mathrm{D}(f \circ \gamma)(0). \tag{3}$$

These constructions can be applied to functions defined on open subsets of M. The functions $f_{1;0}$ and $f_{1;1}$ constructed from a function f on $U \subset M$ are defined on ${\tau_M}^{-1}(U)$. The function $f_{1;0}$ is the composition $f \circ \tau_M$.

The set $\mathrm{T}M$ is a differential manifold. A local chart $(x^\kappa) : M \to \mathbb{R}^m$ induces a local chart

$$(x^\kappa, \dot{x}^\lambda) : \mathrm{T}M \to \mathbb{R}^{2m} \tag{4}$$

of $\mathrm{T}M$. The local coordinates x^κ and $\dot{x}^\lambda$ are the functions ${x^\kappa}_{1;0}$ and ${x^\lambda}_{1;1}$ constructed from the coordinates of M. Note that we are using the symbol x^κ to denote local coordinates of M and also of $\mathrm{T}M$. The diagram

$$\begin{tikzcd} \mathrm{T}M \arrow[d, "\tau_M"'] \\ M \end{tikzcd} \tag{5}$$

is a differential fibration.

Each curve $\gamma : \mathbb{R} \to M$ has a *tangent prolongation*

$$\mathbb{t}\gamma : \mathbb{R} \to \mathrm{T}M : t \mapsto \mathbb{t}\gamma(t + \cdot)(0). \tag{6}$$

The value $\mathbb{t}\gamma(t)$ of the prolongation $\mathbb{t}\gamma$ depends only on the definition of the curve γ in the immediate neighbourhood of $t \in \mathbb{R}$. It follows that a local curve $\gamma : I \to M$ defined on an open subset $I \subset \mathbb{R}$ has a well defined prolongation $\mathbb{t}\gamma : I \to \mathrm{T}M$. The curve γ is obtained from $\mathbb{t}\gamma$ as the projection $\tau_M \circ \mathbb{t}\gamma$.

Let M and N be differential manifolds and let $\alpha : M \to N$ be a differentiable mapping. We have the mapping

$$\mathrm{T}\alpha : \mathrm{T}M \to \mathrm{T}N : \mathbb{t}\gamma(0) \mapsto \mathbb{t}(\alpha \circ \gamma)(0). \tag{7}$$

The diagram

$$\begin{tikzcd} \mathrm{T}M \arrow[r, "\mathrm{T}\alpha"] \arrow[d, "\tau_M"'] & \mathrm{T}N \arrow[d, "\tau_N"'] \\ M \arrow[r, "\alpha"] & N \end{tikzcd} \tag{8}$$

is a differential fibration morphism. If M, N, and O are differential manifolds and $\alpha : M \to N$ and $\beta : N \to O$ are differentiable mappings, then

$$\mathrm{T}(\beta \circ \alpha) = \mathrm{T}\beta \circ \mathrm{T}\alpha. \tag{9}$$

We have introduced a covariant functor T in the category of differential manifolds and differentiable mappings.

We introduce a second equivalence relation in the set $C^\infty(M|\mathbb{R})$ of differentiable mappings from $\mathbb{R}$ to M. Two mappings γ and γ' are equivalent if $\gamma'(0) = \gamma(0)$, $\mathrm{D}(f \circ \gamma')(0) = \mathrm{D}(f \circ \gamma)(0)$, and $\mathrm{D}^2(f \circ \gamma')(0) = \mathrm{D}^2(f \circ \gamma)(0)$ for each differentiable function $f : M \to \mathbb{R}$. The set of equivalence classes will be denoted by T^2M and the equivalence class of a curve $\gamma : \mathbb{R} \to M$ will be denoted by $\mathbb{t}^2\gamma(0)$. The set T^2M is the set of *second tangent vectors* in M or the *second tangent bundle* of M.

We have surjective mappings

$$\tau_{2M} : \mathrm{T}^2M \to M : \mathbb{t}^2\gamma(0) \mapsto \gamma(0) \tag{10}$$

and

$$\tau^1{}_{2M} : \mathrm{T}^2M \to \mathrm{T}M : \mathbb{t}^2\gamma(0) \mapsto \mathbb{t}\gamma(0) \tag{11}$$

satisfying $\tau_M \circ \tau^1{}_{2M} = \tau_{2M}$.

From a function $f : M \to \mathbb{R}$ we construct functions

$$f_{2;0} : \mathrm{T}^2M \to \mathbb{R} : \mathbb{t}^2\gamma(0) \mapsto (f \circ \gamma)(0), \tag{12}$$

$$f_{2;1} : \mathrm{T}^2M \to \mathbb{R} : \mathbb{t}^2\gamma(0) \mapsto \mathrm{D}(f \circ \gamma)(0), \tag{13}$$

and

$$f_{2;2} : \mathrm{T}^2M \to \mathbb{R} : \mathbb{t}^2\gamma(0) \mapsto \mathrm{D}^2(f \circ \gamma)(0). \tag{14}$$

These constructions can be applied to functions defined on open subsets of M.

The set T^2M is a differential manifold. A local chart $(x^\kappa) : M \to \mathbb{R}^m$ induces a local chart

$$(x^\kappa, \dot{x}^\lambda, \ddot{x}^\mu) : \mathrm{T}^2M \to \mathbb{R}^{3m} \tag{15}$$

of $\mathrm{T}M$. The local coordinates x^κ, $\dot{x}^\lambda$, and $\ddot{x}^\mu$ are the functions $x^\kappa{}_{2;0}$, $x^\lambda{}_{2;1}$, and $x^\mu{}_{2;2}$ constructed from the coordinates of M. Note that x^κ denote local coordinates of M and also of T^2M. Diagrams

$$\begin{tikzcd} \mathrm{T}^2M \arrow[d, "\tau_{2M}"'] \\ M \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathrm{T}^2M \arrow[d, "\tau^1{}_{2M}"'] \\ \mathrm{T}M \end{tikzcd} \tag{16}$$

are differential fibrations.

Each curve $\gamma : \mathbb{R} \to M$ has a *second tangent prolongation*

$$\mathbb{t}^2\gamma : \mathbb{R} \to \mathrm{T}^2M : t \mapsto \mathbb{t}^2\gamma(t + \cdot)(0). \tag{17}$$

Second tangent prolongations of local curves can be constructed. Relations $\tau_{2M} \circ \mathbb{t}^2\gamma = \gamma$ and $\tau^1{}_{2M} \circ \mathbb{t}^2\gamma = \mathbb{t}\gamma$ hold.

Let M and N be differential manifolds and let $\alpha : M \to N$ be a differentiable mapping. We have the mapping

$$\mathrm{T}^2\alpha : \mathrm{T}^2M \to \mathrm{T}^2N : \mathbb{t}^2\gamma(0) \mapsto \mathbb{t}^2(\alpha \circ \gamma)(0). \tag{18}$$

Diagrams

$$\begin{tikzcd} \mathrm{T}^2 M \arrow[r, "\mathrm{T}^2\alpha"] \arrow[d, "\tau_{2M}"'] & \mathrm{T}^2 N \arrow[d, "\tau_{2N}"'] \\ M \arrow[r, "\alpha"] & N \end{tikzcd} \tag{19}$$

and

$$\begin{tikzcd} \mathrm{T}^2 M \arrow[r, "\mathrm{T}^2\alpha"] \arrow[d, "{\tau^1}_{2M}"'] & \mathrm{T}^2 N \arrow[d, "{\tau^1}_{2N}"'] \\ \mathrm{T}M \arrow[r, "\mathrm{T}\alpha"] & \mathrm{T}N \end{tikzcd} \tag{20}$$

are morphisms of differential fibrations. If M, N, and O are differential manifolds and $\alpha : M \to N$ and $\beta : N \to O$ are differentiable mappings, then

$$\mathrm{T}^2(\beta \circ \alpha) = \mathrm{T}^2\beta \circ \mathrm{T}^2\alpha. \tag{21}$$

We have introduced a covariant functor T^2 in the category of differential manifolds and differentiable mappings.

2.1.1. Tangent and cotangent vectors

The fibration

$$\begin{tikzcd} \mathrm{T}M \arrow[d, "\tau_M"'] \\ M \end{tikzcd} \tag{22}$$

is a vector fibration. It is called the *tangent fibration*. Since representatives of vectors (curves in M) can not be added the construction of linear operations in fibres of τ_M is somewhat indirect. Let v, v_1, and v_2 be elements of the same fibre $\mathrm{T}_x M = {\tau_M}^{-1}(x)$. We write

$$v = v_1 + v_2 \tag{23}$$

if

$$f_{1,1}(v) = f_{1,1}(v_1) + f_{1,1}(v_2) \tag{24}$$

for each function f on M. Note that

$$f_{1,1}(v) = \langle \mathrm{d}f, v\rangle. \tag{25}$$

We have defined a relation between three elements of a fibre $\mathrm{T}_x M$. This relation will turn into a binary operation if we show that for each pair $(v_1, v_2) \in \mathrm{T}_x M \times \mathrm{T}_x M$ there is an unique vector $v \in \mathrm{T}_x M$ such that $v = v_1 + v_2$. The coordinate construction

$$(x^\kappa \circ \gamma)(t) = x^\kappa(v_1) + (\dot{x}^\kappa(v_1) + \dot{x}^\kappa(v_2))t \tag{26}$$

of a representative γ of v proves existence. Let v and v' be in relations $v = v_1 + v_2$ and $v' = v_1 + v_2$ with v_1 and v_2. Then

$$f_{1,1}(v') = f_{1,1}(v_1) + f_{1,1}(v_2) = f_{1,1}(v) \tag{27}$$

for each function f. It follows that $v' = v$. This proves uniqueness.

Let v and u be elements of $\mathrm{T}_x M$ and let k be a number. We write

$$v = ku \tag{28}$$

if

$$f_{1,1}(v) = k f_{1,1}(u) \tag{29}$$

for each function f on M. The coordinate construction

$$(x^\kappa \circ \gamma)(t) = x^\kappa(u) + k\dot{x}^\kappa(u)t \tag{30}$$

shows that for each $k \in \mathbb{R}$ and $u \in \mathrm{T}_x M$ there is a vector $v \in \mathrm{T}_x M$ such that $v = ku$. If v and v' are two such vectors, then

$$f_{1,1}(v') = k f_{1,1}(u) = f_{1,1}(v). \tag{31}$$

It follows that the vector v is unique.

We have defined operations

$$+ : \mathrm{T}M \underset{(\tau_M, \tau_M)}{\times} \mathrm{T}M \to \mathrm{T}M \tag{32}$$

and

$$\cdot : \mathbb{R} \times \mathrm{T}M \to \mathrm{T}M. \tag{33}$$

The symbol $\mathrm{T}M \underset{(\tau_M, \tau_M)}{\times} \mathrm{T}M$ denotes the *fibre product*

$$\{(v_1, v_2) \in \mathrm{T}M \times \mathrm{T}M;\ \tau_M(v_1) = \tau_M(v_2)\}. \tag{34}$$

A section $X : M \to \mathrm{T}M$ of the tangent fibration τ_M is a *vector field.* The *zero section* will be denoted by O_{τ_M}.

Let $C^\infty(\mathbb{R}|M)$ denote the algebra of differentiable functions on a differential manifold M. In the set $C^\infty(\mathbb{R}|M) \times M$ we introduce an equivalence relation. Two pairs (f, x) and (f', x') are equivalent if $x' = x$ and

$$\mathrm{D}(f' \circ \gamma)(0) = \mathrm{D}(f \circ \gamma)(0) \tag{35}$$

for each differentiable curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = x$. The set of equivalence classes denoted by T^*M is called the *cotangent bundle* of M. Elements of T^*M are called *cotangent vectors.* The equivalence class of (f, x) denoted by $\mathrm{d}f(x)$ is called the *differential* of f at x. The mapping

$$\pi_M : \mathrm{T}^*M \to M : \mathrm{d}f(x) \mapsto x \tag{36}$$

is called the *cotangent bundle projection.*

The cotangent bundle T^*M is a differential manifold. A local chart $(x^\kappa) : M \to \mathbb{R}^m$ induces a local chart $(x^\kappa, p_\lambda) : \mathrm{T}^*M \to \mathbb{R}^{2m}$. The local coordinates x^κ and p_λ are the functions

$$x^\kappa : \mathrm{T}^*M \to \mathbb{R} : \mathrm{d}f(x) \mapsto x^\kappa(x) \tag{37}$$

and

$$p_\lambda : \mathrm{T}^*M \to \mathbb{R} : \mathrm{d}f(x) \mapsto \mathrm{D}(f \circ \gamma_{\lambda,x})(0), \tag{38}$$

where $\gamma_{\lambda,x}$ are curves in M characterized by

$$x^\kappa(\gamma_{\lambda,x}(t)) = x^\kappa(x) + \delta^\kappa{}_\lambda t \tag{39}$$

for t sufficiently close to $0 \in \mathbb{R}$. Note that x^κ is used to denote local coordinates in M and also in T^*M.

The diagram

$$\begin{tikzcd} \mathrm{T}^*M \arrow[d, "\pi_M"'] \\ M \end{tikzcd} \tag{40}$$

is a differential vector fibration. It is called the *cotangent fibration.* The linear operations in fibres of the cotangent fibration have natural definitions

$$+ : \mathrm{T}^*M \underset{(\pi_M,\pi_M)}{\times} \mathrm{T}^*M \to \mathrm{T}^*M : (\mathrm{d}f_1(x), \mathrm{d}f_2(x)) \mapsto \mathrm{d}(f_1 + f_2)(x) \tag{41}$$

and

$$\cdot : \mathbb{R} \times \mathrm{T}^*M \to \mathrm{T}^*M : (k, \mathrm{d}f(x)) \mapsto \mathrm{d}(kf)(x). \tag{42}$$

The mapping

$$\langle\ ,\ \rangle : \mathrm{T}^*M \underset{(\pi_M, \tau_M)}{\times} \mathrm{T}M \to \mathbb{R} : (\mathrm{d}f(a), \mathfrak{t}\gamma(0)) \mapsto \mathrm{D}(f \circ \gamma)(0) \tag{43}$$

is a differentiable, bilinear, and nondegenerate pairing. The symbol $\mathrm{T}^*M \underset{(\pi_M, \tau_M)}{\times} \mathrm{T}M$ is again the fibre product defined as the equalizer

$$\mathrm{T}^*M \underset{(\pi_M, \tau_M)}{\times} \mathrm{T}M = \{(p, v) \in \mathrm{T}^*M \times \mathrm{T}M;\ \pi_M(p) = \tau_M(v)\} \tag{44}$$

of the projections π_M and τ_M. The tangent fibration and the cotangent fibration are a dual pair of vector fibrations.

The *Liouville form* is a 1-form ϑ_M on T^*M defined as

$$\vartheta_M : \mathrm{T}(\mathrm{T}^*M) \to \mathbb{R} : w \mapsto \langle \tau_{\mathrm{T}^*M}(w), \mathrm{T}\pi_M(w) \rangle. \tag{45}$$

A 1-form on M can interpreted as a section of the cotangent projection π_M. If $\mu : M \to \mathrm{T}^*M$, then

$$\begin{aligned} \langle \mu^*\vartheta_M, v \rangle &= \langle \vartheta_M, \mathrm{T}\mu(v) \rangle \\ &= \langle \tau_{\mathrm{T}^*M}(\mathrm{T}\mu(v)), \mathrm{T}\pi_M(\mathrm{T}\mu(v)) \rangle \\ &= \langle \mu(\tau_M(v)), \mathrm{T}(\pi_M \circ \mu)(v) \rangle \\ &= \langle \mu, v \rangle \end{aligned} \tag{46}$$

for each $v \in \mathrm{T}M$. Hence, $\mu^*\vartheta_M = \mu$. The *zero section* of the cotangent fibration will be denoted by O_{π_M}.

The cotangent bundle T^*M together with the 2-form $\omega_M = \mathrm{d}\vartheta_M$ form a symplectic manifold $(\mathrm{T}^*M, \omega_M)$. It follows from the local expressions

$$\vartheta_M = p_\kappa \mathrm{d}x^\kappa \tag{47}$$

and

$$\omega_M = \mathrm{d}p_\kappa \wedge \mathrm{d}x^\kappa \tag{48}$$

that the form ω_M is non degenerate.

2.1.2. Special symplectic structures

Let (P,ω) be a symplectic manifold. A *special symplectic structure* for (P,ω) is a diagram

$$\begin{array}{c}(P,\vartheta)\\ \pi\big\downarrow\\ M\end{array} \tag{49}$$

where $\pi : P \to Q$ is a vector fibration and ϑ is a vertical one-form on P such that $\mathrm{d}\vartheta = \omega$. An additional requirement is the existence of a vector fibration morphism

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & \mathrm{T}^*M \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi_M} \\ M & = & M \end{array} \tag{50}$$

such that $\vartheta = \alpha^*\vartheta_M$. This morphism is necessarily an isomorphism. For each $w \in \mathrm{T}P$ we have

$$\begin{aligned} \langle \vartheta, w\rangle &= \langle \alpha^*\vartheta_M, w\rangle \\ &= \langle \vartheta_M, \mathrm{T}\alpha(w)\rangle \\ &= \langle \tau_{\mathrm{T}^*M}(\mathrm{T}\alpha(w)), \mathrm{T}\pi_M(\mathrm{T}\alpha(w))\rangle \\ &= \langle \alpha(\tau_P(w)), \mathrm{T}(\pi_M\circ\alpha)(w)\rangle \\ &= \langle \alpha(\tau_P(w)), \mathrm{T}\pi(w)\rangle. \end{aligned} \tag{51}$$

It follows that the mapping $\alpha : P \to \mathrm{T}^*M$ is completely characterized by

$$\langle \alpha(p), v\rangle = \langle \vartheta, w\rangle, \tag{52}$$

where $v \in \mathrm{T}_{\pi(p)}M$ and w is any vector in T_pP such that $\mathrm{T}\pi(w) = v$. We conclude that if the morphism (50) associated with a special symplectic structure exists, then it is unique. It can be shown that if the 1-form ϑ interpreted as a mapping $\vartheta : \mathrm{T}P \to \mathbb{R}$ is linear on fibres of the vector fibration $\mathrm{T}\pi : \mathrm{T}P \to \mathrm{T}M$, then the morphism (50) exists. We will usually present a special symplectic structure together with the associated vector fibration isomorphism.

2.1.3. Generating functions and Legendre transformations

Let

$$\begin{tikzcd} (P,\vartheta) \arrow[d, "\pi"'] \\ M \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "\pi"'] & \mathrm{T}^*M \arrow[d, "\pi_M"'] \\ M \arrow[r, equal] & M \end{tikzcd} \tag{53}$$

be a special symplectic structure for a symplectic manifold (P,ω) with its associated vector fibre morphism. Let $S \subset P$ be the image $\mathrm{im}(\sigma)$ of a section $\sigma : M \to P$ of π. If S is a Lagrangian submanifold of (P,ω), then the 1-form $\sigma^*\vartheta$ is closed since $\mathrm{d}\sigma^*\vartheta = \sigma^*\omega = 0$. Let this form be exact. A function $F : M \to \mathbb{R}$ such that $\sigma^*\vartheta = \mathrm{d}F$ is called a *generating function* of S relative to the special symplectic structure (53). From

$$\sigma^*\vartheta = \sigma^*\alpha^*\vartheta_M = (\alpha\circ\sigma)^*\vartheta_M = \alpha\circ\sigma \tag{54}$$

it follows that $\sigma = \alpha^{-1}\circ\mathrm{d}F$. From

$$\mathrm{d}F = (\pi\circ\sigma)^*\mathrm{d}F = \sigma^*\pi^*\mathrm{d}F = \sigma^*\mathrm{d}(F\circ\pi) \tag{55}$$

it follows that $\sigma^*\vartheta = \sigma^*\mathrm{d}(F\circ\pi)$. Consequently $\rho^*\vartheta = \rho^*\mathrm{d}(F\circ\pi)$ for any mapping $\rho : R \to P$ such that $\mathrm{im}(\rho) \subset S$. This is in particular true for the canonical injection $\iota_S : S \to P$. Since the forms ϑ and $\mathrm{d}(F\circ\pi)$ are both vertical we have $\vartheta\circ\iota_S = \mathrm{d}(F\circ\pi)\circ\iota_S$ or $\vartheta|S = \mathrm{d}(F\circ\pi)|S$. The set S can be obtained from its generating function F as the equalizer

$$S = \{p \in P;\ \vartheta(p) = \mathrm{d}(F\circ\pi)(p)\} \tag{56}$$

of the two forms. If $S = \mathrm{im}(\sigma)$ is not Lagrangian we define its *generating form* relative to the special symplectic structure (53) as the 1-form φ on M such that $\sigma^*\vartheta = \varphi$. Relations $\sigma = \alpha^{-1}\circ\varphi$ and

$$S = \{p \in P;\ \vartheta(p) = (\pi^*\varphi)(p)\} \tag{57}$$

replace the corresponding relations of the Lagrangian case.

Let

$$\begin{tikzcd} (P,\vartheta') \arrow[d, "\pi'"'] \\ M' \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\alpha'"] \arrow[d, "\pi'"'] & \mathrm{T}^*M' \arrow[d, "\pi'_M"'] \\ M' \arrow[r, equal] & M' \end{tikzcd} \tag{58}$$

be a second special symplectic structure for (P,ω). The difference $\vartheta' - \vartheta$ is a closed form. Let it be exact and let G be a function on P such that $\vartheta' - \vartheta = \mathrm{d}G$. Let S be the image $\mathrm{im}(\sigma')$ of a section $\sigma' : M' \to P$ of π'. The function

$$F' = (F \circ \pi + G) \circ \sigma' \tag{59}$$

is a generating function of S relative to the new special symplectic structure since

$$\mathrm{d}F' = \sigma'^*\mathrm{d}(F\circ\pi+G) = \sigma'^*\mathrm{d}(F\circ\pi)+\sigma'^*(\vartheta'-\vartheta) = \sigma'^*\mathrm{d}(F\circ\pi)-\sigma'^*\vartheta+\sigma'^*\vartheta' = \sigma'^*\vartheta'. \tag{60}$$

The passage from F to F' is a version of the Legendre transformation. If $S = \mathrm{im}(\sigma) = \mathrm{im}(\sigma')$ is generated by a generating form φ with respect to the special symplectic structure (53), then the form

$$\varphi' = \sigma'^*(\pi^*\varphi + \mathrm{d}G) \tag{61}$$

is the generating form of S relative to the special symplectic structure (58).

2.1.4. Iterated tangent bundles $\mathbf{TT}M$, $\mathbf{TT}^2M$, and $\mathbf{T}^2\mathbf{T}M$

The sets $\mathrm{TT}M$, TT^2M, and $\mathrm{T}^2\mathrm{T}M$ obtained by repeated application of tangent functors are differential manifolds. From a function $f : M \to \mathbb{R}$ we construct sets of functions

$$\{f_{1,0;1,0}, f_{1,0;1,1}, f_{1,1;1,0}, f_{1,1;1,1}\} \tag{62}$$

on $\mathrm{TT}M$,

$$\{f_{1,0;2,0}, f_{1,0;2,1}, f_{1,0;2,2}, f_{1,1;2,0}, f_{1,1;2,1}, f_{1,1;2,2}\} \tag{63}$$

on TT^2M, and

$$\{f_{2,0;1,0}, f_{2,0;1,1}, f_{2,1;1,0}, f_{2,1;1,1}, f_{2,2;1,0}, f_{2,2;1,1}\} \tag{64}$$

on $\mathrm{T}^2\mathrm{T}M$. The functions are obtained by repeated application of the constructions introduced in formulae (2) and (3). The general definition is

$$f_{k',k;i,j} = (f_{k;j})_{k';i} \tag{65}$$

with suitable values of the indices. These constructions apply to local functions as well. By applying these constructions to the local coordinates for a chart $(x^\kappa) : M \to \mathbb{R}^m$ we construct charts

$$({x^\kappa}_{1,0;1,0}, {x^\lambda}_{1,0;1,1}, {x^\mu}_{1,1;1,0}, {x^\nu}_{1,1;1,1}) : \mathrm{TT}M \to \mathbb{R}^{4m}, \tag{66}$$

$$({x^\kappa}_{1,0;2,0}, {x^\lambda}_{1,0;2,1}, {x^\mu}_{1,0;2,2}, {x^\nu}_{1,1;2,0}, {x^\omega}_{1,1;2,1}, {x^\rho}_{1,1;2,2}) : \mathrm{TT}^2M \to \mathbb{R}^{6m}, \tag{67}$$

and

$$(x^\kappa{}_{2,0;1,0}, x^\lambda{}_{2,0;1,1}, x^\mu{}_{2,1;1,0}, x^\nu{}_{2,1;1,1}, x^\omega{}_{2,2;1,0}, x^\rho{}_{2,2;1,1}) : \mathrm{T}^2\mathrm{T}M \to \mathbb{R}^{6m}. \tag{68}$$

Coordinates in $\mathrm{TT}M$ are usually denoted by $(x^\kappa, \dot{x}^\lambda, \delta x^\mu, \delta\dot{x}^\nu)$, coordinates in TT^2M are denoted by $(x^\kappa, \dot{x}^\lambda, \ddot{x}^\mu, \delta x^\nu, \delta\dot{x}^\omega, \delta\ddot{x}^\rho)$, and coordinates in $\mathrm{T}^2\mathrm{T}M$ could be denoted by $(x^\kappa, \dot{x}^\lambda, x'^\mu, \dot{x}'^\nu, x''^\omega, \dot{x}''^\rho)$.

Each element of one of the spaces $\mathrm{TT}M$, TT^2M, and $\mathrm{T}^2\mathrm{T}M$ is conveniently represented by a mapping $\chi : \mathbb{R}^2 \to M$. Mappings

$$\eta : \mathbb{R} \to \mathrm{T}M : s \mapsto \mathbb{t}\chi(s, \cdot)(0) \tag{69}$$

and

$$\zeta : \mathbb{R} \to \mathrm{T}^2M : s \mapsto \mathbb{t}^2\chi(s, \cdot)(0) \tag{70}$$

derived from the mapping χ serve as representatives of elements $\mathbb{t}\eta(0) \in \mathrm{TT}M$, $\mathbb{t}\zeta(0) \in \mathrm{TT}^2M$, and $\mathbb{t}^2\eta(0) \in \mathrm{T}^2\mathrm{T}M$. We denote these elements by $\mathbb{t}^{1,1}\chi(0,0)$, $\mathbb{t}^{1,2}\chi(0,0)$, and $\mathbb{t}^{2,1}\chi(0,0)$ respectively. The mapping χ characterized by

$$x^\kappa(\chi(s,t)) = x^\kappa(x) + t\dot{x}^\kappa(x) + s\delta x^\kappa(x) + st\delta\dot{x}^\kappa(x) \tag{71}$$

for (s,t) sufficiently close to $(0,0) \in \mathbb{R}^2$ is a representative of an element x of $\mathrm{TT}M$. This coordinate construction proves the existence of representatives. The corresponding coordinate constructions of representatives of elements $y \in \mathrm{TT}^2M$ and $z \in \mathrm{T}^2\mathrm{T}M$ are provided by

$$x^\kappa(\chi(s,t)) = x^\kappa(y) + t\dot{x}^\kappa(y) + \tfrac{1}{2}t^2\ddot{x}^\kappa(y) + s\delta x^\kappa(x) + st\delta\dot{x}^\kappa(y) + \tfrac{1}{2}s^2t\delta\dot{x}^\kappa(y) \tag{72}$$

and

$$x^\kappa(\chi(s,t)) = x^\kappa(z) + t\dot{x}^\kappa(z) + sx'^\kappa(z) + st\dot{x}'^\kappa(z) + \tfrac{1}{2}s^2x''^\kappa(z) + \tfrac{1}{2}s^2t\dot{x}''^\kappa(z) \tag{73}$$

respectively. Relations

$$\tau^{k''}{}_{k'\mathrm{T}^kM}(\mathbb{t}^{k',k}\chi(0,0)) = \mathbb{t}^{k'',k}\chi(0,0) \tag{74}$$

and

$$\mathrm{T}^{k'}\tau^{k''}{}_{kM}(\mathbb{t}^{k',k}\chi(0,0)) = \mathbb{t}^{k',k''}\chi(0,0) \tag{75}$$

are easily verified. The definition (65) is equivalent to

$$f_{k',k;i,j}(\mathbb{t}^{k',k}\chi(0,0)) = \mathrm{D}^{(i,j)}(f \circ \chi)(0,0). \tag{76}$$

The equalities

$$\langle \mathrm{d}f_{k;i}, w\rangle = f_{1,k;1,i}(w) \tag{77}$$

for $k = 1$ or $k = 2$, $i \leqslant k$, and $w \in \mathrm{TT}^k$ follow from

$$\begin{aligned} \langle \mathrm{d}f_{k;i}, \mathfrak{t}^{1,k}\chi(0,0)\rangle &= \mathrm{D}(f_{k;i} \circ \zeta)(0) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}(f_{k;i}(\mathfrak{t}^k\chi(s,\cdot)(0)))_{|s=0} \\ &= \frac{\partial^{i+1}}{\partial s \partial t^i}(f(\chi(s,t)(0)))_{|s=0,t=0} \\ &= \mathrm{D}^{(1,i)(f\circ\chi)(0,0)} \\ &= f_{1,k;1,i}(\mathfrak{t}^{1,k}\chi(0,0)) \end{aligned} \tag{78}$$

with

$$\zeta : \mathbb{R} \to \mathrm{T}^k M : s \mapsto \mathfrak{t}^k\chi(s,\cdot)(0). \tag{79}$$

From a mapping $\chi : \mathbb{R}^2 \to M$ we derive the mapping $\widetilde{\chi} : \mathbb{R}^2 \to M$ by setting $\widetilde{\chi}(t,s) = \chi(s,t)$. This construction is used in the definitions of mappings

$$\kappa^{1,1}{}_M : \mathrm{TT}M \to \mathrm{TT}M : \mathfrak{t}^{1,1}\chi(0,0) \mapsto \mathfrak{t}^{1,1}\widetilde{\chi}(0,0), \tag{80}$$

$$\kappa^{1,2}{}_M : \mathrm{TT}^2M \to \mathrm{T}^2\mathrm{T}M : \mathfrak{t}^{1,2}\chi(0,0) \mapsto \mathfrak{t}^{2,1}\widetilde{\chi}(0,0), \tag{81}$$

and

$$\kappa^{2,1}{}_M : \mathrm{T}^2\mathrm{T}M \to \mathrm{TT}^2M : \mathfrak{t}^{2,1}\chi(0,0) \mapsto \mathfrak{t}^{1,2}\widetilde{\chi}(0,0). \tag{82}$$

Diagrams

$$\begin{tikzcd} \mathrm{T}^{k'}\mathrm{T}^kM \arrow[r, "\kappa^{k',k}{}_M"] \arrow[d, "\tau^{k''}{}_{k'\mathrm{T}^kM}"'] & \mathrm{T}^k\mathrm{T}^{k'}M \arrow[d, "\mathrm{T}^k\tau^{k''}{}_{k'M}"'] \\ \mathrm{T}^{k''}\mathrm{T}^kM \arrow[r, "\kappa^{k'',k}{}_M"] & \mathrm{T}^k\mathrm{T}^{k''}M \end{tikzcd} \tag{83}$$

for $k'' \leqslant k'$ are commutative and relations

$$\kappa^{k,k'}{}_M \circ \kappa^{k',k}{}_M = 1_{\mathrm{T}^{k'}\mathrm{T}^kM} \tag{84}$$

are satisfied for all applicable values of k, k', and k''. The special case $\kappa_M = \kappa^{1,1}{}_M$ is the most frequently used. It is known as the canonical involution in $\mathrm{TT}M$.

2.1.5. Derivations

Let $\Omega(M)$ be the exterior algebra of differential forms on a differential manifold M. A linear operator $\mathrm{a} : \Omega(M) \to \Omega(M)$ is called a *derivation* of $\Omega(M)$ of degree p if $\mathrm{a}\mu$ is a form of degree $q+p$ and

$$\mathrm{a}(\mu \wedge \nu) = \mathrm{a}\mu \wedge \nu + (-1)^{pq}\mu \wedge \mathrm{a}\nu \tag{85}$$

when μ is a form of degree q and ν is any form on M. The exterior differential $\mathrm{d} : \Omega(M) \to \Omega(M)$ is a derivation of degree 1. The *commutator*

$$[\mathrm{a}, \mathrm{a}'] = \mathrm{a}\mathrm{a}' - (-1)^{pp'}\mathrm{a}'\mathrm{a} \tag{86}$$

of derivations a and a$'$ of degrees p and p' respectively is a derivation of degree $p + p'$. A derivation a is said to be of *type* i_* if $\mathrm{a}f = 0$ for each function f on M. A derivation a is said to be of *type* d_* if $[\mathrm{a}, \mathrm{d}] = 0$. If i_A is a derivation of type i_*, then $\mathrm{d}_A = [\mathrm{i}_A, \mathrm{d}]$ is a derivation of type d_*. Derivations are local operators: if a is a derivation and μ is a differential form on M vanishing on an open subset $U \subset M$, then $\mathrm{a}\mu$ vanishes on U. A derivation is fully characterized by its action on functions and differentials of functions since each differential form is locally represented as a sum of exterior products of differentials of functions multiplied by functions. A derivation of type d_* is fully characterized by its action on functions.

A *vector-valued p-form* is a linear mapping

$$A : \wedge^p \mathrm{T}M \to \mathrm{T}M. \tag{87}$$

If $w \in \wedge^p \mathrm{T}_a M$, then $A(w) \in \mathrm{T}_a M$. Following Frölicher and Nijenhuis [13] we associate with a vector-valued p-form A a derivation i_A of type i_* and degree $p-1$ and the derivation $\mathrm{d}_A = [\mathrm{i}_A, \mathrm{d}]$. The derivation i_A is characterized by its action on 1-forms. If μ is a 1-form, then $\mathrm{i}_A\mu$ is a p-form and

$$\langle \mathrm{i}_A\mu, w\rangle = \langle \mu, A(w)\rangle \tag{88}$$

for each $w \in \wedge^p \mathrm{T}M$.

For $k = 1$ or $k = 2$, and each $n \in \mathbb{N}$ we define a linear mapping

$$F(k;n) : \mathrm{T}\mathrm{T}^k M \to \mathrm{T}\mathrm{T}^k M : \mathbb{t}^{1,k}\chi(0,0) \to \mathbb{t}^{1,k}\chi^n(0,0), \tag{89}$$

where χ is a mapping from $\mathbb{R}^2$ to M and

$$\chi^n : \mathbb{R}^2 \to M : (s,t) \mapsto \chi(st^n, t). \tag{90}$$

Relations

$$F(k;0) = 1_{\mathrm{T}\mathrm{T}^k M}, \tag{91}$$

$$F(k;n') \circ F(k;n) = F(k;n'+n), \tag{92}$$

and

$$F(k;n) = 0 \quad \text{if } n \geqslant k \tag{93}$$

are easily established. It follows that $F(1;1)$, $F(2;1)$, and $F(2;2)$ are the only non trivial cases. The diagrams

$$\begin{tikzcd} \mathrm{T}\mathrm{T}^k M \arrow[r, "F(k;n)"] \arrow[d, "\tau_{\mathrm{T}^k M}"'] & \mathrm{T}\mathrm{T}^k M \arrow[d, "\tau_{\mathrm{T}^k M}"'] \\ \mathrm{T}^k M \arrow[r, equal] & \mathrm{T}^k M \end{tikzcd} \tag{94}$$

are commutative since $\chi^n(0,\cdot) = \chi(0,\cdot)$ and the diagrams

$$\begin{tikzcd} \mathrm{T}\mathrm{T}^2 M \arrow[r, "F(2;n)"] \arrow[d, "\mathrm{T}\tau^1{}_{2M}"'] & \mathrm{T}\mathrm{T}^2 M \arrow[d, "\mathrm{T}\tau^1{}_{2M}"'] \\ \mathrm{T}\mathrm{T}M \arrow[r, "F(1;n)"] & \mathrm{T}\mathrm{T}M \end{tikzcd} \tag{95}$$

are obviously commutative. The mappings $F(k;n)$ are vector-valued 1-forms.

Proposition 1

$$\langle \mathrm{d}f_{k;i}, F(k;n)(w)\rangle = \frac{i!}{(i-n)!}\langle \mathrm{d}f_{k;i-n}, w\rangle \tag{96}$$

if $i \geqslant n$ and

$$\langle \mathrm{d}f_{k;i}, F(k;n)(w)\rangle = 0 \tag{97}$$

if $i < n$.

Proof : The proof is established by the calculations

$$\begin{aligned}
\langle \mathrm{d}f_{k;i}, F(k;n)(\mathbb{t}^{1,k}\chi(0,0))\rangle &= f_{1,k;1,i}(F(k;n)(\mathbb{t}^{1,k}\chi(0,0))) \\
&= \mathrm{D}^{(1,i)}(f\circ\chi^n)(0,0) \\
&= \frac{\partial^{i+1}}{\partial s\partial t^i}(f(\chi(st^n,t)))_{|s=0,t=0} \\
&= \frac{\partial^i}{\partial t^i}(t^n\frac{\partial}{\partial u}f(\chi(u,t)))_{|u=0,t=0} \\
&= \frac{i!}{(i-n)!}\frac{\partial^{i-n+1}}{\partial u\partial t^{i-n}}(f(\chi(u,t)))_{|u=0,t=0} \\
&= \frac{i!}{(i-n)!}\mathrm{D}^{(1,i-n)}(f\circ\chi)(0,0) \\
&= \frac{i!}{(i-n)!}\langle \mathrm{d}f_{k;i-n}, \mathbb{t}^{1,k}\chi(0,0)\rangle \qquad (98)
\end{aligned}$$

if $i \geqslant n$ and

$$\langle \mathrm{d}f_{k;i}, F(k;n)(\mathbb{t}^{1,k}\chi(0,0))\rangle = \frac{\partial^i}{\partial t^i}\left(t^n\frac{\partial}{\partial u}f(\chi(u,t))\right)_{|u=0,t=0} = 0 \qquad (99)$$

if $i < n$. ■

Here are the non trivial cases of formulae (96) and (97):

$$\langle \mathrm{d}f_{1;0}, F(1;1)(w)\rangle = 0, \qquad (100)$$

$$\langle \mathrm{d}f_{1;1}, F(1;1)(w)\rangle = \langle \mathrm{d}f_{1;0}, w\rangle, \qquad (101)$$

$$\langle \mathrm{d}f_{2;0}, F(2;1)(w)\rangle = 0, \qquad (102)$$

$$\langle \mathrm{d}f_{2;1}, F(2;1)(w)\rangle = \langle \mathrm{d}f_{2;0}, w\rangle, \qquad (103)$$

$$\langle \mathrm{d}f_{2;2}, F(2;1)(w)\rangle = 2\langle \mathrm{d}f_{2;1}, w\rangle, \qquad (104)$$

$$\langle \mathrm{d}f_{2;0}, F(2;2)(w)\rangle = 0, \qquad (105)$$

$$\langle \mathrm{d}f_{2;1}, F(2;2)(w)\rangle = 0, \qquad (106)$$

$$\langle \mathrm{d}f_{2;2}, F(2;2)(w)\rangle = 2\langle \mathrm{d}f_{2;0}, w\rangle. \qquad (107)$$

It follows from formulae (100) and (102) that if $w \in \mathrm{im}(F(1;1))$, then $\langle \mathrm{d}f_{1;0}, w\rangle = 0$ and if $w \in \mathrm{im}(F(2;1))$, then $\langle \mathrm{d}f_{2;0}, w\rangle = 0$ for each function f on M.

Proposition 2

If $w \in \mathrm{TT}M$ and $\langle \mathrm{d}f_{1;0}, w\rangle = 0$ for each function f, then $w \in \mathrm{im}(F(1;1))$ and if $w \in \mathrm{TT}^2M$ and $\langle \mathrm{d}f_{2;0}, w\rangle = 0$ for each function f, then $w \in \mathrm{im}(F(2;1))$.

Proof : Let $(x^\kappa, \dot{x}^\lambda, \delta x^\mu, \delta\dot{x}^\nu) : \mathrm{TT}M \to \mathbb{R}^{4m}$ be a chart of $\mathrm{TT}M$ derived from a chart $(x^\kappa) : M \to \mathbb{R}^m$. If $w \in \mathrm{TT}M$ and $\langle \mathrm{d}f_{1;0}, w\rangle = 0$ for each function f, then $\delta x^\kappa(w) = \langle \mathrm{d}x^\kappa, w\rangle = 0$. A representative χ of w such that

$$(x^\kappa \circ \chi)(s,t) = x^\kappa(w) + \dot{x}^\kappa(w)t + \delta\dot{x}^\kappa(w)st \tag{108}$$

can be chosen. If ζ is the mapping

$$\zeta : \mathbb{R}^2 \to M : (s,t) \mapsto \lim_{u\to t} \chi(su^{-n}, u), \tag{109}$$

then $\chi = \zeta^1$ and $w = F(1;1)(\mathfrak{t}^{1,1}\zeta(0,0))$.

We use in TT^2M coordinates $(x^\kappa, \dot{x}^\lambda, \ddot{x}^\mu, \delta x^\nu, \delta\dot{x}^\omega, \delta\ddot{x}^\rho)$ derived from a chart $(x^\kappa) : M \to \mathbb{R}^m$. If $w \in \mathrm{TT}^2M$ and $\langle \mathrm{d}f_{2;0}, w\rangle = 0$ for each function f, then $\delta x^\kappa(w) = 0$. We choose a representative χ of w such that

$$(x^\kappa \circ \chi)(s,t) = x^\kappa(w) + \dot{x}^\kappa(w)t + \tfrac{1}{2}\ddot{x}^\kappa(w)t^2 + \delta\dot{x}^\kappa(w)st + \tfrac{1}{2}\delta\ddot{x}^\kappa(w)st^2. \tag{110}$$

If ζ is the mapping defined in formula (109), then $\chi = \zeta^1$ and $w = F(2;1)$ $(\mathfrak{t}^{2,1}\zeta(0,0))$. ■

Proposition 3

$\mathrm{im}(F(1;1)) = \ker(F(1;1))$ and $\mathrm{im}(F(2;1)) = \ker(F(2;2))$.

Proof : From $F(1;1) \circ F(1;1) = F(1;2) = 0$ and $F(2;1) \circ F(2;2) = F(2;3) = 0$ we derive the inclusions $\mathrm{im}(F(1;1)) \subset \ker(F(1;1))$ and $\mathrm{im}(F(2;1)) \subset \ker(F(2;2))$. If $F(1;1)(w) = 0$, then

$$\langle \mathrm{d}f_{1;0}, w\rangle = \langle \mathrm{d}f_{1;1}, F(1;1)(w)\rangle = 0. \tag{111}$$

Hence $w \in \mathrm{im}(F(1;1))$. If $F(2;2)(w) = 0$, then

$$\langle \mathrm{d}f_{2;0}, w\rangle = \tfrac{1}{2}\langle \mathrm{d}f_{2;2}, F(2;2)(w)\rangle = 0. \tag{112}$$

Hence $w \in \mathrm{im}(F(2;1))$. ■

Proposition 4

$\ker(\mathrm{T}\tau_M) = \ker(F(1;1))$ and $\ker(\mathrm{T}\tau_{2M}) = \ker(F(2;2))$

Proof : The results follow directly from the identities

$$\langle \mathrm{d}f_{1;1}, F(1;1)(w)\rangle = \langle \mathrm{d}f_{1;0}, w\rangle = \langle \mathrm{d}f, \mathrm{T}\tau_M(w)\rangle \tag{113}$$

for $w \in \mathrm{TT}M$ and

$$\langle \mathrm{d}f_{2;2}, F(2;2)(w)\rangle = 2\langle \mathrm{d}f_{2;0}, w\rangle = 2\langle \mathrm{d}f, \mathrm{T}\tau_{2M}(w)\rangle \tag{114}$$

for $w \in \mathrm{TT}^2M$. ■

The relations $\ker(\mathrm{T}\tau_M) = \mathrm{im}(F(1;1))$ and $\ker(\mathrm{T}\tau_{2M}) = \mathrm{im}(F(2;1))$ follow from the two above propositions.

Let $\Omega_1(M)$ and $\Omega_2(M)$ denote the exterior algebras of differential forms on the tangent bundles $\mathrm{T}M$ and T^2M respectively. We will denote by ${\sigma_2}^1{}_M$ the homomorphism

$${\tau^1}_{2M}{}^* : \Omega_1(M) \to \Omega_2(M). \tag{115}$$

Derivations $\mathrm{i}_{F(k;n)}$ and $\mathrm{d}_{F(k;n)}$ are associated with the vector-valued 1-forms $F(k;n)$. The diagram

$$\begin{tikzcd}
\Omega_1(M) \arrow[r, "\mathrm{i}_{F(1;1)}"] \arrow[d, "{\sigma_2}^1{}_M"'] & \Omega_1(M) \arrow[d, "{\sigma_2}^1{}_M"'] \\
\Omega_2(M) \arrow[r, "\mathrm{i}_{F(2;1)}"] & \Omega_2(M)
\end{tikzcd} \tag{116}$$

is commutative.

The article [12] offers a generalization of the Frölicher and Nijenhuis theory. Let $\varphi : N \to M$ be a differentiable mapping. The mapping $\varphi^* : \Phi(M) \to \Phi(N)$ is a homomorphism of the exterior algebras. A *derivation of degree p relative to* φ^* is a linear operator $\mathrm{a} : \Phi(M) \to \Phi(N)$ such that $\mathrm{a}\mu$ is a form on N of degree $q+p$ and

$$\mathrm{a}(\mu \wedge \nu) = \mathrm{a}\mu \wedge \varphi^*\nu + (-1)^{pq}\varphi^*\mu \wedge \mathrm{a}\nu \tag{117}$$

if μ is a form on M of degree q and ν is any form on M. A derivation of the algebra $\Phi(M)$ is a derivation relative to the identity mapping 1_M. A derivation a relative to φ is said to be of *type* i_* if $\mathrm{a}f = 0$ for each function f on M. A relative derivation a of degree p is said to be of *type* d_* if $\mathrm{ad} - (-1)^p\mathrm{da} = 0$. If i_A is a derivation of type i_* relative to φ, then $\mathrm{d}_A = \mathrm{i}_A\mathrm{d} - (-1)^p\mathrm{di}_A$ is a derivation of type d_* relative to φ. Note that the expressions $\mathrm{ad} - (-1)^p\mathrm{da}$ and $\mathrm{i}_A\mathrm{d} - (-1)^p\mathrm{di}_A$ are not commutators since each of these expressions involves two different exterior differentials d. If a is a derivation of degree p relative to φ^* and $\varphi : O \to N$ is a differentiable mapping, then the operator $\varphi^*\mathrm{a} : \Phi(M) \to \Phi(O)$ is a derivation of degree p relative to $(\varphi \circ \varphi)^*$ since

$$\begin{aligned}
\varphi^*\mathrm{a}(\mu \wedge \nu) &= \varphi^*\mathrm{a}\mu \wedge \varphi^*\varphi^*\nu + (-1)^{pq}\varphi^*\varphi^*\mu \wedge \varphi^*\mathrm{a}\nu \\
&= \varphi^*\mathrm{a}\mu \wedge (\varphi \circ \varphi)^*\nu + (-1)^{pq}(\varphi \circ \varphi)^*\mu \wedge \varphi^*\mathrm{a}\nu
\end{aligned} \tag{118}$$

if μ is a form on M of degree q and ν is any form on M. If a is a derivation of type i_* or d_*, then $\varphi^*\mathrm{a}$ is a derivation of the same type. Relative derivations are again local operators and are completely characterized by their action on functions and differentials of functions.

A *vector-valued p-form relative to* $\varphi : N \to M$ is a linear mapping

$$A : \wedge^p \mathrm{T}N \to \mathrm{T}M \tag{119}$$

such that if $w \in \wedge^p \mathrm{T}_b N$, then $A(w) \in \mathrm{T}_{\varphi(b)}M$. We associate with a vector-valued p-form A relative to φ a derivation i_A relative to φ^* of type i_* and degree $p-1$ and the relative derivation $\mathrm{d}_A = \mathrm{i}_A\mathrm{d} - (-1)^p \mathrm{d}\mathrm{i}_A$. If μ is a 1-form on M, then $\mathrm{i}_A\mu$ is a p-form on N and

$$\langle \mathrm{i}_A\mu, w\rangle = \langle \mu, A(w)\rangle \tag{120}$$

for each $w \in \wedge^p \mathrm{T}N$.

Let $T(0) : \mathrm{T}M \to \mathrm{T}M$ be the identity mapping interpreted as a deformation of the tangent projection $\tau_M : \mathrm{T}M \to M$. We associate with $T(0)$ derivations $\mathrm{i}_{T(0)} : \Omega(M) \to \Omega_1(M)$ and $\mathrm{d}_{T(0)} : \Omega(M) \to \Omega_1(M)$ relative to $\sigma_M = {\tau_M}^*$. The derivation $\mathrm{i}_{T(0)}$ is a derivation of degree -1. If μ is a $(q+1)$-form on M, then $\mathrm{i}_{T(0)}\mu$ is a q-form on $\mathrm{T}M$ and if $w_1, \ldots, w_q$ are elements of $\mathrm{TT}M$ such that $\tau_{\mathrm{T}M}(w_1) = \ldots = \tau_{\mathrm{T}M}(w_q)$, then

$$\langle \mathrm{i}_{T(0)}\mu, w_1 \wedge \ldots \wedge w_q\rangle = \langle \mu, \tau_{\mathrm{T}M}(w_1) \wedge \mathrm{T}\tau_M(w_1) \wedge \ldots \wedge \mathrm{T}\tau_M(w_q)\rangle. \tag{121}$$

Let $X : M \to \mathrm{T}M$ be a vector field. The operator $X^*\mathrm{i}_{T(0)}$ is a derivation of $\Omega(M)$ of type i_* and degree -1. For each 1-form μ on M we have

$$X^*\mathrm{i}_{T(0)}\mu = \mathrm{i}_{T(0)}\mu \circ X = \langle \mu, T(0) \circ X\rangle = \langle \mu, X\rangle = \mathrm{i}_X\mu. \tag{122}$$

Hence, $X^*\mathrm{i}_{T(0)} = \mathrm{i}_X$. The relation $X^*\mathrm{d}_{T(0)} = \mathrm{d}_X$ is established by

$$X^*\mathrm{d}_{T(0)} = X^*(\mathrm{i}_{T(0)}\mathrm{d} + \mathrm{d}\mathrm{i}_{T(0)}) = X^*\mathrm{i}_{T(0)}\mathrm{d} + \mathrm{d}X^*\mathrm{i}_{T(0)} = \mathrm{i}_X\mathrm{d} + \mathrm{d}\mathrm{i}_X = \mathrm{d}_X. \tag{123}$$

Let f be a function on M. For each vector $v = \dot\xi(0) = \mathfrak{t}\xi(0) \in \mathrm{T}M$ we find

$$\begin{aligned} \mathrm{d}_{T(0)}f(v) &= \mathrm{i}_{T(0)}\mathrm{d}f(v) \\ &= \langle \mathrm{d}f, T(0)(v)\rangle \\ &= \langle \mathrm{d}f, v\rangle \\ &= \mathrm{D}(f \circ \xi)(0). \end{aligned} \tag{124}$$

Hence, $\mathrm{i}_{T(0)}\mathrm{d}f = f_{1;1}$, $\mathrm{d}_{T(0)}f = f_{1;1}$, and $\mathrm{d}_{T(0)}\mathrm{d}f = \mathrm{d}f_{1;1}$.

For each vector $w \in \mathrm{TT}M$ there is a mapping $\delta\xi : \mathbb{R} \to \mathrm{T}M$ such that $w = \kappa^{1,1}(\mathbb{t}\delta\xi(0))$. Let $w_1, \ldots, w_q$ be elements of $\mathrm{TT}M$ such that

$$\tau_{\mathrm{T}M}(w_1) = \ldots = \tau_{\mathrm{T}M}(w_q) \tag{125}$$

and let

$$\delta\xi_1 : \mathbb{R} \to \mathrm{T}M, \ldots, \delta\xi_q : \mathbb{R} \to \mathrm{T}M \tag{126}$$

be the mappings such that

$$w_1 = \kappa^{1,1}(\mathbb{t}\delta\xi_1(0)), \ldots, w_q = \kappa^{1,1}(\mathbb{t}\delta\xi_q(0)). \tag{127}$$

We will require that these mappings satisfy the condition

$$\tau_M \circ \delta\xi_1 = \cdots = \tau_M \circ \delta\xi_q. \tag{128}$$

The following construction proves the existence of such mappings. Let $(x^\kappa, \dot{x}^\lambda) : \mathrm{T}M \to \mathbb{R}^{2m}$ be a chart of $\mathrm{T}M$ and $(x^\kappa, \dot{x}^\lambda, \delta x^\mu, \delta\dot{x}^\nu) : \mathrm{TT}M \to \mathbb{R}^{4m}$ a chart of $\mathrm{TT}M$ derived from a chart $(x^\kappa) : M \to \mathbb{R}^m$. Mappings $\delta\xi_1, \ldots, \delta\xi_q$ characterized by

$$(x^\kappa, \dot{x}^\lambda)(\delta\xi_1(t)) = (x^\kappa(w_1) + t\dot{x}^\kappa(w_1), \delta x^\lambda(w_1) + t\delta\dot{x}^\lambda(w_1))$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(x^\kappa, \dot{x}^\lambda)(\delta\xi_q(t)) = (x^\kappa(w_q) + t\dot{x}^\kappa(w_q), \delta x^\lambda(w_q) + t\delta\dot{x}^\lambda(w_1)) \tag{129}$$

for t close to $0 \in \mathbb{R}$ have the required property since $x^\kappa(w_1) = \ldots = x^\kappa(w_q)$ and $\dot{x}^\kappa(w_1) = \ldots = \dot{x}^\kappa(w_q)$. We denote by ξ the mapping $\tau_M \circ \delta\xi_1 = \ldots = \tau_M \circ \delta\xi_q$. The following proposition is stated in terms of the mappings ξ and $\delta\xi_1, \ldots, \delta\xi_q$.

Proposition 5

If $q > 0$ and μ is a q-form on M, then $\mathrm{d}_{T(0)}\mu$ is a q-form on $\mathrm{T}M$ and

$$\langle \mathrm{d}_{T(0)}\mu, w_1 \wedge \ldots \wedge w_q \rangle = \mathrm{D}\langle \mu, \delta\xi_1 \wedge \ldots \wedge \delta\xi_q \rangle(0)\,, \tag{130}$$

where $w_1, \ldots, w_q$ are vectors in $\mathrm{TT}M$ such that $\tau_{\mathrm{T}M}(w_1) = \ldots = \tau_{\mathrm{T}M}(w_q)$.

Proof : Let an operator $\mathrm{a} : \Omega(M) \to \Omega_1(M)$ of degree 0 be defined by

$$\mathrm{a}f = \mathrm{d}_{T(0)}f \tag{131}$$

for each function f on M and

$$\langle \mathrm{a}\mu, w_1 \wedge \ldots \wedge w_q \rangle = \mathrm{D}\langle \mu, \delta\xi_1 \wedge \ldots \wedge \delta\xi_q \rangle(0), \tag{132}$$

if $q > 0$, μ is a q-form on M, and $w_1, \ldots, w_q$ are elements of $\mathrm{TT}M$ such that $\tau_{\mathrm{T}M}(w_1) = \ldots = \tau_{\mathrm{T}M}(w_q)$.

We show that a is a derivation relative to σ_M. If f_1 and f_2 are functions on M, then

$$\mathrm{a}(fg) = \mathrm{d}_{T(0)}(fg) = \mathrm{d}_{T(0)}fg\circ\tau_M + f\circ\tau_M \mathrm{d}_{T(0)}g = \mathrm{a}fg\circ\tau_M + f\circ\tau_M\mathrm{a}g. \quad (133)$$

If f is a function on M and μ is a q-form on M with $q > 0$, then

$$\begin{aligned}\langle \mathrm{a}(f\mu), w_1\wedge\ldots\wedge w_q\rangle &= \mathrm{D}\langle f\mu, \delta\xi_1\wedge\ldots\wedge\delta\xi_q\rangle(0)\\ &= \mathrm{D}((f\circ\xi)\langle\mu, \delta\xi_1\wedge\ldots\wedge\delta\xi_q\rangle)(0)\\ &= \langle \mathrm{d}f, \mathbb{t}\xi(0)\rangle\langle\mu, \mathrm{T}\tau_M(w_1)\wedge\ldots\wedge\mathrm{T}\tau_M(w_q)\rangle\\ &\quad + f(\xi(0))\mathrm{D}\langle\mu, \delta\xi_1\wedge\ldots\wedge\delta\xi_q\rangle(0)\\ &= \mathrm{a}f(\mathbb{t}\xi(0))\langle\sigma_M\mu, w_1\wedge\ldots\wedge w_q\rangle\\ &\quad + \sigma_M f(\mathbb{t}\xi(0))\langle\mathrm{a}\mu, w_1\wedge\ldots\wedge w_q\rangle\\ &= \langle \mathrm{a}f\sigma_M\mu + \sigma_M f\mathrm{a}\mu, w_1\wedge\ldots\wedge w_q\rangle. \qquad (134)\end{aligned}$$

If μ_1 and μ_2 are forms on M of degrees $q_1 > 0$ and $q_2 > 0$, respectively, and $q = q_1 + q_2$, then $(\epsilon(\sigma) = \mathrm{sign}(\sigma))$

$$\begin{aligned}&\langle \mathrm{a}(\mu_1\wedge\mu_2), w_1\wedge\ldots\wedge w_q\rangle = \mathrm{D}\langle\mu_1\wedge\mu_2, \delta\xi_1\wedge\ldots\wedge\delta\xi_q\rangle(0)\\ &= \frac{1}{q_1!q_2!}\sum_{\sigma\in S_q}\epsilon(\sigma)\mathrm{D}\left(\left\langle\mu_1, \delta\xi_{\sigma(1)}\wedge\ldots\wedge\delta\xi_{\sigma(q_1)}\right\rangle\left\langle\mu_2, \delta\xi_{\sigma(q_1+1)}\wedge\ldots\wedge\delta\xi_{\sigma(q)}\right\rangle\right)(0)\\ &= \frac{1}{q_1!q_2!}\sum_{\sigma\in S_q}\epsilon(\sigma)\left(\mathrm{D}\left\langle\mu_1, \delta\xi_{\sigma(1)}\wedge\ldots\wedge\delta\xi_{\sigma(q_1)}\right\rangle(0)\left\langle\mu_2, \delta\xi_{\sigma(q_1+1)}\wedge\ldots\wedge\delta\xi_{\sigma(q)}\right\rangle(0)\right.\\ &\left. + \mathrm{D}\left\langle\mu_1, \delta\xi_{\sigma(1)}\wedge\ldots\wedge\delta\xi_{\sigma(q_1)}\right\rangle(0)\left\langle\mu_2, \delta\xi_{\sigma(q_1+1)}\wedge\ldots\wedge\delta\xi_{\sigma(q)}\right\rangle(0)\right)\\ &= \frac{1}{q_1!q_2!}\sum_{\sigma\in S_q}\epsilon(\sigma)\left(\left\langle\mathrm{a}\mu_1, w_{\sigma(1)}\wedge\ldots\wedge w_{\sigma(q_1)}\right\rangle\left\langle\mu_2, \mathrm{T}\tau_M(w_{\sigma(q_1+1)})\wedge\ldots\wedge\mathrm{T}\tau_M(w_{\sigma(q)})\right\rangle\right.\\ &\left. + \left\langle\mu_1, \mathrm{T}\tau_M(w_{\sigma(1)})\wedge\ldots\wedge\mathrm{T}\tau_M(w_{\sigma(q_1)})\right\rangle\left\langle\mathrm{a}\mu_2, w_{\sigma(q_1+1)}\wedge\ldots\wedge w_{\sigma(q)}\right\rangle\right)\\ &= \frac{1}{q_1!q_2!}\sum_{\sigma\in S_q}\epsilon(\sigma)\left(\left\langle\mathrm{a}\mu_1, w_{\sigma(1)}\wedge\ldots\wedge w_{\sigma(q_1)}\right\rangle\left\langle\sigma_M\mu_2, w_{\sigma(q_1+1)}\wedge\ldots\wedge w_{\sigma(q)}\right\rangle\right.\\ &\left. + \left\langle\sigma_M\mu_1, w_{\sigma(1)}\wedge\ldots\wedge w_{\sigma(q_1)}\right\rangle\left\langle\mathrm{a}\mu_2, w_{\sigma(q_1+1)}\wedge\ldots\wedge w_{\sigma(q)}\right\rangle\right)\\ &= \langle\mathrm{a}\mu_1\wedge\sigma_M\mu_2 + \sigma_M\mu_1\wedge\mathrm{a}\mu_2, w_1\wedge\ldots\wedge w_q\rangle. \qquad (135)\end{aligned}$$

This completes the proof that a is a derivation relative to σ_M.

Let w be an element of $\mathrm{TT}M$. We associate the mapping

$$\delta\xi : \mathbb{R}\to\mathrm{T}M : t\mapsto \mathbb{t}\chi(\cdot, t)(0) \quad (136)$$

with a representative $\chi : \mathbb{R}^2 \to M$ of w. If f is a function on M, then

$$\begin{aligned}\langle \mathrm{a}\mathrm{d}f, w\rangle &= \mathrm{D}\langle \mathrm{d}f, \delta\xi\rangle(0)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{d}f, \delta\xi(t)\rangle_{|t=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{d}f, \mathbb{t}\chi(\cdot,t)(0)\rangle_{|t=0}\\ &= \mathrm{D}^{(1,1)}(f\circ\chi)(0,0)\\ &= f_{1,1;1,1}(\mathbb{t}^{1,1}\chi(0,0))\\ &= \langle \mathrm{d}f_{1;1}, \mathbb{t}^{1,1}\chi(0,0)\rangle\\ &= \langle \mathrm{d}_{T(0)}\mathrm{d}f, w\rangle. \end{aligned} \tag{137}$$

The equality $\mathrm{a}\mathrm{d}f = \mathrm{d}_{T(0)}\mathrm{d}f$ together with $\mathrm{d}_{T(0)}f = \mathrm{a}f$ for each function f imply the equality $\mathrm{d}_{T(0)} = \mathrm{a}$.

The mapping

$$T(1) : \mathrm{T}^2M \to \mathrm{TT}M : \mathbb{t}^2\xi(0) \mapsto \mathbb{t}\mathbb{t}\xi(0) \tag{138}$$

is a vector valued 0-form relative to ${\tau^1}_2M$. We associate with $T(1)$ derivations $\mathrm{i}_{T(1)} : \Omega_1(M) \to \Omega_2(M)$ and $\mathrm{d}_{T(1)} : \Omega_1(M) \to \Omega_2(M)$ relative to ${\sigma_2}^1{}_M$. Derivations $\mathrm{i}_{T(1)}$ and $\mathrm{d}_{T(1)}$ have properties analogous to those of derivations $\mathrm{i}_{T(0)}$ and $\mathrm{d}_{T(0)}$. If μ is a $(q+1)$-form on $\mathrm{T}M$, then $\mathrm{i}_{T(1)}\mu$ is a q-form on T^2M and if $w_1, \ldots, w_q$ are elements of TT^2M such that $\tau_{\mathrm{T}^2M}(w_1) = \ldots = \tau_{\mathrm{T}^2M}(w_q)$, then

$$\langle \mathrm{i}_{T(1)}\mu, w_1\wedge\ldots\wedge w_q\rangle = \langle \mu, \tau_{\mathrm{T}^2M}(w_1)\wedge \mathrm{T}{\tau^1}_{2M}(w_1)\wedge\ldots\wedge \mathrm{T}{\tau^1}_{2M}(w_q)\rangle. \tag{139}$$

Let F be a function on $\mathrm{T}M$. For each element $a = \mathbb{t}^2\xi(0) \in \mathrm{T}^2M$ we have

$$\begin{aligned}\mathrm{d}_{T(1)}F(a) &= \mathrm{i}_{T(1)}\mathrm{d}F(a)\\ &= \langle \mathrm{d}F, T(1)(\mathbb{t}^2\xi(0))\rangle\\ &= \langle \mathrm{d}F, \mathbb{t}\mathbb{t}\xi(0)\rangle\\ &= \mathrm{D}(F\circ\mathbb{t}\xi)(0). \end{aligned} \tag{140}$$

If $w_1, \ldots, w_q$ are elements of TT^2M such that

$$\tau_{\mathrm{T}^2M}(w_1) = \ldots = \tau_{\mathrm{T}^2M}(w_q), \tag{141}$$

then it is possible to choose mappings

$$\delta\xi_1 : \mathbb{R} \to \mathrm{T}M, \ldots, \delta\xi_q : \mathbb{R} \to \mathrm{T}M \tag{142}$$

such that

$$w_1 = \kappa^{2,1}(\mathbb{t}^2\delta\xi_1(0)), \ldots, w_q = \kappa^{2,1}(\mathbb{t}^2\delta\xi_q(0)) \tag{143}$$

and

$$\tau_M \circ \delta\xi_1 = \cdots = \tau_M \circ \delta\xi_q. \tag{144}$$

Let $(x^\kappa, \dot{x}^\lambda) : \mathrm{T}M \to \mathbb{R}^{2m}$ be a chart of $\mathrm{T}M$ and $(x^\kappa, \dot{x}^\lambda, \ddot{x}^\mu, \delta x^\nu, \delta\dot{x}^\omega, \delta\ddot{x}^\pi) : \mathrm{T}\mathrm{T}^2M \to \mathbb{R}^{6m}$ a chart of $\mathrm{T}\mathrm{T}^2M$ derived from a chart $(x^\kappa) : M \to \mathbb{R}^m$. Mappings $\delta\xi_1, \ldots, \delta\xi_q$ such that

$$(x^\kappa, \dot{x}^\lambda)(\delta\xi_1(t)) = \left(x^\kappa(w_1) + t\dot{x}^\kappa(w_1) + \frac{t^2}{2}\ddot{x}^\kappa(w_1), \delta x^\lambda(w_1) + t\delta\dot{x}^\lambda(w_1) + \frac{t^2}{2}\delta\ddot{x}^\lambda(w_1)\right)$$

....

$$(x^\kappa, \dot{x}^\lambda)(\delta\xi_q(t)) = \left(x^\kappa(w_q) + t\dot{x}^\kappa(w_q) + \frac{t^2}{2}\ddot{x}^\kappa(w_q), \delta x^\lambda(w_q) + t\delta\dot{x}^\lambda(w_q) + \frac{t^2}{2}\delta\ddot{x}^\lambda(w_q)\right) \tag{145}$$

for t close to $0 \in \mathbb{R}$ are a correct choice since $x^\kappa(w_1) = \ldots = x^\kappa(w_q)$, $\dot{x}^\kappa(w_1) = \ldots = \dot{x}^\kappa(w_q)$, and $\ddot{x}^\kappa(w_1) = \ldots = \ddot{x}^\kappa(w_q)$. We introduce mappings

$$\delta\dot{\xi}_1 = \kappa^{1,1} \circ \mathbb{t}\delta\xi_q, \ldots, \delta\dot{\xi}_q = \kappa^{1,1} \circ \mathbb{t}\delta\xi_q. \tag{146}$$

The following proposition is stated in terms of these mappings.

Proposition 6

If $q > 0$ and μ is a q-form on $\mathrm{T}M$, then $\mathrm{d}_{T(1)}\mu$ is a q-form on T^2M and

$$\langle \mathrm{d}_{T(1)}\mu, w_1 \wedge \ldots \wedge w_q \rangle = \mathrm{D}\langle \mu, \delta\dot{\xi}_1 \wedge \ldots \wedge \delta\dot{\xi}_q \rangle(0) \tag{147}$$

where $w_1, \ldots, w_q$ are vectors in $\mathrm{T}\mathrm{T}^2M$ such that $\tau_{\mathrm{T}^2M}(w_1) = \ldots = \tau_{\mathrm{T}^2M}(w_q)$.

Proof : The proof of this proposition is analogous to that of Proposition 5. An operator $\mathrm{a} : \Omega_1(M) \to \Omega_2(M)$ of degree 0 is defined by

$$\mathrm{a}g = \mathrm{d}_{T(1)}g \tag{148}$$

for each function g on $\mathrm{T}M$ and

$$\langle \mathrm{a}\mu, w_1 \wedge \ldots \wedge w_q \rangle = \mathrm{D}\langle \mu, \delta\dot{\xi}_1 \wedge \ldots \wedge \delta\dot{\xi}_q \rangle(0), \tag{149}$$

if $q > 0$, μ is a q-form on $\mathrm{T}M$, and $w_1, \ldots, w_q$ are elements of $\mathrm{T}\mathrm{T}^2M$ such that $\tau_{\mathrm{T}M}(w_1) = \ldots = \tau_{\mathrm{T}M}(w_q)$. It is shown that a is a derivation relative to ${\sigma_2}^1{}_M$ by performing essentially the same calculations as in the proof of Proposition 5.

With a representative $\chi : \mathbb{R}^2 \to M$ of an element $w = \mathfrak{t}^{1,2}\chi(0,0)$ of $\mathrm{TT}^2 M$ we associate the mapping

$$\delta\dot{\xi} : \mathbb{R} \to \mathrm{TT}M : t \mapsto \mathfrak{t}^{1,1}\chi(\cdot, t). \tag{150}$$

If f is a function on M, then

$$\begin{aligned}
\langle \mathrm{ad} f_{1;0}, w\rangle &= \mathrm{D}\langle \mathrm{d} f_{1;0}, \delta\dot{\xi}\rangle(0) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{d} f_{1;0}, \delta\dot{\xi}(t)\rangle_{|t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{d} f_{1;0}, \mathfrak{t}^{1,1}\chi(\cdot,t)(0)\rangle_{|t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle f_{1,1;1,0}, \mathfrak{t}^{1,1}\chi(\cdot,t)(0)\rangle_{|t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{D}^{(1,0)}(f\circ\chi)(0,t)\right)_{|t=0} \\
&= \mathrm{D}^{(1,1)}(f\circ\chi)(0,0) \\
&= f_{1,2;1,1}(\mathfrak{t}^{1,2}\chi(0,0)) \\
&= f_{1,2;1,1}(w)) \\
&= \langle \mathrm{d} f_{2;1}, w\rangle \\
&= \langle \mathrm{d}_{T(1)}\mathrm{d} f_{1;0}, w\rangle
\end{aligned} \tag{151}$$

and

$$\begin{aligned}
\langle \mathrm{ad} f_{1;1}, w\rangle &= \mathrm{D}\langle \mathrm{d} f_{1;1}, \delta\dot{\xi}\rangle(0) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{d} f_{1;1}, \delta\dot{\xi}(t)\rangle_{|t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{d} f_{1;1}, \mathfrak{t}^{1,1}\chi(\cdot,t)(0)\rangle_{|t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle f_{1,1;1,1}, \mathfrak{t}^{1,1}\chi(\cdot,t)(0)\rangle_{|t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{D}^{(1,1)}(f\circ\chi)(0,t)\right)_{|t=0} \\
&= \mathrm{D}^{(1,2)}(f\circ\chi)(0,0) \\
&= f_{1,2;1,2}(\mathfrak{t}^{1,2}\chi(0,0)) \\
&= f_{1,2;1,2}(w)) \\
&= \langle \mathrm{d} f_{2;2}, w\rangle \\
&= \langle \mathrm{d}_{T(1)}\mathrm{d} f_{1;1}, w\rangle.
\end{aligned} \tag{152}$$

Equalities $\mathrm{d}_{T(1)} f_{1;0} = \mathrm{a} f_{1;0}$, $\mathrm{d}_{T(1)} f_{1;1} = \mathrm{a} f_{1;1}$, $\mathrm{d}_{T(1)}\mathrm{d} f_{1;0} = \mathrm{ad} f_{1;0}$, and $\mathrm{d}_{T(1)}\mathrm{d} f_{1;1} = \mathrm{ad} f_{1;1}$ for each function f imply the equality $\mathrm{d}_{T(1)} = \mathrm{a}$. ■

Proposition 7

The relation

$$\mathrm{i}_{F(2;1)}\mathrm{d}_{T(1)} - \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)} = {\sigma_2}^1{}_M \mathrm{i}_{F(1;0)} \tag{153}$$

holds.

Proof : We show that the operator $\mathrm{i}_{F(2;1)}\mathrm{d}_{T(1)} - \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)}$ is a derivation of type i_* and degree 0 relative to ${\sigma_2}^1{}_M$. For each function g on $\mathrm{T}M$ we have

$$(\mathrm{i}_{F(2;1)}\mathrm{d}_{T(1)} - \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)})g = 0. \tag{154}$$

For any two forms μ and ν on $\mathrm{T}M$ we have

$$\begin{aligned}
&(\mathrm{i}_{F(2;1)}\mathrm{d}_{T(1)} - \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)})(\mu \wedge \nu)\\
&= \mathrm{i}_{F(2;1)}(\mathrm{d}_{T(1)}\mu \wedge {\sigma_2}^1{}_M\nu + {\sigma_2}^1{}_M\mu \wedge \mathrm{d}_{T(1)}\nu)\\
&-\mathrm{d}_{T(1)}(\mathrm{i}_{F(1;1)}\mu \wedge \nu + \mu \wedge \mathrm{i}_{F(1;1)}\nu)\\
&= \mathrm{i}_{F(2;1)}\mathrm{d}_{T(1)}\mu \wedge {\sigma_2}^1{}_M\nu + \mathrm{d}_{T(1)}\mu \wedge \mathrm{i}_{F(2;1)}{\sigma_2}^1{}_M\nu\\
&+\mathrm{i}_{F(2;1)}{\sigma_2}^1{}_M\mu \wedge \mathrm{d}_{T(1)}\nu + {\sigma_2}^1{}_M\mu \wedge \mathrm{i}_{F(2;1)}\mathrm{d}_{T(1)}\nu\\
&-\mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)}\mu \wedge {\sigma_2}^1{}_M\nu - {\sigma_2}^1{}_M\mathrm{i}_{F(1;1)}\mu \wedge \mathrm{d}_{T(1)}\nu\\
&-\mathrm{d}_{T(1)}\mu \wedge {\sigma_2}^1{}_M\mathrm{i}_{F(1;1)}\nu - {\sigma_2}^1{}_M\mu \wedge \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)}\nu\\
&= \mathrm{i}_{F(2;1)}\mathrm{d}_{T(1)}\mu \wedge {\sigma_2}^1{}_M\nu + {\sigma_2}^1{}_M\mu \wedge \mathrm{i}_{F(2;1)}\mathrm{d}_{T(1)}\nu\\
&-\mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)}\mu \wedge {\sigma_2}^1{}_M\nu - {\sigma_2}^1{}_M\mu \wedge \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)}\nu\\
&= (\mathrm{i}_{F(2;1)}\mathrm{d}_{T(1)} - \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)})\mu \wedge {\sigma_2}^1{}_M\nu\\
&+{\sigma_2}^1{}_M\mu \wedge (\mathrm{i}_{F(2;1)}\mathrm{d}_{T(1)} - \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)})\nu.
\end{aligned} \tag{155}$$

This proves that the operator under consideration is a derivation of the stated type. The equalities

$$(\mathrm{i}_{F(2;1)}\mathrm{d}_{T(1)} - \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)})\mathrm{d}f_{1;0} = \mathrm{i}_{F(2;1)}\mathrm{d}f_{2;1} = \mathrm{d}f_{2;0} = {\sigma_2}^1{}_M\mathrm{i}_{F(1;0)}\mathrm{d}f_{1;0} \tag{156}$$

and

$$\begin{aligned}
(\mathrm{i}_{F(2;1)}\mathrm{d}_{T(1)} - \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)})\mathrm{d}f_{1;1} &= \mathrm{i}_{F(2;1)}\mathrm{d}f_{2;2} - \mathrm{d}_{T(1)}\mathrm{d}f_{1;0}\\
&= 2\mathrm{d}f_{2;1} - \mathrm{d}f_{2;1}\\
&= {\sigma_2}^1{}_M\mathrm{i}_{F(1;0)}\mathrm{d}f_{1;1}
\end{aligned} \tag{157}$$

complete the proof. ∎

2.1.6. The Lagrange differential

We define a linear operator $\boldsymbol{E} : \Omega_1(M) \to \Omega_2(M)$ by the formula

$$\boldsymbol{E} = {\sigma_2}^1{}_M - \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)}. \tag{158}$$

Proposition 8

For each 1-form μ on $\mathrm{T}M$ the 1-form $\boldsymbol{E}\mu$ on T^2M is vertical with respect to the projection $\tau_{2M} : \mathrm{T}^2M \to M$.

Proof : Verticality means that $\langle \boldsymbol{E}\mu, w\rangle = 0$ for each $w \in \ker(\mathrm{T}\tau_{2M})$. Verticality is established by showing that $\mathrm{i}_{F(2;1)}\boldsymbol{E}\mu = 0$ since $\ker(\mathrm{T}\tau_{2M}) = \mathrm{im}(F(2;1))$ and $\langle \mathrm{i}_{F(2;1)}\boldsymbol{E}\mu, v\rangle = \langle \boldsymbol{E}\mu, F(2;1)(v)\rangle$. The equality

$$\begin{aligned}
\mathrm{i}_{F(2;1)}\boldsymbol{E}\mu &= \mathrm{i}_{F(2;1)}({\sigma_2}^1{}_M - \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)})\mu \\
&= ({\sigma_2}^1{}_M\mathrm{i}_{F(1;1)} - \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)}\mathrm{i}_{F(1;1)} - {\sigma_2}^1{}_M\mathrm{i}_{F(1;1)})\mu \\
&= 0
\end{aligned} \tag{159}$$

follows from $\mathrm{i}_{F(1;1)}\mathrm{i}_{F(1;1)}\mu = \mathrm{i}_{F(1;2)}\mu = 0$. We have used formulae (92) and (153) and the commutativity of the diagram (116). ■

The operator $\boldsymbol{P} = \mathrm{i}_{F(1;1)} : \Omega_1 \to \Omega_1$ appears in the decomposition ${\sigma_2}^1{}_M = \boldsymbol{E} + \mathrm{d}_{T(1)}\boldsymbol{P}$ used in the calculus of variations. The decomposition ${\sigma_2}^1{}_M\mu = \boldsymbol{E}\mu + \mathrm{d}_{T(1)}\boldsymbol{P}\mu$ for a 1-form μ on $\mathrm{T}M$ is usually obtained by using local coordinates and integrating by parts. For each 1-form μ on $\mathrm{T}M$ the 1-form $\boldsymbol{P}\mu$ is vertical with respect to the tangent projection $\tau_M : \mathrm{T}M \to M$. This property follows from $\mathrm{i}_{F(1;1)}\boldsymbol{P}\mu = \mathrm{i}_{F(1;1)}\mathrm{i}_{F(1;1)}\mu = \mathrm{i}_{F(1;2)}\mu = 0$.

Let L be a function on $\mathrm{T}M$. Verticality of the form $\boldsymbol{E}\mathrm{d}L$ makes it possible to construct a mapping $\mathcal{E}L : \mathrm{T}^2M \to \mathrm{T}^*M$ such that $\pi_M \circ \mathcal{E}L = \tau_{2M}$. This mapping is characterized by $\langle \boldsymbol{E}\mathrm{d}L, w\rangle = -\langle \mathcal{E}L(\tau_{\mathrm{T}^2M}(w)), \mathrm{T}\tau_{2M}(w)\rangle$ for each $w \in \mathrm{T}\mathrm{T}^2M$. Verticality of the form $\boldsymbol{P}\mathrm{d}L$ implies the existence of a mapping $\mathcal{P}L : \mathrm{T}M \to \mathrm{T}^*M$ such that $\pi_M \circ \mathcal{P}L = \tau_M$. This mapping is characterized by $\langle \boldsymbol{P}\mathrm{d}L, w\rangle = \langle \mathcal{P}L(\tau_{\mathrm{T}M}(w)), \mathrm{T}\tau_M(w)\rangle$ for each $w \in \mathrm{T}\mathrm{T}M$.

The equation $\mathcal{E}L \circ \mathfrak{t}^2\gamma = 0$ is a second order differential equation for a curve $\gamma : I \to M$ known as the *Euler-Lagrange equation* derived from the *Lagrangian* $L : \mathrm{T}M \to \mathbb{R}$. The mapping $\mathcal{P}L$ is called the *Legendre mapping.*

2.1.7. The tangent of a vector fibration and its dual

Let $\varepsilon : E \to M$ be a differential fibration. Local triviality implies the existence of *adapted charts.* An adapted chart $(x^\kappa, e^i) : E \to \mathbb{R}^{m+k}$ associated with a chart $(x^\kappa) : M \to \mathbb{R}^m$ is characterized by the equality $x^\kappa \circ \varepsilon = x^\kappa$. Note that the symbol x^κ is used to denote a coordinate of M and also of E. The chart $(x^\kappa, \dot{x}^\lambda) : \mathrm{T}M \to \mathbb{R}^{2m}$ is an adapted chart for the tangent fibration τ_M. There are two fibrations $\tau_E : \mathrm{T}E \to E$ and $\mathrm{T}\varepsilon : \mathrm{T}E \to \mathrm{T}M$ for the tangent bundle $\mathrm{T}E$. The tangent chart $(x^\kappa, e^i, \dot{x}^\lambda, \dot{e}^j)$ for $\mathrm{T}E$ induced by an adapted chart (x^κ, e^i) is adapted to both fibrations since $(x^\kappa, e^i) \circ \tau_E = (x^\kappa, e^i)$ and $(x^\kappa, \dot{x}^\lambda) \circ \mathrm{T}\varepsilon = (x^\kappa, \dot{x}^\lambda)$.

Let $\varepsilon : E \to M$ and $\varphi : F \to M$ be differential fibrations. The equalizer

$$F \underset{(\varphi,\varepsilon)}{\times} E = \{(f, e) \in F \times E;\ \varphi(f) = \varepsilon(e)\} \tag{160}$$

of the projections φ and ε is called the *fibre product* of F and E. A curve $(\sigma, \rho) : \mathbb{R} \to F \underset{(\varphi,\varepsilon)}{\times} E$ consists of two curves $\rho : \mathbb{R} \to E$ and $\sigma : \mathbb{R} \to F$ such that $\varepsilon \circ \rho = \varphi \circ \sigma$. The mapping

$$\chi : \mathrm{T}(F \underset{(\varphi,\varepsilon)}{\times} E) \to \mathrm{T}F \underset{(\mathrm{T}\varphi,\mathrm{T}\varepsilon)}{\times} \mathrm{T}E : \mathbb{t}(\sigma, \rho)(0) \mapsto (\mathbb{t}\sigma(0), \mathbb{t}\rho(0)) \tag{161}$$

is obviously injective. Let $(x^\kappa, e^i, \dot{x}^\lambda, \dot{e}^j)$ and $(x^\kappa, f^A, \dot{x}^\lambda, \dot{f}^B)$ be tangent charts for $\mathrm{T}E$ and $\mathrm{T}F$ respectively induced by adapted charts (x^κ, e^i) and (x^κ, f^A). If $(w, v) \in \mathrm{T}F \underset{(\mathrm{T}\varphi,\mathrm{T}\varepsilon)}{\times} \mathrm{T}E$, then $(x^\kappa, \dot{x}^\lambda)(w) = (x^\kappa, \dot{x}^\lambda)(v)$ since $\mathrm{T}\varphi(w) = \mathrm{T}\varepsilon(v)$. Curves $\rho : \mathbb{R} \to E$ and $\sigma : \mathbb{R} \to F$ characterized locally by

$$(x^\kappa, e^i) \circ \rho : \mathbb{R} \to \mathbb{R}^{m+k} : s \mapsto (x^\kappa(v) + s\dot{x}^\kappa(v), e^i(v) + s\dot{e}^i(v)) \tag{162}$$

and

$$(x^\kappa, f^A) \circ \rho : \mathbb{R} \to \mathbb{R}^{m+l} : s \mapsto (x^\kappa(w) + s\dot{x}^\kappa(w), f^A(w) + s\dot{f}^A(w)) \tag{163}$$

define a curve $(\sigma, \rho) : \mathbb{R} \to F \underset{(\varphi,\varepsilon)}{\times} E$ such that $\chi(\mathbb{t}(\sigma, \rho)) = (w, v)$. It follows that χ is surjective. We will identify the space $\mathrm{T}(F \underset{(\varphi,\varepsilon)}{\times} E)$ with $\mathrm{T}F \underset{(\mathrm{T}\varphi,\mathrm{T}\varepsilon)}{\times} \mathrm{T}E$. The diagram

$$\begin{tikzcd}
\mathrm{T}F \underset{(\mathrm{T}\varphi,\mathrm{T}\varepsilon)}{\times} \mathrm{T}E \arrow[d, "(\tau_F, \tau_E)"'] \\
F \underset{(\varphi,\varepsilon)}{\times} E
\end{tikzcd} \tag{164}$$

with

$$(\tau_F, \tau_E) : \mathrm{T}F \underset{(\mathrm{T}\varphi,\mathrm{T}\varepsilon)}{\times} \mathrm{T}E \to F \underset{(\varphi,\varepsilon)}{\times} E : (w, v) \mapsto (\tau_F(w), \tau_E(v)) \tag{165}$$

is a vector fibration. If (w_1, v_1) and (w_2, v_2) are elements of $\mathrm{T}F \underset{(\mathrm{T}\varphi,\mathrm{T}\varepsilon)}{\times} \mathrm{T}E$ such that $\tau_F(w_1) = \tau_F(w_2)$ and $\tau_E(v_1) = \tau_E(v_2)$, then $(w_1, v_1) + (w_2, v_2) = (w_1 + w_2, v_1 + v_2)$. If $(w, v) \in \mathrm{T}F \underset{(\mathrm{T}\varphi,\mathrm{T}\varepsilon)}{\times} \mathrm{T}E$ and $k \in \mathbb{R}$, then $k(w, v) = (kw, kv)$. The diagram

$$\begin{tikzcd} \mathrm{T}(F \underset{(\varphi,\varepsilon)}{\times} E) \arrow[r, "\chi"] \arrow[d, "\tau_{F \underset{(\varphi,\varepsilon)}{\times} E}"'] & \mathrm{T}F \underset{(\mathrm{T}\varphi,\mathrm{T}\varepsilon)}{\times} \mathrm{T}E \arrow[d, "(\tau_F, \tau_E)"'] \\ F \underset{(\varphi,\varepsilon)}{\times} E \arrow[r, equal] & F \underset{(\varphi,\varepsilon)}{\times} E \end{tikzcd} \tag{166}$$

is an isomorphism of vector fibrations.

Let

$$\begin{tikzcd} E \arrow[d, "\varepsilon"'] \\ M \end{tikzcd} \tag{167}$$

be a vector fibration with operations

$$+ : E \underset{(\varepsilon,\varepsilon)}{\times} E \to E \tag{168}$$

and

$$\cdot : \mathbb{R} \times E \to E. \tag{169}$$

Let

$$O_\varepsilon : M \to E \tag{170}$$

be the zero section.

The tangent fibration

$$\begin{tikzcd} \mathrm{T}E \arrow[d, "\tau_E"'] \\ E \end{tikzcd} \tag{171}$$

is a vector fibration with operations

$$+ : \mathrm{T}E \underset{(\tau_E,\tau_E)}{\times} \mathrm{T}E \to \mathrm{T}E \tag{172}$$

and

$$\cdot : \mathbb{R} \times \mathrm{T}E \to \mathrm{T}E \tag{173}$$

and the zero section

$$O_{\tau_E} : E \to \mathrm{T}E. \tag{174}$$

The diagram

$$\begin{tikzcd} \mathrm{T}E \arrow[d, "\mathrm{T}\varepsilon"'] \\ \mathrm{T}M \end{tikzcd} \tag{175}$$

is again a vector fibration with operations

$$+^{\mathfrak{t}} : \mathrm{T}E \underset{(\mathrm{T}\varepsilon,\mathrm{T}\varepsilon)}{\times} \mathrm{T}E \to \mathrm{T}E \tag{176}$$

and

$$\cdot^{\mathfrak{t}} : \mathbb{R} \times \mathrm{T}E \to \mathrm{T}E \tag{177}$$

and the zero section

$$O_{\mathrm{T}\varepsilon} : \mathrm{T}M \to \mathrm{T}E. \tag{178}$$

The operation $+^{\mathfrak{t}}$ is obtained from the tangent mapping

$$\mathrm{T}+ : \mathrm{T}(E \underset{(\varepsilon,\varepsilon)}{\times} E) \to \mathrm{T}E \tag{179}$$

by identifying the space $\mathrm{T}(E \underset{(\varepsilon,\varepsilon)}{\times} E)$ with $\mathrm{T}E \underset{(\mathrm{T}\varepsilon,\mathrm{T}\varepsilon)}{\times} \mathrm{T}E$. The operation $\cdot^{\mathfrak{t}}$ is constructed from the tangent mapping

$$\mathrm{T}\cdot : \mathrm{T}(\mathbb{R} \times E) \to \mathrm{T}E. \tag{180}$$

The space $\mathrm{T}(\mathbb{R}\times E)$ is identified with $\mathrm{T}\mathbb{R}\times\mathrm{T}E = \mathbb{R}^2\times\mathrm{T}E$ and the operation $\cdot^{\mathfrak{t}}$ is defined as the mapping

$$\cdot^{\mathfrak{t}} : \mathbb{R} \times \mathrm{T}E \to \mathrm{T}E : (k, v) \mapsto \mathrm{T}\cdot((k,0)\, v). \tag{181}$$

In the diagrams

$$\begin{tikzcd} \mathrm{T}E \arrow[r, "\mathrm{T}\varepsilon"] \arrow[d, "\tau_E"'] & \mathrm{T}M \arrow[d, "\tau_M"] \\ E \arrow[r, "\varepsilon"] & M \end{tikzcd} \qquad \begin{tikzcd} \mathrm{T}E \arrow[r, "\tau_E"] \arrow[d, "\mathrm{T}\varepsilon"'] & E \arrow[d, "\varepsilon"] \\ \mathrm{T}M \arrow[r, "\tau_M"] & M \end{tikzcd} \tag{182}$$

vertical arrows are vector fibrations and horizontal arrows define vector fibration morphisms. The space $\mathrm{T}E$ with its two vector bundle structures forms a *double vector bundle.*

Let

$$\begin{array}{ccc} E & \qquad & F \\ \varepsilon \big\downarrow & & \varphi \big\downarrow \\ M & & M \end{array} \tag{183}$$

represent a vector fibration ε and its dual vector fibration φ. Let

$$\langle\ ,\ \rangle : F \underset{(\varphi,\varepsilon)}{\times} E \to \mathbb{R} \tag{184}$$

be the canonical pairing. We have the double vector bundle structures for $\mathrm{T}E$ and $\mathrm{T}F$. The tangent fibrations

$$\begin{array}{ccc} \mathrm{T}E & \qquad & \mathrm{T}F \\ \mathrm{T}\varepsilon \big\downarrow & & \mathrm{T}\varphi \big\downarrow \\ \mathrm{T}M & & \mathrm{T}M \end{array} \tag{185}$$

are a dual pair of vector fibrations. The *tangent pairing*

$$\langle\ ,\ \rangle^{\mathbb{t}} : \mathrm{T}F \underset{(\mathrm{T}\varphi,\mathrm{T}\varepsilon)}{\times} \mathrm{T}E \to \mathbb{R} \tag{186}$$

is constructed from the tangent mapping

$$\mathrm{T}\langle\ ,\ \rangle : \mathrm{T}(F \underset{(\varphi,\varepsilon)}{\times} E) \to \mathrm{T}\mathbb{R} \tag{187}$$

by identifying the space $\mathrm{T}(F \underset{(\varphi,\varepsilon)}{\times} E)$ with $\mathrm{T}F \underset{(\mathrm{T}\varphi,\mathrm{T}\varepsilon)}{\times} \mathrm{T}E$ and composing the tangent mapping with the second projection of $\mathrm{T}\mathbb{R} = \mathbb{R}^2$ onto $\mathbb{R}$. If (w, v) is an element of $\mathrm{T}F \underset{(\mathrm{T}\varphi,\mathrm{T}\varepsilon)}{\times} \mathrm{T}E$ and (σ, ρ) is a curve in $F \underset{(\varphi,\varepsilon)}{\times} E$ such that $(w, v) = (\mathbb{t}\sigma(0), \mathbb{t}\rho(0))$, then

$$\langle w, v\rangle^{\mathbb{t}} = \langle \mathbb{t}\sigma(0), \mathbb{t}\rho(0)\rangle^{\mathbb{t}} = \mathrm{D}\langle \sigma, \rho\rangle(0). \tag{188}$$

If (σ, ρ) is a curve in $F \underset{(\varphi,\varepsilon)}{\times} E$, then

$$\langle \mathbb{t}\sigma, \mathbb{t}\rho\rangle^{\mathbb{t}} = \mathrm{D}\langle \sigma, \rho\rangle. \tag{189}$$

Linearity of the tangent mapping (187) implies linearity of the tangent pairing. If (w_1, v_1) and (w_2, v_2) are elements of $\mathrm{T}F \underset{(\mathrm{T}\varphi, \mathrm{T}\varepsilon)}{\times} \mathrm{T}E$ such that $\tau_F(w_1) = \tau_F(w_2)$ and $\tau_E(v_1) = \tau_E(v_2)$, then

$$\langle w_1 + w_2, v_1 + v_2 \rangle^{\mathfrak{t}} = \langle w_1, v_1 \rangle^{\mathfrak{t}} + \langle w_2, v_2 \rangle^{\mathfrak{t}}. \tag{190}$$

If $(w, v) \in \mathrm{T}F \underset{(\mathrm{T}\varphi, \mathrm{T}\varepsilon)}{\times} \mathrm{T}E$ and $k \in \mathbb{R}$, then

$$\langle kw, kv \rangle^{\mathfrak{t}} = k \langle w, v \rangle^{\mathfrak{t}}. \tag{191}$$

There are mappings

$$\mu_\varepsilon : E \underset{(\varepsilon, \varepsilon)}{\times} E \to \mathrm{T}E : (e, e') \mapsto \mathfrak{t}\gamma(0) \tag{192}$$

with

$$\gamma : \mathbb{R} \to E : s \mapsto e + se' \tag{193}$$

and

$$\varphi_\varepsilon : E \underset{(\varepsilon, \varepsilon \circ \tau_E)}{\times} \mathrm{T}E \to \mathrm{T}E : (e, \dot{e}) \mapsto \dot{e} - \mu_\varepsilon(\tau_E(\dot{e}), e). \tag{194}$$

The image of μ_ε is the subbundle

$$\mathrm{V}E = \{v \in \mathrm{T}E;\ \mathfrak{t}\varepsilon(v) = 0\} \tag{195}$$

of vertical vectors.

Let e be an element of a vector bundle E and let f and f' be elements of the dual bundle F such that $\varphi(f) = \varphi(f') = \varepsilon(e)$. The vector $\mu_\varphi(f, f') \in \mathrm{T}F$ is the tangent vector of the curve $\sigma : \mathbb{R} \to F : s \mapsto f + sf'$ and $O_{\tau_E}(e) \in \mathrm{T}E$ is the tangent vector of the constant curve $\rho : \mathbb{R} \to E : s \mapsto e$. We have $\mathrm{D}\langle \sigma, \rho \rangle(0) = \langle f', e \rangle$ since $\langle \sigma, \rho \rangle(s) = \langle f, e \rangle + s \langle f', e \rangle$. It follows that

$$\langle \mu_\varphi(f, f'), O_{\tau_E}(e) \rangle = \langle f', e \rangle. \tag{196}$$

2.1.8. The structure of the tangent bundle of the cotangent bundle

As was stated earlier the fibrations

$$\begin{array}{ccc} \mathrm{T}^*M & \qquad\qquad & \mathrm{T}M \\ \Big\downarrow{\scriptstyle \pi_M} & & \Big\downarrow{\scriptstyle \tau_M} \\ M & & M \end{array} \tag{197}$$

for each differential manifold M are a dual pair of vector fibrations with the canonical pairing

$$\langle\ ,\ \rangle : \mathrm{T}^*M \underset{(\pi_M,\tau_M)}{\times} \mathrm{T}M \to \mathbb{R}. \tag{198}$$

The fibrations

$$\begin{array}{ccc} \mathrm{T}^*\mathrm{T}M & \qquad & \mathrm{T}\mathrm{T}M \\ \downarrow{\scriptstyle \pi_{\mathrm{T}M}} & & \downarrow{\scriptstyle \tau_{\mathrm{T}M}} \\ \mathrm{T}M & & \mathrm{T}M \end{array} \tag{199}$$

are again a dual pair with the pairing

$$\langle\ ,\ \rangle : \mathrm{T}^*\mathrm{T}M \underset{(\pi_{\mathrm{T}M},\tau_{\mathrm{T}M})}{\times} \mathrm{T}\mathrm{T}M \to \mathbb{R}. \tag{200}$$

By applying the tangent functor to fibrations (197) we obtain a dual pair of vector fibrations

$$\begin{array}{ccc} \mathrm{T}\mathrm{T}^*M & \qquad & \mathrm{T}\mathrm{T}M \\ \downarrow{\scriptstyle \mathrm{T}\pi_{M}} & & \downarrow{\scriptstyle \mathrm{T}\tau_{M}} \\ \mathrm{T}M & & \mathrm{T}M \end{array} \tag{201}$$

with the pairing

$$\langle\ ,\ \rangle^{\mathbb{t}} : \mathrm{T}\mathrm{T}^*M \underset{(\mathrm{T}\pi_M,\mathrm{T}\tau_M)}{\times} \mathrm{T}\mathrm{T}M \to \mathbb{R}. \tag{202}$$

If $\delta\xi : \mathbb{R} \to \mathrm{T}M$ and $\eta : \mathbb{R} \to \mathrm{T}^*M$ are curves such that $\pi_M \circ \eta = \tau_M \circ \delta\xi$ and if $w = \mathbb{t}\delta\xi(0)$ and $z = \mathbb{t}\eta(0)$, then

$$\langle z, w\rangle^{\mathbb{t}} = \mathrm{D}\langle \eta, \delta\xi\rangle(0). \tag{203}$$

We have two vector bundle structures for the manifold $\mathrm{T}\mathrm{T}M$ and the diagram

$$\begin{array}{ccc} \mathrm{T}\mathrm{T}M & \xleftarrow{\ \kappa_M\ } & \mathrm{T}\mathrm{T}M \\ \downarrow{\scriptstyle \mathrm{T}\tau_M} & & \downarrow{\scriptstyle \tau_{\mathrm{T}M}} \\ \mathrm{T}M & = & \mathrm{T}M \end{array} \tag{204}$$

represents an isomorphism of vector fibrations. This is the diagram (83) with $k' = k = 1$ and $k'' = 0$.

Pairings (200) and (202) permit the introduction of the vector fibration isomorphism

$$\begin{tikzcd} \mathrm{T}\mathrm{T}^*M \arrow[r, "\alpha_M"] \arrow[d, "\mathrm{T}\pi_M"'] & \mathrm{T}^*\mathrm{T}M \arrow[d, "\pi_{\mathrm{T}M}"] \\ \mathrm{T}M \arrow[r, equal] & \mathrm{T}M \end{tikzcd} \tag{205}$$

dual to the vector fibration isomorphism (204). If $w \in \mathrm{TT}M$, $z \in \mathrm{TT}^*M$ and $\tau_{\mathrm{T}M}(w) = \mathrm{T}\pi_M(z)$, then

$$\langle \alpha_M(z), w \rangle = \langle z, \kappa_M(w) \rangle^{\mathbb{t}}. \tag{206}$$

Let d_T and i_T denote the derivations $\mathrm{d}_{T(0)} : \Omega(\mathrm{T}^*M) \to \Omega_1(\mathrm{T}^*M)$ and $\mathrm{i}_{T(0)} : \Omega(\mathrm{T}^*M) \to \Omega_1(\mathrm{T}^*M)$ respectively. The manifold TT^*M with the 2-form $\mathrm{d}_T\omega_M = \mathrm{d}\mathrm{d}_T\vartheta_M$ form a symplectic manifold $(\mathrm{TT}^*M, \mathrm{d}_T\omega_M)$. We construct two natural special symplectic structures for this symplectic manifold.

The following proposition implies that the diagrams

$$\begin{tikzcd} (\mathrm{TT}^*M, \mathrm{d}_T\vartheta_M) \arrow[d, "\mathrm{T}\pi_M"'] \\ \mathrm{T}M \end{tikzcd} \qquad \begin{tikzcd} \mathrm{T}\mathrm{T}^*M \arrow[r, "\alpha_M"] \arrow[d, "\mathrm{T}\pi_M"'] & \mathrm{T}^*\mathrm{T}M \arrow[d, "\pi_{\mathrm{T}M}"] \\ \mathrm{T}M \arrow[r, equal] & \mathrm{T}M \end{tikzcd} \tag{207}$$

constitute a special symplectic structure for the symplectic manifold $(\mathrm{TT}^*M, \mathrm{d}_T\omega_M)$.

Proposition 9

If $z \in \mathrm{TT}^*M$, $w \in \mathrm{TTT}^*M$, and $v \in \mathrm{TT}M$ satisfy relations $\tau_{\mathrm{TT}^*M}(w) = z$ and $\mathrm{TT}\pi_M(w) = v$, then

$$\langle \alpha_M(z), v \rangle = \langle \mathrm{d}_T\vartheta_M, w \rangle. \tag{208}$$

Proof : Let $\chi : \mathbb{R}^2 \to \mathrm{T}^*M$ be a representative of w. The mapping $\varphi = \pi_M \circ \chi$ is a representative of v and $\zeta = \chi(0, \cdot)$ is a representative of z. The vector $\kappa_M(v) = \mathbb{t}^{1,1}\widetilde{\varphi}(0,0)$ is represented by the curve

$$\eta : \mathbb{R} \to \mathrm{T}M : t \mapsto \mathbb{t}\widetilde{\varphi}(t, \cdot)(0) = \mathbb{t}\varphi(\cdot, t)(0). \tag{209}$$

The mapping

$$\delta\zeta : \mathbb{R} \to \mathrm{TT}^*M : t \mapsto \mathbb{t}\chi(\cdot, t)(0) \tag{210}$$

appears in the formula

$$\langle \mathrm{d}_T\vartheta_M, w\rangle = \mathrm{D}\langle \vartheta_M, \delta\zeta\rangle(0) \tag{211}$$

derived in Proposition 5. Relations $\eta = \mathrm{T}\pi_M \circ \delta\zeta$ and $\zeta = \tau_{\mathrm{T}^*M} \circ \delta\zeta$ follow from

$$(\mathrm{T}\pi_M \circ \delta\zeta)(t) = \mathrm{T}\pi_M(\mathbb{t}\chi(\cdot, t)(0)) = \mathbb{t}(\pi_M \circ \chi)(\cdot, t)(0) = \mathbb{t}\varphi(\cdot, t)(0) \tag{212}$$

and

$$(\tau_{\mathrm{T}^*M} \circ \delta\zeta)(t) = \tau_{\mathrm{T}^*M}(\mathbb{t}\chi(\cdot, t)(0)) = \chi(0, t). \tag{213}$$

The calculation

$$\begin{aligned} \langle \alpha_M(z), v\rangle &= \langle z, \kappa_M(v)\rangle^{\mathbb{t}} \\ &= \langle \mathbb{t}\zeta(0), \mathbb{t}\eta(0)\rangle^{\mathbb{t}} \\ &= \mathrm{D}\langle \zeta, \eta\rangle(0) \\ &= \mathrm{D}\langle \vartheta_M, \delta\zeta\rangle(0) \\ &= \langle \mathrm{d}_T\vartheta_M, w\rangle \end{aligned} \tag{214}$$

completes the proof. ■

The mapping $\beta_{(\mathrm{T}^*M,\omega_M)} : \mathrm{TT}^*M \to \mathrm{T}^*\mathrm{T}^*M$ characterized by

$$\langle \beta_{(\mathrm{T}^*M,\omega_M)}(u), v\rangle = \langle \omega_M, u \wedge v\rangle \tag{215}$$

with $u \in \mathrm{TT}^*M$ and $v \in \mathrm{TT}^*M$ such that $\tau_{\mathrm{T}^*M}(u) = \tau_{\mathrm{T}^*M}(v)$ establishes an isomorphism

$$\begin{tikzcd} \mathrm{TT}^*M \arrow[r, "\beta_{(\mathrm{T}^*M,\omega_M)}"] \arrow[d, "\tau_{\mathrm{T}^*M}"'] & \mathrm{T}^*\mathrm{T}^*M \arrow[d, "\pi_{\mathrm{T}^*M}"'] \\ \mathrm{T}^*M \arrow[r, equal] & \mathrm{T}^*M \end{tikzcd} \tag{216}$$

of vector fibrations.

Proposition 10

A special symplectic structure for the symplectic manifold $(\mathrm{TT}^*M, \mathrm{d}_T\omega_M)$ is introduced by the diagrams

$$\begin{tikzcd}
(\mathrm{TT}^*M, \mathrm{i}_T\omega_M) \arrow[d, "\tau_{\mathrm{T}^*M}"'] \\
\mathrm{T}^*M
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{TT}^*M \arrow[r, "\beta_{(\mathrm{T}^*M,\omega_M)}"] \arrow[d, "\tau_{\mathrm{T}^*M}"'] & \mathrm{T}^*\mathrm{T}^*M \arrow[d, "\pi_{\mathrm{T}^*M}"'] \\
\mathrm{T}^*M \arrow[r, equal] & \mathrm{T}^*M
\end{tikzcd}
\tag{217}$$

Proof : We have $\mathrm{d}_T\omega_M = \mathrm{d}\mathrm{i}_T\omega_M$ since $\mathrm{d}\omega_M = 0$. Further

$$\begin{aligned}
\langle {\beta_{(\mathrm{T}^*M,\omega_M)}}^*\vartheta_{\mathrm{T}^*M}, w\rangle &= \langle \vartheta_{\mathrm{T}^*M}, \mathrm{T}\beta_{(\mathrm{T}^*M,\omega_M)}(w)\rangle \\
&= \langle \tau_{\mathrm{T}^*\mathrm{T}^*M}(\mathrm{T}\beta_{(\mathrm{T}^*M,\omega_M)}(w)), \mathrm{T}\pi_{\mathrm{T}^*M}(\mathrm{T}\beta_{(\mathrm{T}^*M,\omega_M)}(w))\rangle \\
&= \langle \beta_{(\mathrm{T}^*M,\omega_M)}(\tau_{\mathrm{TT}^*M}(w)), \mathrm{T}(\pi_{\mathrm{T}^*M}\circ\beta_{(\mathrm{T}^*M,\omega_M)})(w)\rangle \\
&= \langle \beta_{(\mathrm{T}^*M,\omega_M)}(\tau_{\mathrm{TT}^*M}(w)), \mathrm{T}\tau_{\mathrm{T}^*M}(w)\rangle \\
&= \langle \omega_M, \tau_{\mathrm{TT}^*M}(w)\wedge \mathrm{T}\tau_{\mathrm{T}^*M}(w)\rangle \\
&= \langle \mathrm{i}_T\omega_M, w\rangle
\end{aligned}\tag{218}$$

for each $w\in \mathrm{TTT}^*M$. Hence, $\beta^*_{(\mathrm{T}^*M,\omega_M)}\vartheta_{\mathrm{T}^*M} = \mathrm{i}_T\omega_M$. ■

The symplectic form $\mathrm{d}_T\omega_M$ can be obtained as the pull back $\alpha^*_M\omega_{\mathrm{T}M}$ of the symplectic form $\omega_{\mathrm{T}M}$ on $\mathrm{T}^*\mathrm{T}M$. The pull back $\beta^*_{(\mathrm{T}^*M,\omega_M)}\omega_{\mathrm{T}^*M}$ is again the symplectic form $\mathrm{d}_T\omega_M$.

Comparing the Hamiltonian special symplectic structure with the Lagrangian structure we observe that $\mathrm{i}_T\omega_M - \mathrm{d}_T\vartheta_M = -\mathrm{d}\mathrm{i}_T\vartheta_M$. The function $G_M = -\mathrm{i}_T\vartheta_M$ on TT^*M plays the role of the function G introduced in Section 2.4.

An alternative analysis of the structure of TT^*M is offered in a recent *Springer-Verlag* text[1] in Section 6.8 on page 161. We are not in total agreement with this analysis. In particular, we failed to identify the second symplectic structure whose existence is claimed in this publication. We suspect that the claim may be false.

3. Dynamics of autonomous systems

3.1. Motions and variations

Let M be the *configuration manifold* of an autonomous mechanical system. A *configuration* is a point $x\in M$ and a *motion* of the system is a differentiable curve $\xi : I\to M$ defined on an open interval $I\subset\mathbb{R}$. The first and second prolongations of a motion ξ denoted by $\dot\xi$ and $\ddot\xi$ represent the *velocity* and the *acceleration* along the motion.

[1] J. Marsden and T. Ratiu, *Introduction to Mechanics and Symmetry*, Springer (1999).

An *infinitesimal variation* of a motion $\xi : I \to M$ is a differentiable mapping $\delta\xi : I \to \mathrm{T}M$ such that $\tau_M \circ \delta\xi = \xi$. Mappings $\delta\dot{\xi} = \kappa^{1,1}{}_M \circ \mathbb{t}\delta\xi$ and $\delta\ddot{\xi} = \kappa^{2,1}{}_M \circ \mathbb{t}^2\delta\xi$ are the infinitesimal variations of the velocity $\dot{\xi}$ and the acceleration $\ddot{\xi}$. Relations $\tau_M \circ \delta\xi = \xi$, $\tau_{\mathrm{T}M} \circ \delta\dot{\xi} = \dot{\xi}$, and $\tau_{\mathrm{T}^2M} \circ \delta\ddot{\xi} = \ddot{\xi}$ are satisfied. The derivations of additional relations

$$\begin{aligned} \mathrm{T}\tau_M \circ \delta\dot{\xi} &= \mathrm{T}\tau_M \circ \kappa_M \circ \mathbb{t}\delta\xi \\ &= \tau_{\mathrm{T}M} \circ \mathbb{t}\delta\xi \\ &= \delta\xi \end{aligned} \tag{219}$$

and

$$\begin{aligned} \mathrm{T}\tau^1{}_{2M} \circ \delta\ddot{\xi} &= \mathrm{T}\tau^1{}_{2M} \circ \kappa^{2,1}{}_M \circ \mathbb{t}^2\delta\xi \\ &= \kappa_M \circ \tau^1{}_{2\mathrm{T}M} \circ \mathbb{t}^2\delta\xi \\ &= \kappa_M \circ \mathbb{t}\delta\xi \\ &= \delta\dot{\xi} \end{aligned} \tag{220}$$

are based on the commutativity of the diagram (212) with $k' = k = 1$ and $k'' = 0$ and the same diagram with $k = 1$, $k' = 2$, and $k'' = 1$.

3.1.1. Force-momentum trajectories

The fibre product

$$\mathrm{T}^*M \underset{(\pi_M,\pi_M)}{\times} \mathrm{T}^*M \tag{221}$$

is the *force-momentum phase space* Ph of the system. A pair $(f,p) \in Ph$ consists of the *external force* f and the *momentum* p at $x = \pi_M(f) = \pi_M(p)$. A *force-momentum trajectory* of an autonomous system is a curve

$$(\zeta,\eta) : I \to Ph. \tag{222}$$

The two curves $\zeta : I \to \mathrm{T}^*M$ and $\eta : I \to \mathrm{T}^*M$ represent the external force and the momentum along the motion $\xi = \pi_M \circ \zeta = \pi_M \circ \eta$.

3.1.2. The variational principle of dynamics

Let $L : \mathrm{T}M \to \mathbb{R}$ be the Lagrangian of the mechanical system. The *action* is a function A which associates the integral

$$A(\xi,[a,b]) = \int_a^b L \circ \dot{\xi} \tag{223}$$

with a motion $\xi : I \to M$ and an interval $[a,b] \subset I$. The *variation* of the action is the integral

$$\delta A(\xi,[a,b]) = \int_a^b \langle \mathrm{d}L, \delta\dot{\xi}\rangle \tag{224}$$

associated with an infinitesimal variation $\delta\xi : I \to \mathrm{T}M$.

The *dynamics* of the system is a set $\mathcal{D}$ of force-momentum trajectories satisfying the following variational principle. A trajectory $(\zeta,\eta) : I \to Ph$ is in $\mathcal{D}$ if

$$\delta A(\xi,[a,b]) = -\int_a^b \langle \zeta, \delta\xi\rangle + \langle \eta(b), \delta\xi(b)\rangle - \langle \eta(a), \delta\xi(a)\rangle \tag{225}$$

for each $[a,b] \subset I$ and each variation $\delta\xi$ such that $\tau_M \circ \delta\xi = \pi_M \circ \zeta = \pi_M \circ \eta$.

The variation of the action can be converted to an equivalent expression:

$$\begin{aligned}
\delta A(\xi,[a,b]) &= \int_a^b \langle \mathrm{d}L, \delta\dot{\xi}\rangle \\
&= \int_a^b \langle \mathrm{d}L, \mathrm{T}\tau^1{}_{2M} \circ \delta\ddot{\xi}\rangle \\
&= \int_a^b \langle \sigma_2{}^1{}_M \mathrm{d}L, \delta\ddot{\xi}\rangle \\
&= \int_a^b \langle \sigma_2{}^1{}_M \mathrm{d}L - \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)}\mathrm{d}L, \delta\ddot{\xi}\rangle + \int_a^b \langle \mathrm{d}_{T(1)}\mathrm{i}_{F(1;1)}\mathrm{d}L, \delta\ddot{\xi}\rangle \\
&= -\int_a^b \langle \mathcal{E}L \circ \ddot{\xi}, \delta\xi\rangle + \int_a^b \mathrm{D}\langle \mathcal{P}L \circ \dot{\xi}, \delta\xi\rangle \\
&= -\int_a^b \langle \mathcal{E}L \circ \ddot{\xi}, \delta\xi\rangle + \langle (\mathcal{P}L \circ \dot{\xi})(b), \delta\xi(b)\rangle - \langle (\mathcal{P}L \circ \dot{\xi})(a), \delta\xi(a)\rangle.
\end{aligned} \tag{226}$$

By using variations $\delta\xi$ with $\delta\xi(a) = 0$ and $\delta\xi(b) = 0$ we first derive from the variational principle the Euler-Lagrange equations

$$\mathcal{E}L \circ \ddot{\xi} = \zeta \tag{227}$$

in $[a, b]$. Equations

$$(\mathcal{P}L \circ \dot{\xi})(a) = \eta(a) \tag{228}$$

and

$$(\mathcal{P}L \circ \dot{\xi})(b) = \eta(b) \tag{229}$$

follow. These equations are satisfied for each interval $[a, b] \subset I$. It follows that a force-momentum trajectory (ζ, η) is in $\mathcal{D}$ if and only if equations

$$\mathcal{E}L \circ \ddot{\xi} = \zeta \tag{230}$$

and

$$\mathcal{P}L \circ \dot{\xi} = \eta \tag{231}$$

are satisfied in I.

Sets

$$E = \left\{ (f, a) \in \mathrm{T}^*M \underset{(\pi_M, \tau_{2\,M})}{\times} \mathrm{T}^2M;\ f = \mathcal{E}L(a) \right\} \tag{232}$$

and

$$P = \left\{ (p, v) \in \mathrm{T}^*M \underset{(\pi_M, \tau_M)}{\times} \mathrm{T}M;\ p = \mathcal{P}L(v) \right\} \tag{233}$$

are graphs of the Euler-Lagrange and the Legendre maps respectively. The dynamics can be stated in terms of these sets treated as differential equations. Equation (230) means that the mapping (ζ, ξ) is a solution curve of the differential equation E and equation (231) means that the mapping (η, ξ) is a solution curve of the differential equation P. The relation $\xi = \pi_M \circ \zeta = \pi_M \circ \eta$ is always imposed. The Euler-Lagrange equation alone does not provide a complete characterization of dynamics. The equation (231) could be called the *velocity-momentum* relation. It is an essential part of dynamics. The set

$$E_0 = \left\{ a \in \mathrm{T}^2M;\ 0 = \mathcal{E}L(a) \right\} \tag{234}$$

is a version of the Euler-Lagrange equations without external forces. Solution curves are motions of the system with zero external forces.

3.1.3. Lagrangian formulation of dynamics

The version

$$\int_a^b \langle \mathrm{d}L, \delta\dot{\xi} \rangle = -\int_a^b \langle \zeta, \delta\xi \rangle + \int_a^b \mathrm{D}\langle \eta, \delta\xi \rangle \tag{235}$$

of the variational principle is suitable for deriving the infinitesimal limits. Infinitesimal limits are obtained by dividing both sides of the equality by $b-a$ and passing to the limit of $b=a=t\in I$. The resulting equality

$$\langle \mathrm{d}L, \delta\dot{\xi}\rangle = -\langle \zeta, \delta\xi\rangle + \mathrm{D}\langle \eta, \delta\xi\rangle \tag{236}$$

satisfied by the force-momentum trajectory $(\zeta,\eta) : I \to Ph$ for each variation $\delta\xi : I \to \mathrm{TT}M$ such that $\tau_M \circ \delta\xi = \pi_M \circ \zeta$ is a characterization of the dynamics $\mathcal{D}$ equivalent to the original variational principle. The equality

$$\langle \zeta, \delta\xi\rangle = \langle \mu_{\pi_M} \circ (\eta,\zeta), O_{\tau_{\mathrm{T}M}} \circ \delta\xi\rangle^{\mathbb{t}} \tag{237}$$

is derived from formula (196) and the equality

$$\mathrm{D}\langle \eta, \delta\xi\rangle = \langle \mathbb{t}\eta, \mathbb{t}\delta\xi\rangle^{\mathbb{t}} = \langle \dot{\eta}, \kappa_M \circ \delta\dot{\xi}\rangle^{\mathbb{t}} \tag{238}$$

is a version of formula (189). By combining the two equalities we obtain the formula

$$-\langle \zeta, \delta\xi\rangle + \mathrm{D}\langle \eta, \delta\xi\rangle = -\langle \mu_{\pi_M} \circ (\eta,\zeta), O_{\tau_{\mathrm{T}M}} \circ \delta\xi\rangle^{\mathbb{t}} + \langle \dot{\eta}, \kappa_M \circ \delta\dot{\xi}\rangle^{\mathbb{t}}. \tag{239}$$

Relations

$$\tau_{\mathrm{T}^*M} \circ \mu_{\pi_M} \circ (\eta,\zeta) = \tau_{\mathrm{T}^*M} \circ \dot{\eta} = \eta \tag{240}$$

and

$$\tau_{\mathrm{T}M} \circ O_{\tau_{\mathrm{T}M}} \circ \delta\xi = \tau_{\mathrm{T}M} \circ \kappa_M \circ \delta\dot{\xi} \tag{241}$$

permit the use of formula (190). The result is

$$\begin{aligned} -\langle \zeta, \delta\xi\rangle + \mathrm{D}\langle \eta, \delta\xi\rangle &= \langle \dot{\eta} - \mu_{\pi_M} \circ (\eta,\zeta), \kappa_M \circ \delta\dot{\xi}\rangle^{\mathbb{t}} \\ &= \langle \varphi_M \circ (\zeta,\dot{\eta}), \kappa_M \circ \delta\dot{\xi}\rangle^{\mathbb{t}} \\ &= \langle \alpha_M \circ \varphi_M \circ (\zeta,\dot{\eta}), \delta\dot{\xi}\rangle. \end{aligned} \tag{242}$$

We have obtained a characterization of the dynamics $\mathcal{D}$ in terms of a first order differential equation

$$\varphi_M \circ (\zeta,\dot{\eta}) = {\alpha_M}^{-1} \circ \mathrm{d}L \circ \dot{\xi} \tag{243}$$

with $\xi = \pi_M \circ \zeta = \pi_M \circ \eta$. The codomain of $(\zeta,\dot{\eta})$ is the fibre product $\mathrm{T}^*M \underset{(\pi_M, \pi_M\circ\tau_{\mathrm{T}^*M})}{\times} \mathrm{TT}^*M$.

Starting with the identity

$$\int_a^b \langle \mathrm{d}L, \delta\dot{\xi}\rangle = -\int_a^b \langle \mathcal{E}L \circ \ddot{\xi}, \delta\xi\rangle + \int_a^b \mathrm{D}\langle \mathcal{P}L \circ \dot{\xi}, \delta\xi\rangle \tag{244}$$

instead of equation (236) and performing the operations leading from (236) to (243) with η and ζ replaced by $\mathcal{P}L \circ \dot{\xi}$ and $\mathcal{E}L \circ \ddot{\xi}$ respectively we obtain the identity

$$\varphi_M \circ (\mathcal{E}L \circ \ddot{\xi}, \mathbb{t}(\mathcal{P}L \circ \dot{\xi})) = {\alpha_M}^{-1} \circ \mathrm{d}L \circ \dot{\xi}. \tag{245}$$

A useful characterization

$$\mathcal{P}L = \tau_{\mathrm{T}^*M} \circ {\alpha_M}^{-1} \circ \mathrm{d}L \tag{246}$$

of the Legendre mapping follows from this identity.

Equation (243) means that the curve (ζ, η) satisfies the differential equation

$$D = \left\{ (f, w) \in \mathrm{T}^*M \underset{(\pi_M, \pi_M \circ \tau_{\mathrm{T}^*M})}{\times} \mathrm{TT}^*M;\ \varphi_M(f, w) \in D_0 \right\}, \tag{247}$$

where

$$D_0 = \{ w \in \mathrm{TT}^*M;\ \alpha_M(w) = \mathrm{d}L(\tau_{\mathrm{T}^*M}(w)) \}. \tag{248}$$

The set D_0 is a version of the Lagrange equation without external forces. The image of the differential $\mathrm{d}L : \mathrm{T}M \to \mathrm{T}^*\mathrm{T}M$ is a Lagrangian submanifold of $(\mathrm{T}^*\mathrm{T}M, \omega_{\mathrm{T}M})$. Consequently the set $D_0 = {\alpha_M}^{-1}(\mathrm{im}(\mathrm{d}L)) = \mathrm{im}({\alpha_M}^{-1} \circ \mathrm{d}L)$ is a Lagrangian submanifold of $(\mathrm{TT}^*M, \mathrm{d}_T\omega_M)$ and the Lagrangian is its generating function relative to the Lagrangian special symplectic structure

$$\begin{array}{ccc} (\mathrm{TT}^*M, \mathrm{d}_T\vartheta_M) & \qquad & \begin{array}{ccc} \mathrm{TT}^*M & \xrightarrow{\alpha_M} & \mathrm{T}^*\mathrm{T}M \\ {\scriptstyle \mathrm{T}\pi_M}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathrm{T}M}} \\ \mathrm{T}M & = & \mathrm{T}M \end{array} \\ {\scriptstyle \mathrm{T}\pi_M}\big\downarrow & & \\ \mathrm{T}M & & \end{array} \tag{249}$$

for the symplectic manifold $(\mathrm{TT}^*M, \mathrm{d}_T\omega_M)$.

3.1.4. Hamiltonian formulation of dynamics

A Lagrangian is said to be *hyperregular* if the Legendre mapping $\mathcal{P}L = \tau_{\mathrm{T}^*M} \circ {\alpha_M}^{-1} \circ \mathrm{d}L$ is a diffeomorphism. We will denote by Λ the inverse of the Legendre mapping for a hyperregular Lagrangian.

For a hyperregular Lagrangian the set D_0 is the image $\mathrm{im}(Z)$ of the mapping $Z = {\alpha_M}^{-1} \circ \mathrm{d}L \circ \Lambda$. This mapping is a vector field on T^*M since $\tau_{\mathrm{T}^*M} \circ Z = \tau_{\mathrm{T}^*M} \circ {\alpha_M}^{-1} \circ \mathrm{d}L \circ \Lambda = 1_{\mathrm{T}^*M}$. Let a function $H : \mathrm{T}^*M \to \mathbb{R}$ be defined by $H(p) = \langle p, \Lambda(p) \rangle - L(\Lambda(p))$. We show that the function $-H$

is a generating function of D_0 relative to the Hamiltonian special symplectic structure

$$
(\mathrm{TT}^*M, \mathrm{i}_T\omega_M) \xrightarrow{\tau_{\mathrm{T}^*M}} \mathrm{T}^*M
\qquad
\begin{array}{ccc}
\mathrm{TT}^*M & \xrightarrow{\beta_{(\mathrm{T}^*M,\omega_M)}} & \mathrm{T}^*\mathrm{T}^*M \\
\Big\downarrow{\scriptstyle \tau_{\mathrm{T}^*M}} & & \Big\downarrow{\scriptstyle \pi_{\mathrm{T}^*M}} \\
\mathrm{T}^*M & = & \mathrm{T}^*M
\end{array}
\tag{250}
$$

for $(\mathrm{TT}^*M, \mathrm{d}_T\omega_M)$. The generating function of D_0 relative to the Hamiltonian special symplectic structure is the function

$$(L \circ \mathrm{T}\pi_M + G_M) \circ Z, \tag{251}$$

where $G_M = -\mathrm{i}_T\vartheta_M$. From

$$\mathrm{T}\pi_M \circ Z = \mathrm{T}\pi_M \circ {\alpha_M}^{-1} \circ \mathrm{d}L \circ \Lambda = \pi_{\mathrm{T}M} \circ \mathrm{d}L \circ \Lambda = \Lambda \tag{252}$$

and

$$
\begin{aligned}
-G_M(Z(p)) &= \mathrm{i}_T\vartheta_M(({\alpha_M}^{-1} \circ \mathrm{d}L \circ \Lambda)(p)) \\
&= \langle \vartheta_M, ({\alpha_M}^{-1} \circ \mathrm{d}L \circ \Lambda)(p) \rangle \\
&= \langle (\tau_{\mathrm{T}^*M} \circ {\alpha_M}^{-1} \circ \mathrm{d}L \circ \Lambda)(p), (\mathrm{T}\pi_M \circ {\alpha_M}^{-1} \circ \mathrm{d}L \circ \Lambda)(p) \rangle \\
&= \langle p, (\pi_{\mathrm{T}M} \circ \mathrm{d}L \circ \Lambda)(p) \rangle \\
&= \langle p, \Lambda(p) \rangle
\end{aligned}
\tag{253}
$$

it follows that

$$(L \circ \mathrm{T}\pi_M + G_M) \circ Z = -H. \tag{254}$$

The field Z is a Hamiltonian vector field and the function H is a Hamiltonian for this field since

$$\mathrm{i}_Z\omega_M = Z^*\mathrm{i}_T\omega_M = -\mathrm{d}H. \tag{255}$$

The formula

$$Z = -\beta^{-1}_{(\mathrm{T}^*M,\omega_M)} \circ \mathrm{d}H \tag{256}$$

is an equivalent expression of the field Z in terms of the Hamiltonian. A force-momentum trajectory (ζ, η) is in $\mathcal{D}$ if and only if

$$\dot{\eta}(t) = Z(\eta(t)) + \mu_{\pi_M}(\zeta(t), \eta(t)) \tag{257}$$

for each $t \in I$.

3.1.5. Poisson formulation of dynamics

We define the *Poisson tensor* $W_M : \mathrm{T}^*\mathrm{T}^*M \to \mathrm{T}\mathrm{T}^*M$ by $W_M = -\beta^{-1}_{(\mathrm{T}^*M,\omega_M)}$. The *Poisson bracket* of two functions F and G on T^*M is the function $\{F,G\} = \langle \mathrm{d}F, W_M \circ \mathrm{d}G\rangle$. The vector field Z is expressed as $Z = W_M \circ \mathrm{d}H$ and the Lie derivative $\mathrm{d}_Z F = \langle \mathrm{d}F, Z\rangle$ of a function F on T^*M is expressed as the Poisson bracket $\{F, H\}$. A force-momentum trajectory $(\zeta,\eta) : I \to Ph$ is in $\mathcal{D}$ if and only if

$$\mathrm{D}(F \circ \eta)(t) = \{F, H\}(\eta(t)) + \langle \mathrm{d}F, \mu_{\pi_M}(\zeta(t), \eta(t))\rangle \tag{258}$$

for each function F on T^*M and each $t \in I$.

4. Dynamics in the presence of non potential forces

If non potential internal forces are present, then the dynamics is no longer represented by a Lagrangian. Formulations similar to those for the potential case are still possible if the differential $\mathrm{d}L$ of the Lagrangian is replaced by a 1-form λ on $\mathrm{T}M$. This form is typically the difference $\mathrm{d}L - \rho$ of a potential part $\mathrm{d}L$ and a 1-form ρ on $\mathrm{T}M$ vertical with respect to the tangent projection τ_M representing velocity dependent forces. With the help of operators $\boldsymbol{E}$ and $\boldsymbol{P}$ we construct mappings $\mathcal{E}\lambda : \mathrm{T}^2M \to \mathrm{T}^*M$ and $\mathcal{P}\lambda : \mathrm{T}M \to \mathrm{T}^*M$ characterized by $\langle \boldsymbol{E}\lambda, w\rangle = -\langle \mathcal{E}\lambda(\tau_{\mathrm{T}^2M}(w)), \mathrm{T}\tau_{2M}(w)\rangle$ for each $w \in \mathrm{T}\mathrm{T}^2M$ and $\langle \boldsymbol{P}\lambda, w\rangle = \langle \mathcal{P}\lambda(\tau_{\mathrm{T}M}(w)), \mathrm{T}\tau_M(w)\rangle$ for each $w \in \mathrm{T}\mathrm{T}M$. These constructions are possible due to verticality properties of forms $\boldsymbol{E}\lambda$ and $\boldsymbol{P}\lambda$ established in Section 2.7.

4.0.6. The variational principle

The dynamics can be derived from a variational principle even if the action and its variation are no longer defined. The *dynamics* of the system is a set $\mathcal{D}$ of force-momentum trajectories satisfying this variational principle. A trajectory $(\zeta,\eta) : I \to Ph$ is in $\mathcal{D}$ if

$$\int_a^b \langle \lambda, \delta\dot{\xi}\rangle = -\int_a^b \langle \zeta, \delta\xi\rangle + \langle \eta(b), \delta\xi(b)\rangle - \langle \eta(a), \delta\xi(a)\rangle \tag{259}$$

for each $[a,b] \subset I$ and each variation $\delta\xi$ such that $\tau_M \circ \delta\xi = \pi_M \circ \zeta = \pi_M \circ \eta$.

From the equivalent form

$$\int_a^b \langle \lambda, \delta\dot{\xi}\rangle = -\int_a^b \langle \mathcal{E}\lambda \circ \ddot{\xi}, \delta\xi\rangle + \langle (\mathcal{P}\lambda \circ \dot{\xi})(b), \delta\xi(b)\rangle - \langle (\mathcal{P}\lambda \circ \dot{\xi})(a), \delta\xi(a)\rangle \tag{260}$$

of the variational principle equations

$$\mathcal{E}\lambda \circ \ddot{\xi} = \zeta, \tag{261}$$

$$(\mathcal{P}\lambda \circ \dot{\xi})(a) = \eta(a), \tag{262}$$

and

$$(\mathcal{P}\lambda \circ \dot{\xi})(b) = \eta(b) \tag{263}$$

are derived. These equations are satisfied for each interval $[a, b] \subset I$. It follows that a force-momentum trajectory (ζ, η) is in $\mathcal{D}$ if and only if equations

$$\mathcal{E}\lambda \circ \ddot{\xi} = \zeta \tag{264}$$

and

$$\mathcal{P}\lambda \circ \dot{\xi} = \eta \tag{265}$$

are satisfied in I.

4.0.7. Lagrangian formulation of dynamics

The Lagrangian formulation is the infinitesimal limit derived from the version

$$\int_a^b \langle \lambda, \delta\dot{\xi} \rangle = -\int_a^b \langle \zeta, \delta\xi \rangle + \int_a^b \mathrm{D}\langle \eta, \delta\xi \rangle \tag{266}$$

of the variational principle. The first order differential equation

$$\varphi_M \circ (\zeta, \dot{\eta}) = {\alpha_M}^{-1} \circ \lambda \circ \dot{\xi} \tag{267}$$

follow from this principle.

The identity (245) is replaced by

$$\varphi_M \circ (\mathcal{E}L \circ \ddot{\xi}, \mathbb{t}(\mathcal{P}L \circ \dot{\xi})) = {\alpha_M}^{-1} \circ \mathrm{d}L \circ \dot{\xi}. \tag{268}$$

The formula

$$\mathcal{P}\lambda = \tau_{\mathrm{T}^*M} \circ {\alpha_M}^{-1} \circ \lambda \tag{269}$$

follows.

The differential equation

$$D = \left\{ (f, w) \in \mathrm{T}^*M \underset{(\pi_M, \pi_M \circ \tau_{\mathrm{T}^*M})}{\times} \mathrm{TT}^*M;\ \varphi_M(f, w) \in D_0 \right\}, \tag{270}$$

with

$$D_0 = \{ w \in \mathrm{TT}^*M;\ \alpha_M(w) = \lambda(\tau_{\mathrm{T}^*M}(w)) \} \tag{271}$$

can be introduced. The set D_0 is submanifold of $(\mathrm{TT}^*M, \mathrm{d}_T\omega_M)$ but not a Lagrangian submanifold unless λ is closed. The form λ can be considered a *generating form* of D_0 relative to the Lagrangian special symplectic structure since $D_0 = {\alpha_M}^{-1}(\mathrm{im}(\lambda))$.

4.0.8. Hamiltonian formulation of dynamics

As in the potential case we say that the Lagrangian form λ is *hyperregular* if the mapping $\mathcal{P}\lambda$ is a diffeomorphism. In the hyperregular case we denote by $\varLambda$ the inverse of the mapping $\mathcal{P}\lambda$. The set D_0 is the image of the vector field $Z = {\alpha_M}^{-1} \circ \lambda \circ \varLambda$. This field is not necessarily a Hamiltonian vector field. Let a 1-form χ on T^*M be defined by

$$\langle \chi, z \rangle = \langle z, \mathrm{T}\varLambda(z) \rangle^{\mathfrak{t}} - \langle \lambda, \mathrm{T}\varLambda(z) \rangle. \tag{272}$$

We will show that $-\chi$ is the generating form of D_0 relative to the Hamiltonian special symplectic structure.

According to formula (61) adapted to the present case the generating form of D_0 relative to the Hamiltonian special symplectic structure is the form

$$Z^*((\mathrm{T}\pi_M)^*\lambda + \mathrm{d}G_M). \tag{273}$$

The formula $Z^*G_M(p) = -\langle p, \varLambda(p) \rangle$ derived for the potential case is still valid in the non potential case with $\mathrm{d}L$ replaced by λ. The equality

$$\langle Z^*\mathrm{d}G_M, z \rangle = \langle \mathrm{d}Z^*G_M, z \rangle = -\langle v, \mathrm{T}\varLambda(z) \rangle^{\mathfrak{t}} \tag{274}$$

follows from this formula. We have

$$\langle Z^*((\mathrm{T}\pi_M)^*\lambda + \mathrm{d}G_M), z \rangle = \langle \lambda, \mathrm{T}\varLambda(z) \rangle - \langle z, \mathrm{T}\varLambda(z) \rangle^{\mathfrak{t}} = -\langle \chi, z \rangle \tag{275}$$

since

$$\mathrm{T}\pi_M \circ Z = \mathrm{T}\pi_M \circ {\alpha_M}^{-1} \circ \lambda \circ \varLambda = \pi_{\mathrm{T}M} \circ \lambda \circ \varLambda = \varLambda. \tag{276}$$

Hence, the form $-\chi$ defined in (272) is the generating form of D_0 relative to the Hamiltonian special symplectic structure. The formula

$$Z = -\beta^{-1}_{(\mathrm{T}^*M, \omega_M)} \circ \chi \tag{277}$$

follows.

In the special case of $\lambda = \mathrm{d}L - \rho$ we have $\mathcal{P}\lambda = \mathcal{P}L$ since $\mathrm{i}_{F(1;1)}\rho = 0$ due to verticality of ρ. Let $H : \mathrm{T}^*M \to \mathbb{R}$ be the Hamiltonian corresponding to L. This Hamiltonian is defined by $H(p) = \langle p, \varLambda(p) \rangle - L(\varLambda(p))$ and satisfies the relation

$$\langle \mathrm{d}H, z \rangle = \langle z, \mathrm{T}\varLambda(z) \rangle^{\mathfrak{t}} - \langle \mathrm{d}L, \mathrm{T}\varLambda(z) \rangle. \tag{278}$$

By comparing this relation with

$$\langle \chi, z \rangle = \langle z, \mathrm{T}\varLambda(z) \rangle^{\mathfrak{t}} - \langle \mathrm{d}L, \mathrm{T}\varLambda(z) \rangle + \langle \rho, \mathrm{T}\varLambda(z) \rangle \tag{279}$$

we derive the formula

$$\chi = \mathrm{d}H + \varLambda^*\rho. \tag{280}$$

As in the potential case a force-momentum trajectory (ζ, η) is in $\mathcal{D}$ if and only if

$$\dot{\eta}(t) = Z(\eta(t)) + \mu_{\pi_M}(\zeta(t), \eta(t)) \tag{281}$$

for each $t \in I$.

4.0.9. Poisson formulation of dynamics

The vector field Z of the Hamiltonian formulation is expressed as $Z = W_M \circ \chi$. A force-momentum trajectory $(\zeta, \eta) : I \to Ph$ is in $\mathcal{D}$ if and only if

$$\mathrm{D}(F \circ \eta)(t) = \langle \mathrm{d}F, W_M \circ \chi \rangle + \langle \mathrm{d}F, \mu_{\pi_M}(\zeta(t), \eta(t)) \rangle \tag{282}$$

for each function F on T^*M and each $t \in I$.

5. Local expressions

Coordinate definitions of objects add nothing to the clarity of the conceptual structure of a theory. Covariance of a definition with respect to coordinate transformations guarantees the existence of an intrinsic interpretation of the object being defined without providing an interpretation. We have provided intrinsic definitions. Now we give local expressions of most objects introduced earlier in order to facilitate comparison with traditional formulations of mechanics. Local expressions are also used in calculations and appear in examples.

5.1. The tangent and the cotangent fibrations

Coordinates

$$(x^\kappa, \delta x^\lambda) : \mathrm{T}M \to \mathbb{R}^{2m} \tag{283}$$

and

$$(x^\kappa, f_\lambda) : \mathrm{T}^*M \to \mathbb{R}^{2m} \tag{284}$$

induced by coordinates

$$(x^\kappa) : M \to \mathbb{R}^m. \tag{285}$$

Projections:

$$(x^\kappa) \circ \tau_M = (x^\kappa), \tag{286}$$

$$(x^\kappa) \circ \varphi_M = (x^\kappa). \tag{287}$$

Zero sections and linear operations:

$$(x^\kappa, \delta x^\lambda) \circ O_{\tau_M} = (x^\kappa, 0), \tag{288}$$

$$(x^\kappa, \delta x^\lambda)(v + v') = (x^\kappa(v), \delta x^\lambda(v) + \delta x^\lambda(v')) \tag{289}$$

defined if

$$(x^\kappa(v')) = (x^\kappa(v)), \tag{290}$$

$$(x^\kappa, \delta x^\lambda)(kv) = (x^\kappa(v), k\delta x^\lambda(v)), \tag{291}$$

$$(x^\kappa, f_\lambda) \circ O_{\pi_M} = (x^\kappa, 0), \tag{292}$$

$$(x^\kappa, f_\lambda)(f + f') = (x^\kappa(f), f_\lambda(f) + f_\lambda(f')) \tag{293}$$

defined if

$$(x^\kappa(f')) = (x^\kappa(f)), \tag{294}$$

$$(x^\kappa, f_\lambda)(kf) = (x^\kappa(f), kf_\lambda(f)). \tag{295}$$

The canonical pairing:

$$\langle f, v\rangle = f_\kappa(f)\delta x^\kappa(v) \tag{296}$$

defined if

$$x^\kappa(f) = x^\kappa(v). \tag{297}$$

Coordinates

$$(x^\kappa, \dot{x}^\lambda) : \mathrm{T}M \to \mathbb{R}^{2m} \tag{298}$$

and

$$(x^\kappa, p_\lambda) : \mathrm{T}^*M \to \mathbb{R}^{2m} \tag{299}$$

are also used.

5.1.1. The dual pair $\mathbf{TT}M$ and $\mathbf{T^*T}M$

Coordinates:

$$(x^\kappa, \dot{x}^\lambda, \delta x^\mu, \delta\dot{x}^\nu) : \mathrm{TT}M \to \mathbb{R}^{4m}, \tag{300}$$

$$(x^\kappa, \dot{x}^\lambda, a_\mu, b_\nu) : \mathrm{T^*T}M \to \mathbb{R}^{4m}. \tag{301}$$

Projections:

$$(x^\kappa, \dot{x}^\lambda) \circ \tau_{\mathrm{T}M} = (x^\kappa, \dot{x}^\lambda), \tag{302}$$

$$(x^\kappa, \dot{x}^\lambda) \circ \pi_{\mathrm{T}M} = (x^\kappa, \dot{x}^\lambda). \tag{303}$$

Zero sections and linear operations:

$$(x^\kappa, \dot{x}^\lambda, \delta x^\mu, \dot{\delta} x^\nu) \circ O_{\tau_{\mathrm{T}M}} = (x^\kappa, \dot{x}^\lambda, 0, 0), \tag{304}$$

$$(x^\kappa, \dot{x}^\lambda, \delta x^\mu, \delta\dot{x}^\nu)(w{+}w') = (x^\kappa(w), \dot{x}^\lambda(w), \delta x^\mu(w){+}\delta x^\mu(w'), \delta\dot{x}^\nu(w){+}\delta\dot{x}^\nu(w')) \tag{305}$$

defined if

$$(x^\kappa, \dot{x}^\lambda)(w') = (x^\kappa, \dot{x}^\lambda)(w), \tag{306}$$

$$(x^\kappa, \dot{x}^\lambda, \delta x^\mu, \delta\dot{x}^\nu)(kw) = (x^\kappa(w), \dot{x}^\lambda(w), k\delta x^\mu(w), k\delta\dot{x}^\nu(w)), \tag{307}$$

$$(x^\kappa, \dot{x}^\lambda, a_\mu, b_\nu) \circ O_{\pi_{\mathrm{T}M}} = (x^\kappa, \dot{x}^\lambda, 0, 0), \tag{308}$$

$$(x^\kappa, \dot{x}^\lambda, a_\mu, b_\nu)(z + z') = (x^\kappa(z), \dot{x}^\lambda(z), a_\mu(z) + a_\mu(z'), b_\nu(z) + b_\nu(z')) \tag{309}$$

defined if

$$(x^\kappa, \dot{x}^\lambda)(z') = (x^\kappa, \dot{x}^\lambda)(z), \tag{310}$$

$$(x^\kappa, \dot{x}^\lambda, a_\mu, b_\nu)(kz) = (x^\kappa(z), \dot{x}^\lambda(z), ka_\mu(z), kb_\nu(z)). \tag{311}$$

The canonical pairing

$$\langle z, w\rangle = a_\kappa(z)\delta x^\kappa(w) + b_\kappa(z)\delta\dot{x}^\kappa(w) \tag{312}$$

defined if

$$(x^\kappa, \dot{x}^\lambda)(z) = (x^\kappa, \dot{x}^\lambda)(w). \tag{313}$$

5.1.2. The dual pair $\mathrm{TT}M$ and TT^*M

Coordinates:

$$(x^\kappa, \delta x^\lambda, \dot{x}^\mu, \dot{\delta} x^\nu) : \mathrm{TT}M \to \mathbb{R}^{4m}, \tag{314}$$

$$(x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu) : \mathrm{TT}^*M \to \mathbb{R}^{4m}. \tag{315}$$

Coordinates $(x^\kappa, \dot{x}^\lambda)$ in $\mathrm{T}M$ are used.

Projections:

$$(x^\kappa, \dot{x}^\lambda) \circ \mathrm{T}\tau_M = (x^\kappa, \dot{x}^\lambda), \tag{316}$$

$$(x^\kappa, \dot{x}^\lambda) \circ \mathrm{T}\pi_M = (x^\kappa, \dot{x}^\lambda). \tag{317}$$

Zero sections and tangent linear operations:

$$(x^\kappa, \delta x^\lambda, \dot{x}^\mu, \dot{\delta} x^\nu) \circ O_{\mathrm{T}\tau_M} = (x^\kappa, 0, \dot{x}^\mu, 0), \tag{318}$$

$$(x^\kappa, \delta x^\lambda, \dot{x}^\mu, \dot{\delta} x^\nu)(w \mathbin{+^{\mathrm{t}}} w') = (x^\kappa(w), \delta x^\lambda(w) + \delta x^\lambda(w'), \dot{x}^\mu(w), \delta\dot{x}^\nu(w) + \delta\dot{x}^\nu(w')) \tag{319}$$

defined if

$$(x^\kappa, \dot{x}^\lambda)(w') = (x^\kappa, \dot{x}^\lambda)(w), \tag{320}$$

$$(x^\kappa, \delta x^\lambda, \dot{x}^\mu, \dot{\delta} x^\nu)(k \cdot^{\mathrm{t}} w) = (x^\kappa(w), k\delta x^\lambda(w), \dot{x}^\mu(w), k\delta\dot{x}^\nu(w)), \tag{321}$$

$$(x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu) \circ O_{\mathrm{T}\pi_M} = (x^\kappa, 0, \dot{x}^\mu, 0), \tag{322}$$

$$(x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu)(z \mathbin{+^{\mathrm{t}}} z') = (x^\kappa(z), p_\lambda(z) + p_\lambda(z'), \dot{x}^\mu(z), \dot{p}_\nu(z) + \dot{p}_\nu(z')) \tag{323}$$

defined if

$$(x^\kappa, \dot{x}^\lambda)(z') = (x^\kappa, \dot{x}^\lambda)(z), \tag{324}$$

$$(x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu)(k \cdot^{\mathrm{t}} z) = (x^\kappa(z), kp_\lambda(z), \dot{x}^\mu(z), k\dot{p}_\nu(z)). \tag{325}$$

The tangent pairing:

$$\langle z, w\rangle^{\mathfrak{t}} = \dot{p}_\kappa(z)\delta x^\kappa(w) + p_\kappa(z)\delta\dot{x}^\kappa(w) \tag{326}$$

defined if

$$(x^\kappa, \dot{x}^\lambda)(z) = (x^\kappa, \dot{x}^\lambda)(w). \tag{327}$$

Let $\xi : \mathbb{R} \to M$, $\delta\xi : \mathbb{R} \to \mathrm{T}M$, and $\eta : \mathbb{R} \to \mathrm{T}^*M$ be curves such that $\pi_M \circ \eta = \tau_M \circ \delta\xi = \xi$. Let $\dot{\xi} : \mathbb{R} \to \mathrm{T}M$, $\dot{\delta\xi} : \mathbb{R} \to \mathrm{TT}M$, and $\dot{\eta} : \mathbb{R} \to \mathrm{TT}^*M$ be prolongations of these curves.

$$\langle \dot{\eta}(0), \dot{\delta\xi}(0)\rangle^{\mathfrak{t}} = \frac{\mathrm{d}}{\mathrm{d}t}(\eta_\kappa(t)\delta\xi^\kappa(t))_{|t=0} = \dot{\eta}_\kappa(0)\delta\xi^\kappa(0) + \eta_\kappa(0)\dot{\delta\xi}^\kappa(0). \tag{328}$$

5.1.3. Relations between TT^*M and $\mathrm{T}^*\mathrm{T}M$

The local expression

$$(x^\kappa, \dot{x}^\lambda, a_\mu, b_\nu) \circ \alpha_M = (x^\kappa, \dot{x}^\lambda, \dot{p}_\mu, p_\nu) \tag{329}$$

of the isomorphism $\alpha_M : \mathrm{TT}^*M \to \mathrm{T}^*\mathrm{T}M$ dual to the canonical involution $\kappa^{1,1} : \mathrm{TT}M \to \mathrm{TT}M$ defined locally by

$$(x^\kappa, \delta x^\lambda, \dot{x}^\mu, \delta\dot{x}^\nu) \circ \kappa^{1,1} = (x^\kappa, \dot{x}^\lambda, \delta x^\mu, \delta\dot{x}^\nu). \tag{330}$$

Coordinate systems (301), (314), and (315) are used in the local expressions.

5.1.4. The dual pair TT^*M and $\mathrm{T}^*\mathrm{T}^*M$

Coordinates:

$$(x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu) : \mathrm{TT}^*M \to \mathbb{R}^{4m}, \tag{331}$$

$$(x^\kappa, p_\lambda, y_\mu, z^\nu) : \mathrm{T}^*\mathrm{T}^*M \to \mathbb{R}^{4m}. \tag{332}$$

Projections:

$$(x^\kappa, p_\lambda) \circ \tau_{\mathrm{T}^*M} = (x^\kappa, p_\lambda) \tag{333}$$

$$(x^\kappa, p_\lambda) \circ \pi_{\mathrm{T}^*M} = (x^\kappa, p_\lambda) \tag{334}$$

Zero sections and linear operations:

$$(x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu) \circ O_{\tau_{\mathrm{T}^*M}} = (x^\kappa, p_\lambda, 0, 0), \tag{335}$$

$$(x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu)(z+z') = (x^\kappa(z), p_\lambda(z), \dot{x}^\mu(z)+\dot{x}^\mu(z'), \dot{p}_\nu(z)+\dot{p}_\nu)(z')) \tag{336}$$

defined if

$$(x^\kappa, p_\lambda)(z') = (x^\kappa, p_\lambda)(z), \tag{337}$$

$$(x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu)(kz) = (x^\kappa(z), p_\lambda(z), k\dot{x}^\mu(z), k\dot{p}_\nu(z)), \tag{338}$$

$$(x^\kappa, p_\lambda, y_\mu, z^\nu) \circ O_{\pi_{\mathrm{T}^*M}} = (x^\kappa, p_\lambda, 0, 0), \tag{339}$$

$$(x^\kappa, p_\lambda, y_\mu, z^\nu)(b + b') = (x^\kappa(b), p_\lambda(b), y_\mu(b) + y_\mu(b'), z^\nu(b) + z^\nu(b')) \tag{340}$$

defined if

$$(x^\kappa, p_\lambda)(b') = (x^\kappa, p_\lambda)(b), \tag{341}$$

$$(x^\kappa, p_\lambda, y_\mu, z^\nu)(kb) = (x^\kappa(b), p_\lambda(b), ky_\mu(b), kz^\nu(b)). \tag{342}$$

The canonical pairing

$$\langle b, z\rangle = y_\kappa(b)\dot{x}^\kappa(z) + z^\kappa(b)\dot{p}^\kappa(z) \tag{343}$$

defined if

$$(x^\kappa, p_\lambda)(b) = (x^\kappa, p_\lambda)(z). \tag{344}$$

The isomorphism $\beta_{(\mathrm{T}^*M, \omega_M)} : \mathrm{TT}^*M \to \mathrm{T}^*\mathrm{T}^*M$ has a local expression

$$(x^\kappa, p_\lambda, y_\mu, z^\nu) \circ \beta_{(\mathrm{T}^*M, \omega_M)} = (x^\kappa, p_\lambda, \dot{p}_\mu, -\dot{x}^\nu). \tag{345}$$

5.1.5. The bundles T^2M, TT^2M, and $\mathrm{T}^2\mathrm{T}M$

Coordinates:

$$(x^\kappa, \dot{x}^\lambda, \ddot{x}^\mu) : \mathrm{T}^2M \to \mathbb{R}^{3m} \tag{346}$$

$$(x^\kappa, \dot{x}^\lambda, \ddot{x}^\mu, x'^\nu, \dot{x}'^\omega, \ddot{x}'^\pi) : \mathrm{TT}^2M \to \mathbb{R}^{6m} \tag{347}$$

$$(x^\kappa, \dot{x}^\lambda, x'^\mu, \dot{x}'^\nu, x''^\omega, \dot{x}''^\pi) : \mathrm{T}^2\mathrm{T}M \to \mathbb{R}^{6m} \tag{348}$$

Projections:

$$(x^\kappa) \circ \tau_{2M} = (z^\kappa), \tag{349}$$

$$(x^\kappa, \dot{x}^\lambda) \circ \tau^1{}_{2M} = (x^\kappa, \dot{x}^\lambda), \tag{350}$$

$$(x^\kappa, \dot{x}^\lambda, \ddot{x}^\mu) \circ \tau_{\mathrm{T}^2M} = (x^\kappa, \dot{x}^\lambda, \ddot{x}^\mu), \tag{351}$$

$$(x^\kappa, \dot{x}^\lambda) \circ \mathrm{T}\tau_{2M} = (x^\kappa, x'^\lambda), \tag{352}$$

$$(x^\kappa, \dot{x}^\lambda, x'^\mu, \dot{x}'^\nu) \circ \mathrm{T}\tau^1{}_{2M} = (x^\kappa, \dot{x}^\lambda, x'^\mu, \dot{x}'^\nu). \tag{353}$$

Mappings:

$$(x^\kappa, \dot{x}^\lambda, x'^\mu, \dot{x}'^\nu, x''^\omega, \dot{x}''^\pi) \circ \kappa^{2,1} = (x^\kappa, x'^\lambda, \dot{x}^\mu, \dot{x}'^\nu, \ddot{x}^\omega, \ddot{x}'^\pi), \tag{354}$$

$$(x^\kappa, \dot{x}^\lambda, \ddot{x}^\mu, x'^\nu, \dot{x}'^\omega, \ddot{x}'^\pi) \circ \kappa^{1,2} = (x^\kappa, x'^\lambda, x''^\mu, \dot{x}^\nu, \dot{x}'^\omega, \dot{x}''^\pi). \tag{355}$$

5.1.6. Tangent prolongations and tangent mappings

If $\dot{\xi} : \mathbb{R} \to \mathrm{T}M$ and $\ddot{\xi} : \mathbb{R} \to \mathrm{T}^2M$ are prolongations of a curve $\xi : \mathbb{R} \to M$ and $x^\kappa \circ \xi = \xi^\kappa$, then

$$(x^\kappa, \dot{x}^\lambda) \circ \dot{\xi} = (\xi^\kappa, \dot{\xi}^\lambda) = (\xi^\kappa, \mathrm{D}\xi^\lambda). \tag{356}$$

and

$$(x^\kappa, \dot{x}^\lambda, \ddot{x}^\mu) \circ \ddot{\xi} = (\xi^\kappa, \dot{\xi}^\lambda, \ddot{\xi}^\mu) = (\xi^\kappa, \mathrm{D}\xi^\lambda, \mathrm{D}^2\xi^\mu). \tag{357}$$

Let (x^κ) and (y^i) be charts of manifolds M and N, let $(x^\kappa \dot{x}^\lambda)$ and $(y^i, \dot{y}^j)$ be induced charts of $\mathrm{T}M$ and $\mathrm{T}N$, and let $(x^\kappa \dot{x}^\lambda, \ddot{x}^\mu)$ and $(y^i, \dot{y}^j, \ddot{y}^k)$ be induced charts of T^2M and T^2N. A differentiable mapping $\alpha : M \to N$ is represented locally by a set of functions $\alpha^i : \mathbb{R}^m \to \mathbb{R}$ defined by $\alpha^i = y^i \circ \alpha \circ (x^\kappa)^{-1}$. The tangent mapping $\mathrm{T}\alpha$ and the second tangent mapping $\mathrm{T}^2\alpha$ have local representations

$$(y^i, \dot{y}^j) \circ \mathrm{T}\alpha = \left(\alpha^i(x^\mu), \frac{\partial \alpha^j}{\partial x^\kappa}(x^\mu)\dot{x}^\kappa\right) \tag{358}$$

and

$$(y^i, \dot{y}^j, \ddot{y}^k) \circ \mathrm{T}^2\alpha = \left(\alpha^i(x^\mu), \frac{\partial \alpha^j}{\partial x^\kappa}(x^\mu)\dot{x}^\kappa, \frac{\partial^2 \alpha^k}{\partial x^\kappa \partial x^\lambda}(x^\mu)\dot{x}^\kappa \dot{x}^\lambda + \frac{\partial \alpha^k}{\partial x^\kappa}(x^\mu)\ddot{x}^\kappa\right). \tag{359}$$

5.1.7. Vector valued forms and derivations

Local expressions

$$(x^\kappa, \dot{x}^\lambda) \circ T(0) = (x^\kappa, \dot{x}^\lambda), \tag{360}$$

$$(x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu) \circ T = (x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu), \tag{361}$$

$$(x^\kappa, \dot{x}^\lambda, x'^\mu, \dot{x}'^\nu) \circ T(1) = (x^\kappa, \dot{x}^\lambda, \dot{x}^\mu, \ddot{x}^\nu), \tag{362}$$

$$(x^\kappa, \dot{x}^\lambda, x'^\mu, \dot{x}'^\nu) \circ F(1;1) = (x^\kappa, \dot{x}^\lambda, 0, \dot{x}^\nu), \tag{363}$$

$$(x^\kappa, \dot{x}^\lambda, \ddot{x}^\mu, x'^\nu, \dot{x}'^\omega, \ddot{x}'^\pi) \circ F(2;1) = (x^\kappa, \dot{x}^\lambda, \ddot{x}^\mu, 0, x'^\omega, \dot{x}'^\pi), \tag{364}$$

$$(x^\kappa, \dot{x}^\lambda, \ddot{x}^\mu, x'^\nu, \dot{x}'^\omega, \ddot{x}'^\pi) \circ F(2;2) = (x^\kappa, \dot{x}^\lambda, \ddot{x}^\mu, 0, 0, 2x'^\pi) \tag{365}$$

of vector valued forms

$$T(0) : \mathrm{T}M \to \mathrm{T}M, \tag{366}$$

$$T : \mathrm{T}\mathrm{T}^*M \to \mathrm{T}\mathrm{T}^*M, \tag{367}$$

$$T(1) : \mathrm{T}^2M \to \mathrm{T}\mathrm{T}M, \tag{368}$$

$$F(1;1) : \mathrm{TT}M \to \mathrm{TT}M, \tag{369}$$

$$F(2;1) : \mathrm{TT}^2M \to \mathrm{TT}^2M, \tag{370}$$

and

$$F(2;2) : \mathrm{TT}^2M \to \mathrm{TT}^2M. \tag{371}$$

Corresponding derivations:

$$\mathrm{d}_{T(0)}x^\kappa = \mathrm{i}_{T(0)}\mathrm{d}x^\kappa = \dot{x}^\kappa, \tag{372}$$

$$\mathrm{d}_T x^\kappa = \mathrm{i}_T\mathrm{d}x^\kappa = \dot{x}^\kappa,\ \mathrm{d}_T p_\kappa = \mathrm{i}_T\mathrm{d}p_\kappa = \dot{p}_\kappa, \tag{373}$$

$$\mathrm{d}_{T(1)}x^\kappa = \mathrm{i}_{T(1)}\mathrm{d}x^\kappa = \dot{x}^\kappa,\ \mathrm{d}_{T(1)}\dot{x}^\kappa = \mathrm{i}_{T(1)}\mathrm{d}\dot{x}^\kappa = \ddot{x}^\kappa, \tag{374}$$

$$\mathrm{i}_{F(1;1)}\mathrm{d}x^\kappa = 0,\ \mathrm{i}_{F(1;1)}\mathrm{d}\dot{x}^\kappa = \mathrm{d}x^\kappa, \tag{375}$$

$$\mathrm{i}_{F(2;1)}\mathrm{d}x^\kappa = 0,\ \mathrm{i}_{F(2;1)}\mathrm{d}\dot{x}^\kappa = \mathrm{d}x^\kappa,\ \mathrm{i}_{F(2;1)}\mathrm{d}\ddot{x}^\kappa = \mathrm{d}\dot{x}^\kappa, \tag{376}$$

and

$$\mathrm{i}_{F(2;2)}\mathrm{d}x^\kappa = 0,\ \mathrm{i}_{F(2;2)}\mathrm{d}\dot{x}^\kappa = 0,\ \mathrm{i}_{F(2;2)}\mathrm{d}\ddot{x}^\kappa = 2\mathrm{d}x^\kappa. \tag{377}$$

5.1.8. Louville, symplecting and Poisson structures

On T^*M:

$$\vartheta_M = p_\kappa \mathrm{d}x^\kappa, \tag{378}$$

$$\omega_M = \mathrm{d}p_\kappa \wedge \mathrm{d}x^\kappa. \tag{379}$$

On $\mathrm{T}^*\mathrm{T}M$:

$$\vartheta_{\mathrm{T}M} = a_\kappa \mathrm{d}x^\kappa + b_\kappa \mathrm{d}\dot{x}^\kappa, \tag{380}$$

$$\omega_{\mathrm{T}M} = \mathrm{d}a_\kappa \wedge \mathrm{d}x^\kappa + \mathrm{d}b_\kappa \wedge \mathrm{d}\dot{x}^\kappa. \tag{381}$$

On $\mathrm{T}^*\mathrm{T}^*M$:

$$\vartheta_{\mathrm{T}^*M} = y_\kappa \mathrm{d}x^\kappa + z^\kappa \mathrm{d}p_\kappa, \tag{382}$$

$$\vartheta_{\mathrm{T}^*M} = \mathrm{d}y_\kappa \wedge \mathrm{d}x^\kappa + \mathrm{d}z^\kappa \wedge \mathrm{d}p_\kappa. \tag{383}$$

On TT^*M:

$$\mathrm{d}_T\vartheta_M = \dot{p}_\kappa \mathrm{d}x^\kappa + p_\kappa \mathrm{d}\dot{x}^\kappa, \tag{384}$$

$$\mathrm{i}_T\omega_M = \dot{p}_\kappa \mathrm{d}x^\kappa - \dot{x}^\kappa \mathrm{d}p_\kappa, \tag{385}$$

$$\mathrm{d}_T\omega_M = \mathrm{d}\dot{p}_\kappa \wedge \mathrm{d}x^\kappa + \mathrm{d}p_\kappa \wedge \mathrm{d}\dot{x}^\kappa. \tag{386}$$

The function G_M on TT^*M:

$$G_M = \mathrm{i}_T\vartheta_M = p_\kappa \dot{x}^\kappa. \tag{387}$$

Local expression

$$\{F, G\} \circ (x^\kappa, p_\lambda)^{-1} = \frac{\partial \mathcal{F}}{\partial x^\kappa}\frac{\partial \mathcal{G}}{\partial p_\kappa} - \frac{\partial \mathcal{G}}{\partial x^\kappa}\frac{\partial \mathcal{F}}{\partial p_\kappa} \tag{388}$$

of the Poisson bracket of functions F and G on T^*M with local expressions

$$\mathcal{F} = F \circ (x^\kappa, p_\lambda)^{-1} \tag{389}$$

and

$$\mathcal{G} = G \circ (x^\kappa, p_\lambda)^{-1} \tag{390}$$

5.1.9. Other constructions

The mapping

$$\mu_{\pi_M} : \mathrm{T}^*M \underset{(\pi_M, \pi_M)}{\times} \mathrm{T}^*M \to \mathrm{T}\mathrm{T}^*M \tag{391}$$

has a local expression

$$(x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu) \circ \mu_{\pi_M} = (x^\kappa, p_\lambda, 0, f_\nu) \tag{392}$$

in terms of coordinates

$$(x^\kappa, f_\lambda, p_\mu) : \mathrm{T}^*M \underset{(\pi_M, \pi_M)}{\times} \mathrm{T}^*M \to \mathbb{R}^{3m} \tag{393}$$

and (315).

The mapping

$$\varphi_{\pi_M} : \mathrm{T}^*M \underset{(\pi_M, \pi_M \circ \tau_{\mathrm{T}^*M})}{\times} \mathrm{T}\mathrm{T}^*M \to \mathrm{T}\mathrm{T}^*M \tag{394}$$

is defined locally by

$$(x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu) \circ \varphi_{\pi_M} = (x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu - f_\nu) \tag{395}$$

in terms of coordinates

$$(x^\kappa, f_\lambda, p_\mu, \dot{x}^\nu, \dot{p}_\omega) : \mathrm{T}^*M \underset{(\pi_M, \pi_M \circ \tau_{\mathrm{T}^*M})}{\times} \mathrm{T}\mathrm{T}^*M \to \mathbb{R}^{5m} \tag{396}$$

and (315).

6. Local description of dynamics

6.1. The variational principle

We associate mappings

$$\xi^\kappa = x^\kappa \circ \delta\dot{\xi} \tag{397}$$

$$\dot{\xi}^\kappa = \dot{x}^\kappa \circ \delta\dot{\xi} \tag{398}$$

$$\delta\xi^\kappa = \delta x^\kappa \circ \delta\dot{\xi} \tag{399}$$

$$\delta\dot{\xi}^\kappa = \delta\dot{x}^\kappa \circ \delta\dot{\xi} \tag{400}$$

defined in terms of coordinates

$$(x^\kappa, \dot{x}^\lambda, \delta x^\mu, \delta\dot{x}^\nu) : \mathrm{TT}M \to \mathbb{R}^{4m}, \tag{401}$$

with a variation

$$\delta\dot{\xi} : I \to \mathrm{TT}M. \tag{402}$$

We will denote by $\mathcal{L}$ the local expression

$$L \circ (x^\kappa, \dot{x}^\lambda)^{-1} : \mathbb{R}^{2m} \to \mathbb{R} \tag{403}$$

of the Lagrangian.

The variation of the action

$$\int_a^b \mathcal{L}(\xi^\mu(t), \dot{\xi}^\nu(t))\mathrm{d}t \tag{404}$$

is expressed as the integral

$$\int_a^b \left(\frac{\partial \mathcal{L}}{\partial x^\kappa}(\xi^\mu(t), \dot{\xi}^\nu(t))\delta\xi^\kappa(t) + \frac{\partial \mathcal{L}}{\partial \dot{x}^\kappa}(\xi^\mu(t), \dot{\xi}^\nu(t))\delta\dot{\xi}^\kappa(t) \right) \mathrm{d}t. \tag{405}$$

If non potential internal forces represented by the form

$$\rho = \rho_\kappa(x^\mu, \dot{x}^\nu)\mathrm{d}x^\kappa \tag{406}$$

are present, then the differential of the Lagrangian is replaced by

$$\lambda = \frac{\partial \mathcal{L}}{\partial x^\kappa}(x^\mu, \dot{x}^\nu)\mathrm{d}x^\kappa + \frac{\partial \mathcal{L}}{\partial \dot{x}^\kappa}(x^\mu, \dot{x}^\nu)\mathrm{d}\dot{x}^\kappa - \rho_\kappa(x^\mu, \dot{x}^\nu)\mathrm{d}x^\kappa \tag{407}$$

and the variation takes the form

$$\int_a^b \left(\frac{\partial \mathcal{L}}{\partial x^\kappa}(\xi^\mu(t),\dot{\xi}^\nu(t))\delta\xi^\kappa(t) + \frac{\partial \mathcal{L}}{\partial \dot{x}^\kappa}(\xi^\mu(t),\dot{\xi}^\nu(t))\delta\dot{\xi}^\kappa(t) - \rho_\kappa(\xi^\mu(t),\dot{\xi}^\nu(t))\delta\xi^\kappa(t)\right) \mathrm{d}t. \tag{408}$$

Integration by parts results in

$$\int_a^b \left(\frac{\partial \mathcal{L}}{\partial x^\kappa}(\xi^\mu(t),\dot{\xi}^\nu(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}^\kappa}(\xi^\mu(t),\dot{\xi}^\nu(t),\ddot{\xi}^\omega(t)) - \rho_\kappa(\xi^\mu(t),\dot{\xi}^\nu(t))\right)\delta\xi^\kappa(t)\mathrm{d}t$$
$$+\frac{\partial \mathcal{L}}{\partial \dot{x}^\kappa}(\xi^\mu(b),\dot{\xi}^\nu(b))\delta\xi^\kappa(b) - \frac{\partial \mathcal{L}}{\partial \dot{x}^\kappa}(\xi^\mu(a),\dot{\xi}^\nu(a))\delta\xi^\kappa(a). \tag{409}$$

The expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}^\kappa}(\xi^\mu(t),\dot{\xi}^\nu(t),\ddot{\xi}^\omega(t)) \tag{410}$$

stands for

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\kappa}(\xi^\mu(t),\dot{\xi}^\nu(t))\right) = \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\kappa \partial x^\lambda}(\xi^\mu(t),\dot{\xi}^\nu(t))\dot{\xi}^\lambda(t) + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\kappa \partial \dot{x}^\lambda}(\xi^\mu(t),\dot{\xi}^\nu(t))\ddot{\xi}^\lambda(t) \tag{411}$$

The variational principle requires that the variation (409) be equal to

$$-\int_a^b \zeta_\kappa(t)\delta\xi^\kappa(t)\mathrm{d}t + \eta_\kappa(b)\delta\xi^\kappa(b) - \eta_\kappa(a)\delta\xi^\kappa(a), \tag{412}$$

where

$$\eta_\kappa = p_\kappa \circ \eta \tag{413}$$

and

$$\zeta_\kappa = f_\kappa \circ \zeta \tag{414}$$

are mappings derived from a force-momentum trajectory

$$(\zeta, \eta) : I \to Ph. \tag{415}$$

The Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}^\kappa}(\xi^\mu(t),\dot{\xi}^\nu(t),\ddot{\xi}^\omega(t)) - \frac{\partial \mathcal{L}}{\partial x^\kappa}(\xi^\mu(t),\dot{\xi}^\nu(t)) + \rho_\kappa(\xi^\mu(t),\dot{\xi}^\nu(t)) = \zeta_\kappa(t) \tag{416}$$

in $[a,b]$ and the momentum-velocity relations

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\kappa}(\xi^\mu(t),\dot{\xi}^\nu(a)) = \eta_\kappa(a), \tag{417}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^{\kappa}}(\xi^{\mu}(t), \dot{\xi}^{\nu}(b)) = \eta_{\kappa}(b) \tag{418}$$

are derived from this variational principle. The variational principle is to be satisfied in each interval $[a,b] \subset I$. It follows that the Euler-Lagrange equations and the relation

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^{\kappa}}(\xi^{\mu}(t), \dot{\xi}^{\nu}(t)) = \eta_{\kappa}(t) \tag{419}$$

are satisfied in I.

6.1.1. The Lagrangian formulation

Lagrange equations

$$\eta_{\kappa}(t) = \frac{\partial \mathcal{L}}{\partial \dot{x}^{\kappa}}(\xi^{\mu}(t), \dot{\xi}^{\nu}(t)) \tag{420}$$

and

$$\dot{\eta}_{\kappa}(t) - \zeta_{\kappa}(t) = \frac{\partial \mathcal{L}}{\partial x^{\kappa}}(\xi^{\mu}(t), \dot{\xi}^{\nu}(t)) - \rho_{\kappa}(\xi^{\mu}(t), \dot{\xi}^{\nu}(t)) \tag{421}$$

are derived from the infinitesimal form

$$\begin{aligned}&\frac{\partial \mathcal{L}}{\partial x^{\kappa}}(\xi^{\mu}(t), \dot{\xi}^{\nu}(t))\delta\xi^{\kappa}(t) + \frac{\partial \mathcal{L}}{\partial \dot{x}^{\kappa}}(\xi^{\mu}(t), \dot{\xi}^{\nu}(t))\delta\dot{\xi}^{\kappa}(t) - \rho_{\kappa}(\xi^{\mu}(t), \dot{\xi}^{\nu}(t))\delta\xi^{\kappa}(t) \\ &= -\zeta_{\kappa}(t)\delta\xi^{\kappa}(t) + \dot{\eta}_{\kappa}(t)\delta\xi^{\kappa}(t) + \eta_{\kappa}(t)\delta\dot{\xi}^{\kappa}(t)\end{aligned} \tag{422}$$

of the variational principle.

Equations describing the dynamics in the potential case are obtained by setting $\rho = 0$.

6.1.2. The Hamiltonian formulation

The Legendre mapping $\mathcal{P}\lambda$ is represented locally by

$$(x^{\kappa}, p_{\lambda}) \circ \mathcal{P}\lambda = \left(x^{\kappa}, \frac{\partial \mathcal{L}}{\partial \dot{x}^{\lambda}}(x^{\mu}, \dot{x}^{\nu})\right). \tag{423}$$

It is convenient to introduce functions

$$\Pi_{\lambda} = p_{\lambda} \circ \mathcal{P}\lambda \circ (x^{\mu}, x^{\nu})^{-1} = \frac{\partial \mathcal{L}}{\partial \dot{x}^{\lambda}}. \tag{424}$$

If the Legendre mapping is a diffeomorphism, then the inverse diffeomorphism Λ is represented locally by

$$(x^{\kappa}, \dot{x}^{\lambda}) \circ \Lambda = (x^{\kappa}, \Lambda^{\lambda}(x^{\mu}, p_{\nu})), \tag{425}$$

where Λ^κ are the functions

$$\Lambda^\kappa = \dot{x}^\kappa \circ \Lambda \circ (x^\mu, p_\nu)^{-1}. \tag{426}$$

Relations

$$\Pi_\lambda \left(x^\mu(p), \Lambda^\nu(x^\rho(p), p_\sigma(p))\right) = p_\lambda(p) \tag{427}$$

and

$$\Lambda^\lambda \left(x^\mu(v), \Pi_\nu(x^\rho(v), \dot{x}^\sigma(v))\right) = \dot{x}^\lambda(v) \tag{428}$$

are satisfied.

In the potential case we have a Hamiltonian $H : \mathrm{T}^* \to \mathbb{R}$ represented locally by the function

$$\mathcal{H} = H \circ (x^\kappa, p_\lambda)^{-1}. \tag{429}$$

This function is obtained from the formula

$$\mathcal{H}(x^\kappa, p_\lambda) = p_z k \Lambda^\kappa(x^\mu, p_\nu) - \mathcal{L}(x^\kappa, \Lambda^\lambda(x^\mu, p_\nu)). \tag{430}$$

In the non potential case there is the Hamiltonian form

$$\begin{aligned} \chi &= \chi_\kappa(x^\mu, p_\nu)\mathrm{d}x^\kappa + \chi^\kappa(x^\mu, p_\nu)\mathrm{d}p_\kappa \\ &= \frac{\partial \mathcal{H}}{\partial p_\kappa}(x^\mu, p_\nu)\mathrm{d}p_\kappa + \frac{\partial \mathcal{H}}{\partial x^\kappa}(x^\mu, p_\nu)\mathrm{d}x^\kappa + \rho_\kappa(x^\mu, \Lambda^\nu(x^\rho, p_\sigma))\mathrm{d}x^\kappa \\ &= \Lambda^\kappa(x^\mu, p_\nu)\mathrm{d}p_\kappa - \frac{\partial \mathcal{L}}{\partial x^\kappa}(x^\mu, \Lambda^\nu(x^\rho, p_\sigma))\mathrm{d}x^\kappa + \rho_\kappa(x^\mu, \Lambda^\nu(x^\rho, p_\sigma))\mathrm{d}x^\kappa \end{aligned} \tag{431}$$

obtained from formula (280). The function $\mathcal{H}$ is the local expression of the Hamiltonian H associated with L.

The vector field $Z = {\alpha_M}^{-1} \circ \lambda \circ \Lambda = -\beta^{-1}_{(\mathrm{T}^* M, \omega_M)} \circ \chi$ is expressed by

$$\begin{aligned} &(x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu) \circ Z = (x^\kappa, p_\lambda, \chi^\mu(x^\rho, p_\sigma), -\chi_\nu(x^\rho, p_\sigma)) \\ &= \left(x^\kappa, p_\lambda, \frac{\partial \mathcal{H}}{\partial p_\mu}(x^\rho, p_\sigma), -\frac{\partial \mathcal{H}}{\partial x^\nu}(x^\rho, p_\sigma) - \rho_\nu(x^\rho, \Lambda^\sigma(x^\omega, p_\pi))\right) \\ &= \left(x^\kappa, p_\lambda, \Lambda^\mu(x^\rho, p_\sigma), \frac{\partial \mathcal{L}}{\partial x^\nu}(x^\rho, \Lambda^\sigma(x^\omega, p_\pi)) - \rho_\nu(x^\rho, \Lambda^\sigma(x^\omega, p_\pi))\right). \end{aligned} \tag{432}$$

A force-momentum trajectory (ζ, η) satisfies Hamilton's equations

$$\dot{\xi}^\kappa(t) = \frac{\partial \mathcal{H}}{\partial p_\kappa}(\xi^\mu(t), \eta_\nu(t)) \tag{433}$$

and

$$\dot{\eta}_\kappa(t) - \zeta_\kappa(t) = -\frac{\partial \mathcal{H}}{\partial x^\kappa}(\xi^\mu(t), \eta_\nu(t)) - \rho_\kappa(\xi^\mu(t), \dot{\xi}^\nu(t)). \tag{434}$$

6.1.3. The Poisson formulation

Hamilton's equations are equivalent to equations

$$\begin{aligned}&\frac{\partial \mathcal{F}}{\partial x^\kappa}(\xi^\mu(t),\eta_\nu(t))\dot{\xi}^\kappa+\frac{\partial \mathcal{F}}{\partial p_\kappa}(\xi^\mu(t),\eta_\nu(t))\dot{\eta}_\kappa\\&=\left(\frac{\partial \mathcal{F}}{\partial x^\kappa}\frac{\partial \mathcal{H}}{\partial p_\kappa}-\frac{\partial \mathcal{H}}{\partial x^\kappa}\frac{\partial \mathcal{F}}{\partial p_\kappa}\right)(\xi^\mu(t),\eta_\nu(t))\\&+\frac{\partial \mathcal{F}}{\partial p_\kappa}(\xi^\mu(t),\eta_\nu(t))(\zeta_\kappa(t)-\rho_\kappa(\xi^\mu(t),\eta_\nu(t)))\end{aligned} \tag{435}$$

satisfied for each function F on T^*M with local expression

$$\mathcal{F}=F\circ(x^\kappa,p_\lambda)^{-1}. \tag{436}$$

7. An example

An aircraft is travelling in a vertical plane M. The force of gravity and the force due to air viscosity are the internal forces. The jet propulsion force and the aerodynamic forces acting on the wings, the rudder, and the elevator are controlled by the pilot and are considered external forces.

An Euclidean affine chart

$$(x^h,x^v):M\to\mathbb{R}^2 \tag{437}$$

induces charts

$$(x^h,x^v,\dot{x}^h,\dot{x}^v):\mathrm{T}M\to\mathbb{R}^4, \tag{438}$$

$$(x^h,x^v,\dot{x}^h,\dot{x}^v,\ddot{x}^h,\ddot{x}^v):\mathrm{T}M\to\mathbb{R}^6, \tag{439}$$

and

$$(x^h,x^v,f_h,f_v,p_h,p_v):\mathrm{T}^*M\underset{(\pi_M,\pi_M)}{\times}\mathrm{T}^*M\to\mathbb{R}^6. \tag{440}$$

A force-momentum trajectory (ζ,η) has a local representation

$$(\xi^h,\xi^v,\zeta_h,\zeta_v,\eta_h,\eta_v)=(x^h,x^v,f_h,f_v,p_h,p_v)\circ(\zeta,\eta). \tag{441}$$

The mapping $\xi=\pi_M\circ\zeta=\pi_M\circ\eta$ and its prolongations $\dot{\xi}$ and $\ddot{\xi}$ have local representations

$$(\xi^h,\xi^v)=(x^h,x^v)\circ\xi, \tag{442}$$

$$(\xi^h,\xi^v,\dot{\xi}^h,\dot{\xi}^v)=(x^h,x^v,\dot{x}^h,\dot{x}^v)\circ\dot{\xi}, \tag{443}$$

and

$$(\xi^h,\xi^v,\dot{\xi}^h,\dot{\xi}^v,\ddot{\xi}^h,\ddot{\xi}^v)=(x^h,x^v,\dot{x}^h,\dot{x}^v,\ddot{x}^h,\ddot{x}^v)\circ\ddot{\xi}. \tag{444}$$

The form

$$\lambda = \mathrm{d}L - \rho = m\dot{x}^h \mathrm{d}\dot{x}^h + m\dot{x}^v \mathrm{d}\dot{x}^v - mg\mathrm{d}x^v - \gamma_h \dot{x}^h \mathrm{d}x^h - \gamma_v \dot{x}^v \mathrm{d}x^v \tag{445}$$

is constructed from the Lagrangian

$$L = \frac{m}{2}((\dot{x}^h)^2 + (\dot{x}^v)^2) - mgx^v \tag{446}$$

and the form

$$\rho = \gamma_h \dot{x}^h \mathrm{d}x^h + \gamma_v \dot{x}^v \mathrm{d}x^v \tag{447}$$

representing the non potential force of viscosity.

The dynamics in a time interval $[0, T]$ is governed by the Euler-Lagrange equations

$$m\ddot{\xi}^h(t) + \gamma_h \dot{\xi}^h(t) = \zeta_h(t), \tag{448}$$

$$m\ddot{\xi}^v(t) + \gamma_v \dot{\xi}^v(t) + mg = \zeta_v(t), \tag{449}$$

and the momentum-velocity relations

$$\eta_h(0) = m\dot{\xi}^h(0), \;\; \eta_v(0) = m\dot{\xi}^v(0), \;\; \eta_h(T) = m\dot{\xi}^h(T), \;\; \eta_v(T) = m\dot{\xi}^v(T) \tag{450}$$

at the boundary. In the absence of external forces and with initial conditions

$$(\xi^h, \xi^v, \dot{\xi}^h, \dot{\xi}^v)(0) = (0, 0, v_0, 0) \tag{451}$$

we obtain the trajectory

$$\xi^h(t) = \frac{mv_0}{\gamma_h}\left(1 - \exp\left(-\frac{\gamma_h}{m}t\right)\right), \tag{452}$$

$$\xi^v(t) = \frac{m^2 g}{{\gamma_v}^2}\left(1 - \exp\left(-\frac{\gamma_v}{m}t\right)\right) - \frac{mg}{\gamma_v}t, \tag{453}$$

and the momenta

$$\eta_h(0) = mv_0, \;\; \eta_v(0) = 0, \;\; \eta_h(T) = \frac{mv_0}{\gamma_h}\exp\left(-\frac{\gamma_h}{m}T\right),$$

$$\eta_v(T) = -\frac{m^2 g}{\gamma_v}\left(1 - \exp\left(-\frac{\gamma_h}{m}T\right)\right) \tag{454}$$

at the boundary. In order to maintain a horizontal trajectory

$$\xi^h(t) = v_0 t, \;\; \xi^v(t) = 0 \tag{455}$$

with constant velocity it is necessary to supply external forces

$$\zeta_h(t) = \gamma_h v_0, \;\; \zeta_v(t) = mg. \tag{456}$$

The external forces are true forces of control. We have described a very simple situation. In reality these forces can not be chosen in advance since they may have to compensate the effects of varying weather conditions and allow changes of the trajectory necessary due to unforeseeable circumstances.

REFERENCES

[1] C.-M. Marle, *Rend. Sem. Mat. Univ. Pol. Torino* **54**, 353 (1996).

[2] C.-M. Marle, *C. R. Acad. Sci. Paris* **311**, 839 (1990).

[3] C.-M. Marle, *Commun. Math. Phys.* **174**, 295 (1995).

[4] X. Gràcia, J. Marìn-Solano, M.-C. Mũnoz-Lecanda, Some Geometric Aspects of Variational Calculus in Constrained Systems, `math-ph/0004019`.

[5] J. Schwinger, *Lectures on Quantum Dynamics*, Les Houches Summer School (1955).

[6] J.E. Marsden, T.S. Ratiu, J. Scheurle, *J. Math. Phys.* **41**, 3379 (2000).

[7] J.F. Cariñena, *Rend. Sem. Mat. Univ. Pol. Torino* **54**, 245 (1996).

[8] J.F. Cariñena, C. López, *Rev. Math. Phys.* **3**, 379 (1991).

[9] A. Ibort, M. de León, E.A. Lacomba, D.M. de Diego, P. Pitanga *J. Phys. A: Math. Gen.* **30**, 5835 (1997).

[10] M. de León, D.M. de Diego, *J. Math. Phys.* **37**, 3389 (1996).

[11] M. de León, J.C. Marrero, D.M. de Diego, *J. Phys. A: Math. Gen.* **30**, 1167 (1997).

[12] G. Pidello, W.M. Tulczyjew, *Ann. Mat. Pura Appl.* **147**, 250 (1987).

[13] A. Frölicher, A. Nijenhuis, *Indag. Math.* **18**, 338 (1956).