CHAOS ENFORCED INSTANTON TUNNELLING IN ONE-DIMENSIONAL MODEL WITH PERIODIC POTENTIAL

V. Kuvshinov, A. Kuzmin and R. Shulyakovsky

Institute of Physics, National Academy of Sciences of Belarus
Scarina av., 68, 220072 Minsk, Belarus
e-mail: kuvshino@dragon.bas-net.by
avkuzmin@dragon.bas-net.by
shul@dragon.bas-net.by

(Received February 7, 2002; revised version received April 10, 2002)

The influence of chaos on properties of dilute instanton gas in quantum mechanics is studied. We demonstrate on the example of one-dimensional periodic potential that small perturbation leading to chaos squeezes instanton gas and increases the rate of instanton tunnelling.

PACS numbers: 03.65.Xp, 05.45.Mt

1. Introduction

Semiclassical properties of classically chaotic dynamical systems (quantum chaos [1]) is a rapidly developing field of research. One of the attractive phenomenon obtained in this direction is chaos assisted tunnelling ([2] and references therein).

In this work we analytically investigate influence of the perturbation leading to chaos on the rate of instanton [3] transitions on the example of one-dimensional periodic potential. Instantons represent alternative way to describe quantum tunnelling and play an important role not only in quantum mechanics (recent example [4]), but mainly in the modern theories of particle physics, *e.g.* electroweak theory, QCD, SUSY, *etc.* (for review see [5]). The search of instantons, initiated in DESY, is one of the targets of modern QCD [6]. The main problem in this direction is an extremely small probability of instanton-induced events. It is of importance to study if chaos can increase probability of instanton tunnelling. We demonstrate on the particular example of simple quantum mechanical system that it is really possible.

Hamiltonian of the system under consideration is taken in the form

$$\tilde{H} = \frac{1}{2}\tilde{p}^2 + \omega_0^2 \cos x - \varepsilon x \sum_{n=-\infty}^{+\infty} \delta(t - n\tilde{T})\,, \tag{1}$$

$\tilde{T}$ is the real time period of perturbation. The systems with spatially periodic potential are well-studied in solid-state physics [7] and instanton physics [8]. Perturbation used in (1) was widely exploited in the systems exhibiting quantum chaos [9].

For applying instanton technique we have to consider classical solutions of Hamilton's equations in *imaginary* (Euclidean) time. Hamiltonian (1) has the same form (translated on π) in Euclidean time as in real one.

In Euclidean time classical Hamiltonian of the system looks as follows $H = H_0 + V$,

$$H_0 = \tfrac{1}{2}p^2 - \omega_0^2 \cos x\,, \tag{2}$$

$$V = \varepsilon x \sum_{n=-\infty}^{+\infty} \delta(\tau - nT)\,. \tag{3}$$

Here H_0 is non-perturbed Hamiltonian of the system and V is the Euclidean potential of the perturbation. In these expressions variables without tilde denote corresponding quantities in imaginary time.

2. Instanton gas in non-perturbed system

Let us consider at first non-perturbed Hamiltonian (1), *i.e.* put $\varepsilon = 0$. The system is characterised by degenerate vacuum structure on the classical level:

$$x_n^{\mathrm{vac}}(t) \equiv |n\rangle = \pi n, \quad n = \pm 1, \pm 3\,, \ldots\,. \tag{4}$$

As is well-known any tunnelling transition in this system can be represented in terms of path integral in imaginary time:

$$\langle m|\mathrm{e}^{-H_0 \Gamma}|n\rangle = \int [Dx]\mathrm{e}^{-S[x]}\,, \tag{5}$$

where Γ is a time of the transition between wells, $S[x]$ denotes Euclidean action. The main contribution to (5) is given by instantons. One-instanton configurations are classical solutions of Euclidean equations of motion and describe tunnelling between neighbouring vacua. They can be easily found as well as one-instanton action (see, for example [8])

$$x^{\mathrm{inst}}(\tau, \tau_0) = \pm\left(\arctan\left[\mathrm{e}^{\omega_0(\tau-\tau_0)}\right] - \pi\right)\,, S[x^{\mathrm{inst}}(\tau, \tau_0)] \equiv S^{\mathrm{inst}} = 8\omega_0\,, \tag{6}$$

where arbitrary parameter τ_0 is a centre of the instanton; sings '+' and '−' correspond to instantons and anti-instantons.

Multi-instantons are not exact classical solutions, but they give leading contribution to the amplitude of tunnelling (5) between distant wells

$$x^{(n)}(\tau) = \sum_{i=1}^{n} x^{\mathrm{inst}}(\tau, \tau_i)\,, \qquad S[x^{(n)}(\tau)] = nS^{\mathrm{inst}}\,. \tag{7}$$

Here we suppose that time intervals between centres of single instantons τ_i are not too close to each other (dilute instanton gas approximation) [8].

Let us focus our attention on the amplitude $A_q(\Gamma)$ of q one-instanton transitions during time interval Γ. The difference between instantons and anti-instantons is not essential in these calculations. Using standard instanton technique [8] it is easy to obtain Poisson distribution of q for the amplitude in the Gauss approximation:

$$A_q(\Gamma) = N\frac{1}{q!}\mathrm{e}^{-qS^{\mathrm{inst}}}\left(\Gamma\sqrt{S^{\mathrm{inst}}}\right)^q, \tag{8}$$

where the factor N provides correct normalization, $A_0(\Gamma)$ means absence of instantons. The average number of instantons for the time Γ is

$$\langle q\rangle = \mathrm{e}^{-S^{\mathrm{inst}}}\Gamma\sqrt{S^{\mathrm{inst}}}\,. \tag{9}$$

Thus in imaginary time we have a gas which consists of instantons (6) with average time interval between them η_0 and average density ρ_0:

$$\eta_0 = \frac{\Gamma}{\langle q\rangle} = \frac{\mathrm{e}^{S^{\mathrm{inst}}}}{\sqrt{S^{\mathrm{inst}}}}, \qquad \rho_0 = \eta_0^{-1} = \mathrm{e}^{-S^{\mathrm{inst}}}\sqrt{S^{\mathrm{inst}}} = 2\sqrt{2\omega_0}\mathrm{e}^{-8\omega_0}\,. \tag{10}$$

Here, as usual, we neglected the instanton size.

Let us suppose a large value of one-instanton action (6). It corresponds to high energy barriers. In this case we obtain strongly rarefied instanton gas.

It should be noted that result (10) was obtained by using both exact (6) and approximate (7) classical solutions. Taking into account only exact solutions leads to zero instanton density ρ_0 (or infinite interval η_0). Finite density can appear in non-perturbed system (1) only due to approximate solutions contribution.

3. Squeezing of instanton gas due to small perturbation

Now we consider the case with $\varepsilon \neq 0$ and estimate an average interval between instanton transitions (inverse density of instantons in instanton gas) for the perturbed system. For this purpose we represent the perturbation (3) in the form

$$V = \frac{\varepsilon}{T} x \left(2 \sum_{m=-\infty}^{+\infty} \cos\left(m\nu\tau\right) + 1 \right). \tag{11}$$

Here $\nu \equiv 2\pi/T$.

Perturbation (11) destroys separatrix of non-perturbed system (2) and in its place stochastic layer appears [10]. We primarily estimate the width of stochastic layer. We use the method reviewed in [11]. We apply it to the system (2)–(3) which was not considered in that works, although non-perturbed Hamiltonian (2) and perturbation (3) were used independently from each other.

Exact equation of motion for the action variable is

$$\dot{I} = \frac{dI}{dH_0}\dot{H}_0 = -\frac{\varepsilon}{\omega T}\dot{x} \left(2 \sum_{m=0}^{+\infty} \cos\left(m\nu\tau\right) + 1 \right), \tag{12}$$

Here I denotes the action variable and $\omega \equiv dH_0/dI$ is a frequency of non-linear oscillations (see [10]).

Consider behaviour of the system near separatrix (in imaginary time). Dependence of velocity $\dot{x}$ on time has the form of rare soliton-like impulses. Each impulse corresponds to rapid transition between two neighbouring peaks of potential. Long time interval between two impulses is the time to get over a peak. It is seen from equation (12) that the action variable changes mainly during the impulse of velocity. Introduce phase of external force φ defined as $\dot{\varphi} = \nu$.

To study chaotic behaviour of the system we transform differential equation (12) to discrete mapping

$$\begin{cases} \overline{I} = I + \frac{\varepsilon}{T\omega(I)} C(I, \varphi), \\ \overline{\varphi} = \varphi + \frac{\pi\nu}{\omega(\overline{I})}, \end{cases} \tag{13}$$

where $(\overline{I}, \overline{\varphi})$ denote values of action and phase variables just after the impulse of velocity, (I, φ) are the same quantities after previous impulse and

$$C(I, \varphi) = -\int_{\Delta\tau} d\tau \dot{x} \left(2 \sum_{m=0}^{+\infty} \cos\left(m\varphi(t)\right) + 1 \right). \tag{14}$$

Here we integrate over time interval of velocity's impulse $\Delta\tau$. From (13) we obtain (for small ε)

$$\overline{\varphi} \simeq \varphi + \frac{\pi\nu}{\omega(I)} - \frac{\varepsilon\nu^2}{2\omega^3}\frac{d\omega}{dI} C(I, \varphi). \tag{15}$$

Therefore, parameter of local instability (defined in [11]) is

$$K = \left| \frac{\delta \overline{\varphi}}{\delta \varphi} - 1 \right| = \frac{\pi \varepsilon \nu}{T \omega^3} \left| \frac{d\omega}{dI} \right| C_0 \,, \quad C_0 \equiv \left| \frac{\partial C}{\partial \varphi} \right| . \tag{16}$$

In the vicinity of separatrix C_0 can be calculated as

$$C_0 \simeq 4\pi \frac{\sinh\left(\pi \nu / 2\omega_0\right)}{\cosh^2\left(\pi \nu / 2\omega_0\right)} \,, \tag{17}$$

Using (17) and estimation for $|d\omega/dI|$ at $|H - H_{\mathrm{s}}| \ll H_{\mathrm{s}}$ we can represent the parameter of local instability (16) in the form

$$K \simeq \frac{2\varepsilon \nu^2}{\omega_0} \frac{\sinh\left(\pi \nu / 2\omega_0\right)}{\cosh^2\left(\pi \nu / 2\omega_0\right)} \frac{1}{|H - H_{\mathrm{s}}|} . \tag{18}$$

Here $H_{\mathrm{s}} \equiv \omega_0^2$ is the energy of non-perturbed system on the separatrix.

Condition $K \geq 1$ means that dynamics of the system is locally unstable. Local instability leads to mixing and chaos [11]. Thus condition $K = 1$ gives the estimation for the width of the stochastic layer as follows

$$|H_b - H_{\mathrm{s}}| \approx \frac{4\varepsilon \nu^2}{\omega_0} \mathrm{e}^{-\pi \nu / 2\omega_0} \,, \tag{19}$$

under the assumption that $\nu > \omega_0$. Here H_b is an estimated value of energy on the boundary of the stochastic layer. A set of trajectories appearing in the stochastic layer can be considered as new multi-instanton-like exact solutions of Euclidean equations of motion for perturbed system. These solutions demonstrate finite intervals between instanton transitions. The reason is that for majority of trajectories in stochastic layer finite time is needed to pass from one potential well to another. Thus density of instanton gas strongly increases in comparison with density at $\varepsilon = 0$.

Average time interval between instantons in the instanton gas for perturbed system can be estimated in the following way

$$\eta \approx \frac{\pi}{\omega_{\mathrm{av}}} . \tag{20}$$

Here $\omega_{\mathrm{av}} \equiv \omega([H_b + H_{\mathrm{s}}]/2)$. Near the separatrix the following approximation for the frequency can be used [11]

$$\omega(H) \equiv \frac{dH_0}{dI} \approx \frac{\pi}{\sqrt{2}} \frac{\sqrt{H_{\mathrm{s}} + H}}{\ln \frac{16(H_{\mathrm{s}} + H)}{H - H_{\mathrm{s}}}} \,, \quad |H - H_{\mathrm{s}}| \ll H_{\mathrm{s}} . \tag{21}$$

Thus the interval between instanton transitions is

$$\eta \approx \frac{\pi\nu}{2\omega_0^2} + \frac{1}{\omega_0} \ln \frac{8\omega_0^3}{\varepsilon\nu^2}\,, \tag{22}$$

and the density of instantons in instanton gas for the perturbed system can be estimated as follows

$$\rho = \frac{1}{\eta} = \left(\frac{\pi\nu}{2\omega_0^2} + \frac{1}{\omega_0} \ln \frac{8\omega_0^3}{\varepsilon\nu^2} \right)^{-1} . \tag{23}$$

We see that, contrary to the case of non-perturbed system, for perturbed system the density of instanton gas is large if we take into account only exact solutions of Euclidean equations of motion. Consideration of approximate solutions only increases the density of instanton gas and does not lead to qualitative changes. Comparing of densities for perturbed ρ (23) and non-perturbed ρ_0 (10) systems gives

$$\frac{\rho}{\rho_0} \approx \frac{1}{\sqrt{2}} \frac{\omega_0 \sqrt{\omega_0} \mathrm{e}^{8\omega_0}}{\pi\nu + 2\omega_0 \ln \frac{8\omega_0^3}{\varepsilon\nu^2}}\,. \tag{24}$$

This ratio is large at large enough ω_0 (or large enough one-instanton action) and small but nonzero ε. Thus we obtain that small perturbation (3) can strongly increase the density of instanton gas. For calculation of ρ only exact classical solutions of Euclidean equations of motion were taken into account, while to get non-zero value of ρ_0 we had to consider approximate solutions. Thus the limit $\varepsilon \to 0$ cannot be applied directly in (24). Nevertheless we have correspondence between results obtained for perturbed and non-perturbed systems in this limit. Namely, ρ (see (23)) at $\varepsilon = 0$ and ρ_0 are equal to zero if we take into account only exact solutions of Euclidean equations of motion for *both* cases.

Let us apply now our formal computation to the physical phenomena in real time. Classical instanton solutions in imaginary time (6-7) describe quantum tunnelling transitions in real time. Some observables can be directly expressed through the instanton density ρ [8]. In particular, rate of the tunnelling between neighbour potential wells (the number of tunnelling transitions per unit of time) and probability of tunnelling are proportional to squared density of instantons ρ^2. Spectrum and width of the lowest energy zone ΔE read

$$E_\theta \approx \tfrac{1}{2}\omega_0 - 2\rho\cos\theta\,, \quad 0 \le \theta \le \pi\,, \qquad \Delta E \approx 4\rho\,. \tag{25}$$

Thus squeezing of the instanton gas and increase of the density ρ in imaginary time mean that small perturbation (3) leading to chaos can essentially enhance the tunnelling rate and lead to the widening of the energy zone in comparison with non-perturbed system (2). Both these results are consequences of the perturbation leading to destruction of the single non-perturbed instanton solution and appearance of manifold of chaotic perturbed instantons. An increase of the number of instanton solutions provides a larger number of variants for particle to reach one vacuum from another that results in the increase of the rate of tunnelling. The decrease of the lifetime of the particle in a certain vacuum of the system means the widening of the energy zone that is obtained in (25).

On the other hand our results can be considered as a demonstration on the simple model of application of the instanton method to the problem of chaos assisted tunnelling.

4. Conclusion

In this work we have demonstrated on the example of one-dimensional periodic potential that small perturbation leading to chaotic behaviour of the system strongly influences the properties of instanton gas. Our estimations show that classical chaos can greatly increase the density of instanton gas and rate of instanton tunnelling.

Both instanton solutions and chaotic behaviour can exist in complex systems like field theories. The relation between chaos and instantons in such theories is not trivial. Theory of strong interactions (QCD) is the most interesting example, where the investigation of instanton gas (or instanton liquid) could shed light on the structure of hadrons [5, 12].

REFERENCES

[1] M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics,* Springer, New York 1990; *Chaos and Quantum Physics,* eds. M.-J. Giannoni, A. Voros, J. Zinn-Justin, North-Holland, Amsterdam 1991; M. Brack, R.K. Bhaduri, *Semiclassical Physics,* Addison-Wesley, 1997.

[2] W.A. Lin, L.E. Ballentine, *Phys. Rev. Lett.* **65**, 2927 (1990); O. Bohigas, S. Tomsovic, D. Ullmo, *Phys. Rep.* **223**, 43 (1993); S.D. Frischat, E. Doron, *Phys. Rev.* **E57**, 1421 (1998).

[3] A. Belavin, A. Polyakov, A. Schwarz, Yu. Tyupkin, *Phys. Lett.* **B59**, 85 (1975); G. 't Hooft, *Phys. Rev. Lett.* **37**, 8 (1976); *Phys. Rev.* **D14**, 3432 (1976).

[4] K.-I. Aoki, A. Horikoshi, M. Taniguchi, H. Terao, `hep-th/9812050`.

[5] T. Schaefer, E. Shuryak, *Rev. Mod. Phys.* **70**, 323 (1998).

[6] S. Moch, A. Ringwald, F. Schrempp, *Nucl. Phys.* **B507**, 134 (1997); A. Ringwald, F. Schrempp, *Phys. Lett.* **B438**, 217 (1998); *J. Phys.* **G25**, 1297 (1999); *Phys. Lett.* **B495**, 249 (1999); *Comput. Phys. Commun.* **132**, 267 (2000); *Phys. Lett.* **B503**, 331 (2001).

[7] C. Kittel, *Introduction to Solid State Physics,* New-York 1959.

[8] R. Rajaraman, *Solitons and Instantons,* North-Holland, Amsterdam 1982.

[9] G.P. Berman, G.M. Zaslavsky, *Physica* **111A**, 17 (1982).

[10] A.J. Lichtenberg, M.A. Lieberman, *Regular and Stochastic Motion,* Springer Publishing, 1983.

[11] G.M. Zaslavsky, *Phys. Rept.* **80**, 157 (1981); G.M. Zaslavsky, R.Z. Sagdeev, *Introduction in Nonlinear Physics*, Moskow 1988, chapter 5 (in Russian).

[12] C. Callan, R. Dashen, D. Gross, *Phys. Rev.* **D17**, 2717 (1978); *Phys. Rev.* **D19**, 1826 (1979).