ON A TOPOLOGICAL $\mathcal{N} = 4$ YANG–MILLS THEORY

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We show, starting from simple differential geometric example, that the partition function of a twisted $\mathcal{N} = 4$ Yang–Mills theory on certain manifold X is localized on instanton moduli space. Moreover, it equals to the Euler characteristic of this moduli space.

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1. Introduction

Topological Quantum Field Theories of Co-homological type (CQFT) were first proposed by Witten and Atiyah in the eighties [5]. The construction called *twisting* applied on $\mathcal{N} = 2$ Yang–Mills (YM) theory has led to the theory whose correlation functions of observables are topological invariants of four manifolds which were identified with Donaldson's polynomial invariants. These ideas were then applied by Witten to two dimensional σ-models [6]. The concept of CQFT has given a physical method for computing invariants [7] in several interesting cases.

However, there exists another, physically more important super-symmetric extension of YM theory in four dimensions. This is $\mathcal{N} = 4$ YM theory which has the maximal number of independent super-symmetries in $d = 4$. It is believed that this theory has exactly vanishing β-function. Moreover, the relevance of $\mathcal{N} = 4$ YM follows from the Maldacena's conjecture, *i.e.* it serves as the dual picture in description of IIB-type string theory on the $AdS_5 \times S^5$ background.

Thus there is a natural question of constructing CQFT from $\mathcal{N} = 4$ YM using this twisting method. This paper is devoted to the analysis of certain topological features of such a twisted theory.

The paper is organized as follows. In Sec. 2 we review the twisting method. In Sec. 3 we deal with localization of a twisted $\mathcal{N} = 4$ Yang–Mills theory partition function, conclusions are in the last section.

2. Twisting procedure

Twisting method may be in general described as follows. Consider an Euclidean super-symmetric Yang–Mills theory on a four dimensional manifold X. Then the symmetry group is locally $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{I}$ where the first two factors stay for rotational group and I is the internal symmetry group which "rotates" the super-charges. Now we want to change the rotational group in order to obtain at least one scalar supercharge which will serve as a BRST cohomology operator. We choose for instance $\mathrm{SU}(2)_\mathrm{L}$ fixed and replace $\mathrm{SU}(2)_\mathrm{R}$ with the $\mathrm{diag}\,(\mathrm{SU}(2)_\mathrm{R} \times \mathrm{K})$, where K is a SU(2) copy embedded in internal symmetry group I. We review this procedure in the case of $\mathcal{N} = 2, 4$ Yang–Mills theory.

2.1. Case of $\mathcal{N} = 2$ Yang–Mills theory

In this theory we have the field content $A_\mu, \lambda^A_\alpha, \bar{\lambda}^A_{\dot{\alpha}}, \phi$, which means ordinary gauge field, two anti-commuting gauginos and scalar field respectively. The corresponding Lagrangian can be found in [5]. Here we focus on the internal group I which is (neglecting U(1) R-symmetry) SU(2). The present super-charges $Q^A_\alpha, \bar{Q}^A_{\dot{\alpha}}$ transform under $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{SU}(2)$ as $(\mathbf{2}, \mathbf{1}, \mathbf{2})$, $(\mathbf{1}, \mathbf{2}, \mathbf{2})$.

Since $\mathrm{I} = \mathrm{SU}(2)$ there is only one (trivial) possibility for embedding a SU(2) copy. The new rotational group will be simply $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{diag}\,(\mathrm{SU}(2)_\mathrm{R} \times \mathrm{I})$. The super-charges split as $Q^A_\alpha \to Q_\mu$, $\bar{Q}^A_{\dot{\alpha}} \to Q \oplus Q_{\mu\nu}$ and transform under new (twisted) rotational group as $(\mathbf{2}, \mathbf{2})$, $(\mathbf{1}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{3})$. The fields decompose as $A_\mu \to A_\mu$, $\lambda^A_\alpha \to \psi_\mu$, $\bar{\lambda}^A_{\dot{\alpha}} \to \eta \oplus \chi_{\mu\nu}$ and $\phi \to \phi$ and transform under new rotational group as $(\mathbf{2}, \mathbf{2})$, $(\mathbf{2}, \mathbf{2})$, $(\mathbf{1}, \mathbf{1})$, $(\mathbf{1}, \mathbf{3})$ and $(\mathbf{1}, \mathbf{1})$.

The scalar supercharge Q is identified with BRST operator and its action is nilpotent up to a gauge transformation. However, it is possible to modify it to obtain strictly nilpotent one [1] and then one may define observables as its cohomology classes. The resulting theory has $T_{\mu\nu} = \{Q, \Lambda_{\mu\nu}\}$ (Q-exact energy-momentum tensor) and it is a kind of CQFT mentioned in the introduction.

2.2. Case of $\mathcal{N} = 4$ Yang–Mills theory

Now, we focus on the case of $\mathcal{N} = 4$. Here the situation is more complicated. The $\mathrm{I} = \mathrm{SU}(4)$ now and there are several possibilities how to embed a SU(2) copy in it. If one needs to have at least one scalar supercharge, then there are only three possibilities which are characterized by the following four dimensional representations of SU(2): $\mathbf{4} \to (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2})$,

$\mathbf{4} \to (\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1})$, $\mathbf{4} \to (\mathbf{2},\mathbf{2})$. Correspondingly, we have three inequivalent CQFT. The second embedding leads to the theory of non-Abelian adjoint monopoles [2], the third one is called amphi-chiral theory since it is untouched changing the orientation. We will focus on the topological character of the first one.

Let us recall the field content of $\mathcal{N}=4$ YM. In $\mathcal{N}=1$ formalism there are three chiral superfields Φ_p containing scalars B_p, gauginos $\lambda_{p\alpha}$ and a vector superfield V which contains gauge field A_μ and fourth gaugino $\lambda_{4\alpha}$ and $\bar{\lambda}_{4\dot{\alpha}}$. Scalars B_p are usually considered as elements of anti-symmetric tensor

$$\phi_{\mu\nu} = \begin{pmatrix} 0 & -B^{\dagger 3} & B^{\dagger 2} & -B_1 \\ B^{\dagger 3} & 0 & -B^{\dagger 1} & -B_2 \\ -B^{\dagger 2} & B^{\dagger 1} & 0 & -B_3 \\ B_1 & B_2 & B_3 & 0 \end{pmatrix} . \tag{1}$$

All fields are in adjoint representation of gauge group. Supercharges Q^A_α and $\bar{Q}^A_{\dot{\alpha}}$ transform under $\mathrm{SU(2)_L}\times\mathrm{SU(2)_R}\times\mathrm{SU(4)}$ as $(\mathbf{2},\mathbf{1},\mathbf{4})$ and $(\mathbf{1},\mathbf{2},\bar{\mathbf{4}})$. Under the twisted rotational group $\mathrm{SU(2)_L}\times\mathrm{diag}\,(\mathrm{SU(2)_R}\times \mathrm{K})$, where K is determined by $(\mathbf{1},\mathbf{2})\oplus(\mathbf{1},\mathbf{2})$, the supercharges decompose as $Q^A_\alpha \to Q^i \oplus Q^i_{\alpha\beta}$ and $\bar{Q}_{A\dot{\alpha}} \to \bar{Q}^i_{\alpha\dot{\alpha}}$. The field decomposition is $A_\mu \to A_\mu$, $\lambda_{A\alpha} \to \psi^{i\alpha}{}_{\dot{\alpha}}$, $\bar{\lambda}^A_{\dot{\alpha}} \to \chi_{i\beta\alpha} \oplus \eta_i$ and $\phi_{\mu\nu} \to \varphi_{ij} \oplus G_{\alpha\beta}$. There is still a subgroup I' of I which commutes with realization of K. It is of course isomorphic to SU(2) and plays the role of internal symmetry group of the twisted theory.

The action of twisted theory is quite complicated [2]. Its bosonic part, not involving φ_{11}, is

$$\mathcal{S}^{\mathrm{bos}}_{\mathrm{twist}} = \frac{1}{2\mathrm{e}^2}\int\limits_X d^4x\sqrt{|g|}$$
$$\times\,\mathrm{Tr}\left[\left(F^+_{\mu\nu}+\frac{1}{4}\left[B_\mu{}^\delta, B_{\nu\delta}\right]+\frac{1}{2}\left[C, B_{\mu\nu}\right]\right)^2 + \left(\mathcal{D}^\gamma B_{\mu\gamma}+\mathcal{D}_\mu C\right)^2\right], \tag{2}$$

where $C = i\varphi_{12}$, F^+ is the self-dual part of F. The corresponding equations of motion are

$$\begin{aligned} F^+_{\mu\nu} + \frac{1}{4}\left[B_\mu{}^\delta, B_{\nu\delta}\right] + \frac{1}{2}\left[C, B_{\mu\nu}\right] &= 0\,, \\ \mathcal{D}^\gamma B_{\mu\gamma} + \mathcal{D}_\mu C &= 0\,, \end{aligned} \tag{3}$$

and for $B_{\mu\nu} = 0$ they reduce to $F^+_{\mu\nu} = 0$, *i.e.* A_μ is an anti-self-dual connection and $\mathcal{D}_\mu C = 0$, *i.e.* C is covariantly constant. We are going to show that these equations are important for localization of partition function of the theory.

3. Localization

We start with a simple differential geometric model.

Suppose now $n = 2m$ and we have an n-dimensional compact manifold M with spin structure, *i.e.* we have an isomorphism of TM with bundle $P \times_{\mathrm{Spin}(n)} V$ associated with a principal $\mathrm{Spin}(n)$ bundle P over M and standard representation of $\mathrm{Spin}(n)$ on $V = \mathbb{R}^n$. Hence $P \times V$ is a principal $\mathrm{Spin}(n)$ bundle over TM which is equal to π^*P, π is the TM projection. The forms $\Omega(TM)$ can be identified with *basic forms* of $\Omega(P \times V)$, *i.e.* the forms which satisfy $\mathrm{R}_g^*\omega = \omega$, $i_X\omega = 0$ for all g in $\mathrm{Spin}(n)$ and all X in its Lie algebra. In particular, the identification isomorphism is given by pull-back with respect to the projection $f\colon\ P \times V \to TM$.

Next, suppose we have the spin connection θ in P. This connection may be pulled back to $P \times V$. We denote this connection by the same symbol. If one considers its curvature

$$\Omega = d\theta + \theta \wedge \theta\,, \tag{4}$$

then one can define a form on $\Omega(P \times V)_{\mathrm{basic}} \simeq \Omega(TM)$ putting

$$U = \pi^{-m}\mathrm{e}^{-x^2} \sum_{\mathrm{I}\ (\mathrm{even})} \varepsilon\,(\mathrm{I},\mathrm{I}')Pf\left(\frac{\Omega_{\mathrm{I}}}{2}\right)(dx + \theta x)^{\mathrm{I}'}\,. \tag{5}$$

Here I runs over all subsets of $\hat{n}$ with even cardinality, I' is its complement in $\hat{n}$, $Pf(\Omega/2)$ is the Pfaffian of the sub-matrix of $\Omega/2$ with indices in I, x are coordinates on V and $\varepsilon(\mathrm{I},\mathrm{I}')$ is the sign of permutation transforming $\mathrm{I}\,\mathrm{I}'$ to $1\dots n$; see [4] for further details. In fact, one can show that this form is closed [3] and integrates to 1 over the fibers as is easily seen from the identity

$$\pi^{-m}\int\limits_{\mathbb{R}^n} \mathrm{e}^{-x^2}\, d^n x = 1\,. \tag{6}$$

U represents the Thom class of TM. Of course, one needs to have a compact support for such Thom class representative, but this can be reached using some orientation preserving diffeomorphism of TM onto some disc bundle for instance; details can be found in [3,4]. The form U may be expressed as

$$U = \pi^{-m}\int \mathcal{D}\eta\,\mathrm{e}^{-x^2+\frac{1}{2}\eta^t\Omega\eta+(dx+\theta x)^t\eta}\,, \tag{7}$$

where $\int\mathcal{D}\eta$ is the Berezin integration with respect to odd variables $\eta_1,\dots\eta_n$, $(dx+\theta x)^t$ denotes the transpose.

If we denote by $s : M \to TM$ section with isolated zeroes then s^*U represents the Euler class of M and we have the well-known formula

$$\int\limits_M s^*U = \sum_i \mathrm{Ind}_i(s)\,. \tag{8}$$

This can be proved in greater generality for an arbitrary oriented vector bundle over a compact manifold, considering the family of sections $\{s_t\}_{t\in\mathbb{R}^+}$, $s_t := ts$ for such s. All the forms s_t^*U represent the same cohomology class and thus for $t\to 0$ we obtain the Euler class by the definition. The integral $\int s_t^*U$ remains the same for all t and for $t\to+\infty$ one obtains right-hand side of (8). However, in the case of TM the famous Hopf theorem identifies right-hand side of (8) with the Euler characteristic $\chi(M)$. Finally if we choose such a section s then we have from (7)

$$\chi(M) = \pi^{-m}\int\limits_M\int \mathcal{D}\eta\,\mathrm{e}^{-s^2+\frac{1}{2}\eta^t(s^*\Omega)\eta+(\nabla s)^t\eta}\,, \tag{9}$$

which works even for $s=0$ from the reasoning above.

The right-hand side part of (9) may be interpreted as the partition function Z of a "quantum mechanical" model with bosons and fermions represented by the coordinates on M and Grassmann variables $\{\eta_i\}$. The partition function Z of this "quantum" theory is then simply localized on the finite subset of isolated points in M.

Suppose now, that the section s has the zero locus in the form $\cup_{i=1}^q M_i$, $\dim M_i = n_i$. Moreover, let s has nondegenerate behavior in normal directions with respect to all M_i. This means we can define coordinates and trivialization near M_i such that u_j, $j = 1\ldots n-n_i$ are normal to M_i, $s^k = A^k_j u_j$, $j,k = 1\ldots n-n_i$ for some matrix A and $s^k=0$, $k>n-n_i$. One usually thinks about A as to be the full $n\times n$ matrix with other components zero. In this situation we can calculate right-hand side of (9). Once again we introduce $s_t = ts$ family of section where the parameter t may be interpreted as a coupling constant. As we said above, the integral is independent of t and for $t\to+\infty$ it will be a sum of contributions from all M_i. If one does the integration over normal M_i coordinates and then over tangent ones, the right-hand side of (9) will be

$$Z = \sum_i \pm\chi(V_i)\,, \tag{10}$$

where V_i is bundle of η zero modes over M_i and the sign denotes the relative orientation of V_i and TM_i.

The formula (10) has one disadvantage. It contains $\pm$ dependence. To eliminate this we can use a technique from localization theory. Suppose we are in the situation where $s^k = 0$, $k = 1 \ldots n^{'}$, $n^{'} < n$, so now $\dim M_i = n - n^{'}$. We extend the manifold M in order to depend on additional $\{z_k\}_{k=1}^{n}$ variables and introduce the additional functions

$$r_j = \sum_k z_k \frac{\partial s^k}{\partial u^j} + O(z^2),\, j = 1 \ldots n\,. \tag{11}$$

Now let us consider topological invariant of $s^k = r_j = 0$. Then we have a theorem

Theorem 3.1 *Suppose we have a theory whose partition function is localized on the sub-manifold defined by* $s^k = r_j = 0$*, situation is nondegenerate and (11) holds. Moreover, let all the solutions of* $s^k = r_j = 0$ *be localized at* $z = 0$*. Then*

$$Z = \chi(M^{'})\,, \tag{12}$$

where $M^{'}$ *is the space of solutions of* $s^k = 0$*.*

The proof is straightforward. The condition (11) ensures two things. Firstly, all signs for the r-extended system are positive and secondly, $V_i = TM_i$ in this case. So we can proceed as in deriving (10) and we obtain (12). The addition of r_j function may seem to be a little bit uncomfortable, but it indeed reflects the situation in our twisted $\mathcal{N} = 4$.

However, we have to generalize our model for the case of gauge invariant theory. Consider a compact g dimensional gauge group G that acts on M without fixed points. Consider a $n = 2m + g$ dimensional manifold M and a bundle V over M of rank $2m$ which is oriented and equipped with an inner product. In this situation some of r_j could be removed using gauge invariance. Next, we must have r_j tangent to the pull-back of $T(M/G) \to M/G$ to M if we want to divide simply by G and use what we just have done. We start with n invariant r_j functions and take into account a new adjoint valued function C^h on M. If $r_j = -C^h T_{hj}$, where we denote by T_{hj} the components of the vector field induced on M by the element T_h of some chosen basis in $\underline{G}$, then such r_j are removed "dividing" by G since they are tangent to the gauge orbits. This C thus plays the role of projection of r_j onto the part orthogonal to the gauge orbits (using the metric on M; the parts tangent to gauge orbits are thus removed). Let us put

$$r_j^{'} := r_j + C^h T_{hj}\,. \tag{13}$$

Then we can divide by G and use Theorem 3.1. Thus we have generalized

Theorem 3.2 *Suppose we have a theory whose partition function is localized on the sub-manifold defined by $s^k = r_j^{'} = 0$, the situation is nondegenerate and (11) holds. Moreover, let all the solutions of $s^k = r_j^{'} = 0$ be localized at $z = C = 0$ and are nondegenerate as before. Then after integrating out C*

$$Z = \chi\left(\frac{M^{'}}{G}\right), \tag{14}$$

where $M^{'}$ is the space of solutions of $s^k = 0$.

Finally, let us consider our twisted $\mathcal{N}=4$. The role of M plays the space of connections on a G bundle over the space of the theory X. The bundle V is identified with the bundle of self-dual adjoint valued two forms. Equations (3) play the role of sections $s_{\mu\nu}$ and $r_\mu^{'}$ which correspond to the previous s^k, $r_j^{'}$, respectively. The independent variables are now A_μ, $B_{\mu\nu}$, C which play the role of the previous u_i, z_j and C^h. The bosonic action (2) may be rewritten as

$$\begin{aligned}
\mathcal{S}^{bos.}_{twist} &= \frac{\|s\|^2 + \|r^{'}\|^2}{2\mathrm{e}^2} = \frac{1}{2\mathrm{e}^2}\int_X d^4x\sqrt{|g|} \\
&\times \mathrm{Tr}\left[F^{+\,2}_{\mu\nu} + \frac{(\mathcal{D}_\gamma B_{\mu\nu})^2}{4} + (\mathcal{D}_\mu C)^2 + \frac{[B_{\mu\delta}, B_{\nu\delta}][B_{\mu\beta}, B_{\nu\beta}]}{16}\right. \\
&\left. + \frac{[C, B_{\mu\nu}]^2}{4} + B_{\mu\nu}\left(\frac{(g_{\mu\delta}g_{\nu\gamma} - g_{\mu\gamma}g_{\nu\delta})R}{6} + W^+_{\mu\nu\delta\beta}\right)\frac{B_{\delta\gamma}}{4}\right],
\end{aligned} \tag{15}$$

where W^+ is the self-dual part of Weyl tensor, R is scalar curvature. If the curvature is such that

$$B_{\mu\nu}\left(\frac{(g_{\mu\delta}g_{\nu\gamma} - g_{\mu\gamma}g_{\nu\delta})\,R}{6} + W^+_{\mu\nu\delta\beta}\right)\frac{B_{\delta\gamma}}{4} > 0\,, \tag{16}$$

whenever $B_{\mu\nu} \neq 0$, then $s_{\mu\nu} = r_\mu^{'} = 0$ implies $B_{\mu\nu}=0$. Thus for such a manifold X we have a localization condition for B. However, we need one for C, too. $B_{\mu\nu} = 0$ implies $\mathcal{D}_\mu C = 0$ immediately. This means that if $C \neq 0$ then we have a reducible instanton and correspondingly our action on M will not be without fixed points. Therefore, we need some further restrictions on X. If we restrict ourselves to manifolds with $b_2^+ > 0$ (positive dimension of the space of self-dual harmonic two forms), then such instantons do not exist there. $b_2^+ > 0$ trivially for any Kähler manifold since the Kähler form itself is self-dual. Thus we have $C = 0$ too, and we may use Theorem 3.2 again. We therefore conclude with the following statement which is our final generalization of (9):

Theorem 3.3 *Let X be a four manifold with $b_2^+ > 0$ such that (16) is satisfied for every $B_{\mu\nu} \neq 0$. Then the partition function Z of $\mathbf{4} \to (\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2})$ twisted Yang–Mills theory on X is equal to the Euler characteristics of the moduli space of instantons.*

4. Conclusions

Our understanding of non-perturbative aspects of Yang–Mills theories made the strong progress in 1994 when Seiberg and Witten formulated their hypothesis about the form of Wilsonian effective action. This hypothesis was then successfully tested via instanton calculus. The localization theory applied on $\mathcal{N} = 4$ Yang–Mills theory is one of the most powerful methods for testing such aspects. Moreover, the computational method for certain class of correlation functions in $\mathcal{N} = 2$ Yang–Mills theory, using the moduli space differential analysis, has been published recently [1]. There is a natural question if there is a possibility of generalizing these results to the $\mathcal{N} = 4$ case on special types of manifolds. Then it is clear that such a thing would have great importance for AdS/CFT correspondence, which is very popular nowadays. We are concerned with these generalizations in present research.

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REFERENCES

[1] D. Bellisai, F. Fucito, A. Tanzini, G. Travaglini, hep-th/0003272.

[2] J.M.F. Labastida, C. Lozano, hep-th/9702106.

[3] V. Mathai, D. Quillen *Topology*, **25** 1, 85, 1985.

[4] O. Vana, math-ph/0111023.

[5] E. Witten *Commun. Math. Phys.* **117**, 353 1988.

[6] E. Witten *Commun. Math. Phys.* **118**, 411 1988.

[7] E. Witten *J. Math. Phys.* **35**, 5101 1994.