

SIMPLE ANALYTIC EVALUATION OF THE INTEGRALS IN THE GAUSSIAN MODEL OF BOSE–EINSTEIN CORRELATIONS*

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A generating function for products of Hermite polynomials is used to significantly simplify the evaluation of the integrals $G_n(p, p')$ occurring in the Gaussian model of multiple particle production. These integrals are crucial for studies of multi-particle effects in Bose–Einstein correlations.

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1.

Bose–Einstein correlations in multiple particle production processes are a field of intense study. For recent reviews see [1–4]. An important subfield is the study of multi-particle effects on multiplicity and momenta distributions. Here exactly soluble models are a convenient starting point. A soluble model, which is particularly popular, is the Gaussian model [5–13]. In this model all the integrations are Gaussian and consequently can be performed analytically. The generic integral is

$$G_n(p, p') = \int \rho(p, k_1) dk_1 \rho(k_1, k_2) \cdots dk_{n-1} \rho(k_{n-1}, p'), \quad (1)$$

where

$$\rho(p, p') = \frac{1}{\sqrt{2\pi\Delta^2}} \exp\left(-\frac{K^2}{2\Delta^2} - \frac{1}{2}R^2q^2\right), \quad (2)$$

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with

$$K = \frac{p + p'}{2}, \quad q = p - p'. \quad (3)$$

The constants R^2 and Δ^2 are real and positive. For simplicity we start with the one-dimensional case. In the Gaussian model the transition to three dimensions is trivial. It is seen that the integrals G_n are Gaussian and consequently can be performed analytically. One needs, however, a closed formula valid for any n . This is more difficult to obtain. Zimányi and Csörgö [6, 7] following some suggestions from Pratt [5] constructed a system of recurrence relations for three series of parameters and after a rather lengthy calculation got a formula for the integrals, which can be written in a fairly simple form [12]. Wiedemann [13] applied successfully the MATHEMATICA program directly to the integrals G_n . In the present note we point out that a very short derivation of the formula for the integrals G_n can be given, if a mathematical identity for Hermite polynomials is made use of. This work is an extension of the method used in [8, 9].

2.

An identity, valid for $|z| < \frac{1}{2}$, is

$$\begin{aligned} \Phi(x, y, z) &\equiv \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x) H_n(y) \\ &= \frac{1}{\sqrt{1-4z^2}} \exp \left(-4z \frac{z(x^2 + y^2) - xy}{1-4z^2} \right). \end{aligned} \quad (4)$$

A convenient starting point to check this formula is the definition of the Hermite polynomials in the form [14]¹

$$H_n(z) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{+\infty} (z + it)^n e^{-t^2} dt. \quad (5)$$

After substituting it into (4) the summation over n gives an exponential and it is enough to perform a double Gaussian integral to get the right-hand side. The function $\Phi(x, y, z)$ is defined by this formula.

¹ In this edition of the tables the power of two is mistakenly written as $n/2$ instead of n . In order to check the formula it is enough to expand the binomial into powers of z , integrate term by term and compare the coefficients with the formulae given in [15].

3.

For any orthonormal set of functions $\psi_n(p)$ and any set of nonnegative numbers λ_n satisfying the normalization condition

$$\sum_n \lambda_n = 1, \quad (6)$$

the function

$$\rho(p, p') = \sum_n \psi_n(p) \lambda_n \psi_n^*(p') \quad (7)$$

can be interpreted as a well-defined single-particle density matrix in the momentum representation. The functions ψ_n are its eigenfunctions and the numbers λ_n are the corresponding eigenvalues. Let us choose in particular for the functions ψ_n the energy eigenfunctions in the momentum representation of a one-dimensional harmonic oscillator and for the numbers λ_n a geometric progression [8, 9]:

$$\psi_n(p) = \sqrt{\frac{\alpha}{\sqrt{\pi} 2^n n!}} \exp\left(-\frac{\alpha^2 p^2}{2}\right) H_n(\alpha p), \quad \lambda_n = (1-z)z^n. \quad (8)$$

It is assumed here that $|z| < 1$, so that the eigenvalues λ_n are correctly normalized. The parameter α is positive. Its interpretation in terms of the mechanical parameters of the oscillator is of no importance here. This choice corresponds to a single particle density matrix which using (4) can be written in the form

$$\rho(p, p') = \frac{\alpha(1-z)}{\sqrt{\pi}} e^{-\frac{1}{2}\alpha^2(p^2+p'^2)} \Phi(\alpha p, \alpha p', \frac{z}{2}), \quad (9)$$

or substituting the definition of the function Φ and replacing the parameters p, p' by the parameters K, q according to (3)

$$\rho(K, q) = \frac{\alpha}{\sqrt{\pi}} \sqrt{\frac{1-z}{1+z}} \exp\left(-\alpha^2 K^2 \frac{1-z}{1+z} - \frac{\alpha^2 q^2}{4} \frac{1+z}{1-z}\right). \quad (10)$$

This density matrix differs only by notation from the density matrix (2). The correspondence between the parameters is

$$\alpha = \sqrt{\frac{R}{\Delta}}, \quad z = \frac{v-1}{v+1}. \quad (11)$$

Following [12] we have introduced the notation

$$v = 2\Delta R \geq 1, \quad (12)$$

where the inequality follows from the uncertainty principle.

4.

In order to calculate the function $G_n(p, p')$ let us note first that it is the p, p' matrix element of the n -th power of the matrix ρ . Therefore, it has the same eigenfunctions and its eigenvalues are n -th powers of the corresponding eigenvalues of matrix ρ . Thus

$$\begin{aligned} G_n(p, p') &= \sum_{k=0}^{\infty} \psi_k(p) \psi_k(p') \lambda_k^n \\ &= \frac{\alpha(1-z)^n}{\sqrt{\pi}} e^{-\frac{1}{2}\alpha^2(p^2+p'^2)} \Phi(\alpha p, \alpha p', \frac{z^n}{2}). \end{aligned} \quad (13)$$

Substituting the definition of function Φ and changing variables from p, p' to K, q we find

$$G_n(K, q) = \frac{\alpha(1-z)^n}{\sqrt{\pi(1-z^{2n})}} \exp\left(-\alpha^2 K^2 \frac{1-z^n}{1+z^n} - \frac{1}{4}\alpha^2 q^2 \frac{1+z^n}{1-z^n}\right). \quad (14)$$

It is convenient to rewrite this formula in the form [12]

$$G_n(K, q) = \frac{c_n}{\sqrt{2\pi\Delta^2}} \exp\left(-\frac{K^2}{2\Delta_n^2} - \frac{1}{2}R_n^2 q^2\right). \quad (15)$$

By comparison with the previous form $\Delta_n = a_n \Delta^2$, $R_n^2 = a_n R^2$ with

$$a_n = \frac{1}{v} \frac{1+z^n}{1-z^n} = \frac{1}{v} \frac{(v+1)^n + (v-1)^n}{(v+1)^n - (v-1)^n} \quad (16)$$

and

$$c_n = \frac{2^n \sqrt{v}}{\sqrt{(v+1)^{2n} - (v-1)^{2n}}}. \quad (17)$$

5.

Let us add some comments.

- As mentioned, the transition from the one-dimensional case to d -dimensions is very simple. The density matrix (2) gets replaced by a product

$$\rho(\vec{p}, \vec{p}') = \prod_{i=1}^d \frac{1}{\sqrt{2\pi\Delta_i^2}} \exp\left(-\frac{K_i^2}{2\Delta_i^2} - \frac{1}{2}R_i^2 q^2\right). \quad (18)$$

Consequently, the integrals G_m from (1) factorize and one obtains for each G_m a product of expressions (15), in general with different values of Δ^2 and v for each factor.

- For the harmonic oscillator the energy eigenfunctions in the momentum representation and in the coordinate representation have the same form. Besides the replacement of p by x , the interpretation of the parameter α changes, but this is irrelevant for the present analysis. Thus, the results given in the present paper can be used, after some changes in notation, also for the study of Bose–Einstein correlations in ordinary space.
- Since the energy eigenvalues of the harmonic oscillator are equidistant, the corresponding Boltzmann factors form a geometrical progression. Thus, formula (9) as well as the corresponding formula in the coordinate representation, can be used to calculate the canonical density matrix for the harmonic oscillator. According to Landau and Lifszyc [16], this calculation for the diagonal elements of the density matrix in the coordinate representation has been first performed by F. Bloch in year 1932.

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