SEARCH FOR FERMION UNIVERSALITY OF THE DIRAC COMPONENT OF NEUTRINO MASS MATRIX*

Wojciech Królikowski

Institute of Theoretical Physics, Warsaw University
Hoża 69, 00–681 Warszawa, Poland

(Received March 1, 2002)

It is conjectured that a diagonal and degenerate 3×3 active–active component (*i.e.*, lefthanded component) dominates in the effective 6×6 mass matrix for six Majorana neutrinos, three active and three (conventional) sterile, while its 3×3 active-sterile component (*i.e.*, Dirac component) arises through a bimaximal-mixing unitary transformation from a structure *similar* to the 3×3 mass matrices for charged leptons as well as up and down quarks. In such a texture, three neutrino masses are nearly degenerate, $m_1 \simeq m_2 \simeq m_3$, though their mass-squared differences appear hierarchical, $\Delta m^2_{21} \ll \Delta m^2_{32} \simeq \Delta m^2_{31}$, whereas the remaining three neutrino masses can be constructed to vanish, $m_4 = m_5 = m_6 = 0$, or to be, as in Appendix A, degenerate in square with the previous masses, $m_1 = |m_4|, m_2 = |m_5|, m_3 = |m_6|$, in contrast to the familiar seesaw mechanism (in both cases). Appendices B and C are devoted to the author's idea of the *algebraic compositeness* of fundamental particles, resulting into three generations of Standard Model fermions and two generations of new bosons.

PACS numbers: 12.15.Ff, 14.60.Pq, 12.15.Hh

1. Introduction

As is well known, three Dirac neutrinos are $\nu^{(\mathrm{D})}_\alpha = \nu_{\alpha \mathrm{L}} + \nu_{\alpha \mathrm{R}} (\alpha = e, \mu, \tau)$, while three Majorana active neutrinos and three Majorana (conventional) sterile neutrinos become $\nu^{(\mathrm{a})}_\alpha = \nu_{\alpha \mathrm{L}} + (\nu_{\alpha \mathrm{L}})^{\mathrm{c}}$ and $\nu^{(\mathrm{s})}_\alpha = \nu_{\alpha \mathrm{R}} + (\nu_{\alpha \mathrm{R}})^{\mathrm{c}}$ $(\alpha = e, \mu, \tau)$, respectively. The neutrino mass term in the Lagrangian gets generically the form

$$-\mathcal{L}_{\mathrm{mass}} = \frac{1}{2} \sum_{\alpha\beta} \left(\overline{\nu^{(\mathrm{a})}_\alpha}, \overline{\nu^{(\mathrm{s})}_\alpha} \right) \left(\begin{array}{cc} M^{(\mathrm{L})}_{\alpha\beta} & M^{(\mathrm{D})}_{\alpha\beta} \\ M^{(\mathrm{D})*}_{\beta\alpha} & M^{(\mathrm{R})}_{\alpha\beta} \end{array} \right) \left(\begin{array}{c} \nu^{(\mathrm{a})}_\beta \\ \nu^{(\mathrm{s})}_\beta \end{array} \right) . \qquad (1)$$

* Supported in part by the Polish State Committee for Scientific Research (KBN), grant 5 P03B 119 20 (2001–2002).

If $M^{(\mathrm{L})}_{\alpha\beta}$ and $M^{(\mathrm{R})}_{\alpha\beta}$ are not all zero, then in nature there are realized six Majorana neutrino mass fields ν_i or states $|\nu_i\rangle (i = 1,2,3,4,5,6)$ connected with six Majorana neutrino flavor fields ν_α or states $|\nu_\alpha\rangle (\alpha = e,\mu,\tau,e_\mathrm{s},\mu_\mathrm{s},\tau_\mathrm{s})$ through the unitary transformation

$$\nu_\alpha = \sum_i U_{\alpha i}\nu_i \;\text{ or }\; |\nu_\alpha\rangle = \sum_i U^*_{\alpha i}|\nu_i\rangle\,, \tag{2}$$

where we passed to the notation $\nu_\alpha \equiv \nu^{(\mathrm{a})}_\alpha$ and $\nu_{\alpha_\mathrm{s}} \equiv \nu^{(\mathrm{s})}_\alpha$ for $\alpha = e,\,\mu,\,\tau$. Of course, $\nu^{(\mathrm{a})}_{\alpha\mathrm{L}} = \nu_{\alpha\mathrm{L}}$, $\nu^{(\mathrm{a})}_{\alpha\mathrm{R}} = (\nu_{\alpha\mathrm{L}})^\mathrm{c}$ and $\nu_{\alpha_\mathrm{s}\mathrm{R}} \equiv \nu^{(\mathrm{s})}_{\alpha\mathrm{R}} = \nu_{\alpha\mathrm{R}}$, $\nu_{\alpha_\mathrm{s}\mathrm{L}} \equiv \nu^{(\mathrm{s})}_{\alpha\mathrm{L}} = (\nu_{\alpha\mathrm{R}})^\mathrm{c}$ for $\alpha = e,\,\mu,\,\tau$. Thus, the neutrino 6×6 mass matrix $M = (M_{\alpha\beta})$ $(\alpha,\beta = e,\,\mu,\,\tau,\,e_\mathrm{s},\,\mu_\mathrm{s},\,\tau_\mathrm{s})$ is of the form

$$M = \begin{pmatrix} M^{(\mathrm{L})} & M^{(\mathrm{D})} \\ M^{(\mathrm{D})\dagger} & M^{(\mathrm{R})} \end{pmatrix}. \tag{3}$$

The neutrino 6×6 mixing matrix $U = (U_{\alpha i})$ $(i = 1,2,3,4,5,6)$ appearing in Eqs. (2) is, at the same time, the unitary 6×6 diagonalizing matrix,

$$U^\dagger M U = M_\mathrm{d} \equiv \mathrm{diag}(m_1,\, m_2,\, m_3,\, m_4,\, m_5,\, m_6)\,, \tag{4}$$

if the representation is used, where the charged-lepton 3×3 mass matrix is diagonal. This will be assumed henceforth.

2. Model of neutrino texture

In this paper we study the model of neutrino texture, where the 3×3 submatrices in Eq. (3) are

$$\begin{aligned} M^{(\mathrm{L})} &= \overset{0}{m}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ M^{(\mathrm{D})} &= \overset{0}{m}\begin{pmatrix} \frac{t_{14}}{\sqrt{2}} & \frac{t_{25}}{\sqrt{2}} & 0 \\ -\frac{t_{14}}{2} & \frac{t_{25}}{2} & \frac{t_{36}}{\sqrt{2}} \\ \frac{t_{14}}{2} & -\frac{t_{25}}{2} & \frac{t_{36}}{\sqrt{2}} \end{pmatrix}, \\ M^{(\mathrm{R})} &= \overset{0}{m}\begin{pmatrix} t^2_{14} & 0 & 0 \\ 0 & t^2_{25} & 0 \\ 0 & 0 & t^2_{36} \end{pmatrix} \end{aligned} \tag{5}$$

with $\overset{0}{m} > 0$ being a mass scale and t_{ij} $(ij = 14,\, 25,\, 36)$ denoting three dimensionless parameters.

One can show that the unitary diagonalizing matrix U for the mass matrix M defined in Eqs. (3) and (5) is of the form

$$U = \overset{1}{U}\overset{0}{U}\,, \qquad \overset{1}{U} = \begin{pmatrix} U^{(3)} & 0^{(3)} \\ 0^{(3)} & 1^{(3)} \end{pmatrix}, \qquad \overset{0}{U} = \begin{pmatrix} C^{(3)} & -S^{(3)} \\ S^{(3)} & C^{(3)} \end{pmatrix}, \tag{6}$$

where

$$U^{(3)} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}, \; 1^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$C^{(3)} = \begin{pmatrix} c_{14} & 0 & 0 \\ 0 & c_{25} & 0 \\ 0 & 0 & c_{36} \end{pmatrix}, \; S^{(3)} = \begin{pmatrix} s_{14} & 0 & 0 \\ 0 & s_{25} & 0 \\ 0 & 0 & s_{36} \end{pmatrix} \tag{7}$$

and

$$\frac{s_{ij}}{c_{ij}} = t_{ij} \tag{8}$$

with $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$, so that $t_{ij} = \tan\theta_{ij}$ $(ij = 14,\, 25,\, 36)$. Such a diagonalizing matrix leads to the mass spectrum

$$\begin{aligned} m_1 &= \overset{0}{m}\left(1 + t_{14}^2\right), \quad m_4 = 0\,, \\ m_2 &= \overset{0}{m}\left(1 + t_{25}^2\right), \quad m_5 = 0\,, \\ m_3 &= \overset{0}{m}\left(1 + t_{36}^2\right), \quad m_6 = 0 \end{aligned} \tag{9}$$

which can be described equivalently by the equalities

$$c_{14}^2 m_1 = c_{25}^2 m_2 = c_{36}^2 m_3 = \overset{0}{m}\,, \qquad m_4 = m_5 = m_6 = 0\,. \tag{10}$$

The easiest way to prove this theorem is to start with the diagonalizing matrix U given in Eqs. (6) and (7), and then to construct the mass matrix M defined in Eqs. (3) and (5) by making use of the formula $M_{\alpha\beta} = \sum_i U_{\alpha i} m_i U^*_{\beta i}$, where the mass spectrum (9) or (10) is to be taken into account.

We can see from Eqs. (5), (6) and (7) that our neutrino texture corresponds to the mixing angles giving $c_{12} = 1/\sqrt{2} = s_{12}$, $c_{23} = 1/\sqrt{2} = s_{23}$ and $c_{13} = 1$, $s_{13} = 0$, while c_{ij}, s_{ij} $(ij = 14,\, 25,\, 36)$ are to be determined from the experiment.

In this neutrino texture, where the mass matrix M is given in Eqs. (3) and (5), an interesting role is played by the unitarily transformed mass matrix $\overset{0}{M}$ defined as

$$\overset{0}{M} = \overset{1}{U}{}^{\dagger} M \overset{1}{U} \,. \tag{11}$$

Then, writing

$$\overset{0}{M} = \begin{pmatrix} \overset{0}{M}{}^{(\mathrm{L})} & \overset{0}{M}{}^{(\mathrm{D})} \\ \overset{0}{M}{}^{(\mathrm{D})\dagger} & \overset{0}{M}{}^{(\mathrm{R})} \end{pmatrix} , \tag{12}$$

we obtain

$$\begin{aligned} \overset{0}{M}{}^{(\mathrm{L})} = \left(\overset{1}{U}{}^{\dagger} M \overset{1}{U}\right)^{(\mathrm{L})} &= U^{(3)\dagger} M^{(\mathrm{L})} U^{(3)} = M^{(\mathrm{L})} , \\ \overset{0}{M}{}^{(\mathrm{D})} = \left(\overset{1}{U}{}^{\dagger} M \overset{1}{U}\right)^{(\mathrm{D})} &= U^{(3)\dagger} M^{(\mathrm{D})} = \overset{0}{m} \begin{pmatrix} t_{14} & 0 & 0 \\ 0 & t_{25} & 0 \\ 0 & 0 & t_{36} \end{pmatrix} , \\ \overset{0}{M}{}^{(\mathrm{R})} = \left(\overset{1}{U}{}^{\dagger} M \overset{1}{U}\right)^{(\mathrm{R})} &= M^{(\mathrm{R})} . \end{aligned} \tag{13}$$

Thus, the Dirac 3×3 component $\overset{0}{M}{}^{(\mathrm{D})}$ of the mass matrix $\overset{0}{M}$ (transformed unitarily from M by means of the factor $\overset{1}{U}$ of the mixing matrix U) becomes diagonal and so, may get a *hierarchical* structure *similar* to the Dirac mass matrices for charged leptons and quarks, all dominated by their diagonal parts. The transforming factor $\overset{1}{U}$ given in Eq. (6) works effectively thanks to its 3×3 submatrix $U^{(3)}$ that is just the familiar *bimaximal mixing matrix* [1], specific for neutrinos, describing satisfactorily the observed oscillations of solar ν_e's and atmospheric ν_μ's. Note that

$$\overset{0}{U}{}^{\dagger} \overset{0}{M} \overset{0}{U} = \overset{0}{M}_{\mathrm{d}} = M_{\mathrm{d}} = \mathrm{diag}(m_1 , m_2 , m_3 , m_4 , m_5 , m_6) \,, \tag{14}$$

where the factor $\overset{0}{U}$ of the mixing matrix U is defined in Eq. (6). Here, $\overset{0}{M} = \left(\overset{0}{M}_{ij}\right)$, $\overset{0}{U} = \left(\overset{0}{U}_{ij}\right)$ and $\overset{1}{U} = \left(\overset{1}{U}_{\alpha i}\right)$, as $M = (M_{\alpha\beta})$ and $U = (U_{\alpha i})$. With the use of $\overset{0}{M}$ given in Eq. (11) the neutrino mass term (1) in the Lagrangian can be written as $-\mathcal{L}_{\mathrm{mass}} = \frac{1}{2}\sum_{\alpha\beta} \overline{\nu}_\alpha M_{\alpha\beta} \nu_\beta = \frac{1}{2}\sum_{ij} \overline{\overset{0}{\nu}}_i \overset{0}{M}_{ij} \overset{0}{\nu}_j$, where $\nu_\alpha = \sum_i U_{\alpha i} \nu_i = \sum_i \overset{1}{U}_{\alpha i} \overset{0}{\nu}_i$, but $\overset{0}{\nu}_i = \sum_j \overset{0}{U}_{ij} \nu_j$ are not neutrino mass fields, in contrast to ν_i: in fact, $\overset{0}{M} |\overset{0}{\nu}_i\rangle = m_i |\overset{0}{\nu}_i\rangle$, while $M|\nu_i\rangle = m_i |\nu_i\rangle$ ($\overset{0}{M}$ being a unitary transform of the full neutrino mass matrix M).

Specifically, the Dirac 3×3 component $\overset{0}{M}{}^{(\mathrm{D})}$ of the neutrino mass matrix $\overset{0}{M}$ (where the bimaximal mixing characteristic for neutrinos is transformed out unitarily) may be conjectured in a *fermion universal form* that was shown to work very well for the mass matrix of charged leptons [2] and neatly for mass matrices of up and down quarks [3] (obviously, in those three cases of charged fundamental fermions there exist only Dirac-type mass matrices). Then, for neutrinos we get [4]

$$\overset{0}{M}{}^{(\mathrm{D})} = \frac{1}{29}\begin{pmatrix} \mu\varepsilon & 2\alpha & 0 \\ 2\alpha & 4\,\mu\,(80+\varepsilon)/9 & 8\sqrt{3}\,\alpha \\ 0 & 8\sqrt{3}\,\alpha & 24\,\mu\,(624+\varepsilon)/25 \end{pmatrix}, \tag{15}$$

where $\mu > 0$, $\alpha > 0$ and $\varepsilon > 0$ are some neutrino parameters. Since already for charged leptons $\varepsilon^{(e)} = 0.172329$ is small [2], we will put for neutrinos $\varepsilon \to 0$. We will also conjecture that for neutrinos α/μ is negligible, as for charged leptons the small $\left(\alpha^{(e)}/\mu^{(e)}\right)^2 = 0.023^{+0.029}_{-0.025}$ [2] gives the prediction $m_\tau = m_\tau^{\mathrm{exp}} = 1777.03^{+0.30}_{-0.26}$ MeV [5] when $m_e = m_e^{\mathrm{exp}}$ and $m_\mu = m_\mu^{\mathrm{exp}}$ are used as inputs, while with $\left(\alpha^{(e)}/\mu^{(e)}\right)^2 = 0$ the prediction becomes $m_\tau = 1776.80$ MeV. In such a case, from Eqs. (13) and (15) we can conclude that

$$\overset{0}{m}t_{14} = \frac{\mu}{29}\varepsilon \to 0\,,\ \overset{0}{m}t_{25} = \frac{\mu}{29}\frac{4\times80}{9} = 1.23\mu\,,\ \overset{0}{m}t_{36} = \frac{\mu}{29}\frac{24\times624}{25} = 20.7\mu \tag{16}$$

in Eqs. (5), (8) and (9), and

$$\begin{aligned} c_{14} &\to 1\,, & c_{25} &= \frac{1}{\sqrt{1+1.50\,(\mu/\overset{0}{m})^2}}\,, & c_{36} &= \frac{1}{\sqrt{1+427\,(\mu/\overset{0}{m})^2}}\,, \\ s_{14} &\to 0\,, & s_{25} &= \frac{1.23\mu/\overset{0}{m}}{\sqrt{1+1.50\,(\mu/\overset{0}{m})^2}}\,, & s_{36} &= \frac{20.7\,\mu/\overset{0}{m}}{\sqrt{1+427(\mu/\overset{0}{m})^2}} \end{aligned} \tag{17}$$

in Eqs. (7) and (8). Hence, from Eqs. (9) and (16)

$$m_1 \to \overset{0}{m}\,, \quad m_2 = \overset{0}{m} + 1.50\frac{\mu^2}{\overset{0}{m}}\,, \quad m_3 = \overset{0}{m} + 427\frac{\mu^2}{\overset{0}{m}}\,. \tag{18}$$

3. Neutrino oscillations

Accepting the formulae (16) and making tentatively the conjecture that $\mu \ll \overset{0}{m}$, we can operate with the approximation, where $0 \le t_{ij} \ll 1$ or

$0 \leq s_{ij} \ll c_{ij}$ $(ij = 14,\, 25,\, 36)$. Then, we get the case of nearly degenerate spectrum of m_1, m_2, m_3: $m_1 \simeq m_2 \simeq m_3 \simeq \overset{0}{m}$, but with hierarchical mass-squared differences $\Delta m^2_{21} \ll \Delta m^2_{32} \simeq \Delta m^2_{31}$, where

$$\begin{aligned} \Delta m^2_{21} &= 2\overset{0}{m}{}^2\,(t^2_{25} - t^2_{14}) = 3.01\,\mu^2\,, \\ \Delta m^2_{32} &= 2\overset{0}{m}{}^2\,(t^2_{36} - t^2_{25}) = 850\;\;\mu^2\,, \\ \Delta m^2_{31} &= 2\overset{0}{m}{}^2\,(t^2_{36} - t^2_{14}) = 853\;\;\mu^2 \end{aligned} \tag{19}$$

due to Eqs. (9) and (16).

Notice that the option $\overset{0}{m} \ll \mu$, opposite to our conjecture $\mu \ll \overset{0}{m}$, leads to $t_{ij} \gg 1$ or $0 \leq c_{ij} \ll s_{ij}$ $(ij = 14, 25, 36)$. Then, we obtain the case of hierarchical spectrum of m_1, m_2, m_3: $m_1 \ll m_2 \ll m_3$ with mass-squared differences $\Delta m^2_{21} \ll \Delta m^2_{32} \simeq \Delta m^2_{31}$, where

$$\begin{aligned} \Delta m^2_{21} &= \overset{0}{m}{}^2\,(t^4_{25} - t^4_{14}) = 2.26\,\frac{\mu^4}{\overset{0}{m}{}^2}, \\ \Delta m^2_{32} &= \overset{0}{m}{}^2\,(t^4_{36} - t^4_{25}) = 1.82 \times 10^5\,\frac{\mu^4}{\overset{0}{m}{}^2}, \\ \Delta m^2_{31} &= \overset{0}{m}{}^2\,(t^4_{36} - t^4_{14}) = 1.82 \times 10^5\,\frac{\mu^4}{\overset{0}{m}{}^2} \end{aligned} \tag{19'}$$

due to Eqs. (9) and (16). In this case, the component $M^{(\mathrm{R})}$ of the neutrino mass matrix dominates over $M^{(\mathrm{D})}$ (as μ over $\overset{0}{m}$) that dominates in turn over $M^{(\mathrm{L})}$ (as μ over $\overset{0}{m}$): this is the situation, where the familiar seesaw mechanism can formally work in spite of the fact that entries of $M^{(\mathrm{R})}$ are very small, in particular due to $m_4 = m_5 = m_6 = 0$ (not as in the popular seesaw, where they are as large as the GUT scale). With the SuperKamiokande result $\Delta m^2_{32} \sim 3 \times 10^{-3}$ eV^2 we get in this option $\Delta m^2_{21} \sim 3.7 \times 10^{-8}$ eV^2 and $\mu^4/\overset{0}{m}{}^2 \sim 1.6 \times 10^{-8}\,\mathrm{eV}^2$ or $\mu^2/\overset{0}{m} \sim 1.3 \times 10^{-4}$ eV *i.e.*, $\mu \sim 1.3 \times 10^{-4}(\overset{0}{m}/\mu)$ eV $\ll 1.3 \times 10^{-4}$ eV. In contrast, in the case of our conjecture $\mu \ll \overset{0}{m}$, the component $M^{(\mathrm{L})}$ dominates over $M^{(\mathrm{D})}$ which dominates in turn over $M^{(\mathrm{R})}$ and so, we obtain for Δm^2_{21} and μ the much larger values given later on in Eqs. (26) and (24), respectively; also the value of $\Delta m^2_{25} = m^2_2 \simeq \overset{0}{m}{}^2$ appearing in Eq. (28) is much larger.

The familiar formulae for probabilities of neutrino oscillations $\nu_\alpha \to \nu_\beta$ on the energy shell,

$$P(\nu_\alpha \to \nu_\beta) = |\langle \nu_\beta | e^{iPL} | \nu_\alpha \rangle|^2 = \delta_{\beta\alpha} - 4 \sum_{j>i} U^*_{\beta j} U_{\beta i} U_{\alpha j} U^*_{\alpha i} \sin^2 x_{ji} \tag{20}$$

with

$$x_{ji} = 1.27\frac{\Delta m_{ji}^2 L}{E}\,, \quad \Delta m_{ji}^2 = m_j^2 - m_i^2\,, \quad p_i \simeq E - \frac{m_i^2}{2E}\,, \tag{21}$$

valid when a possible CP violation can be ignored (then $U^*_{\alpha i} = U_{\alpha i}$), give in the accepted approximation of $\Delta m_{21}^2 \ll \Delta m_{32}^2 \simeq \Delta m_{31}^2 \ll \overset{0}{m}{}^2$ that

$$\begin{aligned} P(\nu_e \to \nu_e)_{\rm sol} &= 1 - c_{25}^2 \sin^2(x_{21})_{\rm sol} - \tfrac{1}{2}(1 + c_{25}^2)s_{25}^2\,, \\ P(\nu_\mu \to \nu_\mu)_{\rm atm} &= 1 - \tfrac{1}{2}(1 + c_{25}^2)c_{36}^2 \sin^2(x_{32})_{\rm atm} \\ &\quad -\tfrac{1}{8}(1 + c_{25}^2 + 2c_{36}^2)(s_{25}^2 + 2s_{36}^2)\,, \\ P(\nu_\mu \to \nu_e)_{\rm LSND} &= \tfrac{1}{2}s_{25}^4 \sin^2(x_{25})_{\rm LSND}\,, \\ P(\bar\nu_e \to \bar\nu_e)_{\rm Chooz} &= 1 - (1 + c_{25}^2)s_{25}^2 \sin^2(x_{25})_{\rm Chooz} \end{aligned} \tag{22}$$

for solar ν_e's, atmospheric ν_μ's, LSND accelerator ν_μ's ($\bar\nu_\mu$'s) and Chooz reactor $\bar\nu_e$'s, respectively. The first two Eqs. (22) differ from the familiar two–flavor oscillation formulae (used often in analyzes of neutrino oscillations) by some additive terms that, fortunately, are small enough because of $s_{ij}^2 \ll c_{ij}^2$ consistent with $\mu^2 \ll \overset{0}{m}{}^2$.

From the second formula (22) describing atmospheric ν_μ's we infer due to the SuperKamiokande result [6] that

$$\tfrac{1}{2}(1 + c_{25}^2)c_{36}^2 \equiv \sin^2 2\theta_{\rm atm} \sim 1\,, \qquad \Delta m_{32}^2 \equiv \Delta m_{\rm atm}^2 \sim 3 \times 10^{-3}{\rm eV}\,, \tag{23}$$

what gives

$$\mu^2 \sim 3.5 \times 10^{-6}{\rm eV}^2 \;\text{ or }\; \mu \sim 1.9 \times 10^{-3}{\rm eV}\,, \tag{24}$$

when Eq. (19) is used. The nearly maximal atmospheric oscillation amplitude $\sin^2 2\theta_{\rm atm} \sim 1$ implies $c_{25}^2 \sim 1$ and $c_{36}^2 \sim 1$, which is consistent with $\mu^2 \ll \overset{0}{m}{}^2$. For an illustration, taking $\sin^2 2\theta_{\rm atm} \gtrsim 0.85$, we get from Eqs. (17) $(\mu/\overset{0}{m})^2 \lesssim 4.1 \times 10^{-4}$ and so, $\overset{0}{m}{}^2 \gtrsim 8.3 \times 10^{-3}{\rm eV}^2$ or $\overset{0}{m} \gtrsim 9.3 \times 10^{-2}{\rm eV}$ due to Eq. (24). Thus, $\sin^2 2\theta_{\rm atm}$ should be much larger than 0.85 in order to have $\overset{0}{m}{}^2 \gg \Delta m_{32}^2 \sim 3 \times 10^{-3}{\rm eV}^2$. If *e.g.* $\overset{0}{m} \sim 1$ eV, then $\sin^2 2\theta_{\rm atm} \sim 0.998$.

Making use of the estimate (24) in Eqs. (18) we obtain

$$m_1 \to \overset{0}{m}\,, \quad m_2 \sim \overset{0}{m} + 5.3 \times 10^{-6}\,\frac{{\rm eV}^2}{\overset{0}{m}}\,, \quad m_3 \sim \overset{0}{m} + 1.5 \times 10^{-3}\,\frac{{\rm eV}^2}{\overset{0}{m}}\,. \tag{25}$$

The first formula (22) referring to solar ν_e's *predicts* with the use of Eqs. (19) and (24) that

$$\sin^2 2\theta_{\rm sol} \equiv c_{25}^2 \sim 1\,, \quad \Delta m_{\rm sol}^2 \equiv \Delta m_{21}^2 \sim 3.01\mu^2 \sim 1.1 \times 10^{-5}\,{\rm eV}^2\,. \tag{26}$$

Such a prediction for solar ν_e's is not inconsistent with the Large Mixing Angle (LMA) solution [7], though the solar oscillation amplitude in this solution seems to be a bit smaller than the SuperKamiokande atmospheric oscillation amplitude (in contrast to the inequality $c_{25}^2 > \frac{1}{2}(1+c_{25}^2)c_{36}^2$, where $c_{25}^2 > c_{36}^2$ due to Eqs. (17); however the small additive terms $\frac{1}{2}(1+c_{25}^2)s_{25}^2 < \frac{1}{8}(1+c_{25}^2+2c_{36}^2)(s_{25}^2+2s_{36}^2)$ may compensate effectively such an inequality).

From the third formula (22) we can see that in our texture there is *predicted* a very small version of the original LSND effect for accelerator ν_μ's ($\bar{\nu}_\mu$'s) [8] with the oscillation amplitude

$$\sin^2 2\theta_{\rm LSND} \equiv \frac{1}{2}s_{25}^4 = \frac{1.13(\mu/\overset{0}{m})^4}{[1+1.50(\mu/\overset{0}{m})^2]^2} \sim 1.4\times 10^{-11}\left(\frac{\rm eV}{\overset{0}{m}}\right)^4 , \qquad (27)$$

where Eqs. (17) and (24) are used (with $\mu^2 \ll \overset{0}{m}{}^2$). The mass-squared scale for such a version of the LSND effect is equal to

$$\Delta m_{\rm LSND}^2 \equiv \Delta m_{25}^2 = m_2^2 = \overset{0}{m}{}^2 + 3.01\mu^2 \sim \overset{0}{m}{}^2 + 1.1\times 10^{-5}\,{\rm eV}^2 , \qquad (28)$$

where Eq. (19) is applied. Note that $\Delta m_{\rm LSND}^2$ differs by the term $\overset{0}{m}{}^2$ from the solar mass-squared scale $\Delta m_{\rm sol}^2$ given in Eq. (26). If *e.g.* $\overset{0}{m} = O(10^{-1}\ {\rm eV}) - O(1\ {\rm eV})$ (still with $\mu^2 \ll \overset{0}{m}{}^2$), then $\sin^2 2\theta_{\rm LSND} = O(10^{-7}) - O(10^{-11})$ and $\Delta\ m_{\rm LSND}^2 = O(10^{-2}\ {\rm eV}^2) - O(1\ {\rm eV}^2)$.

The fourth formula (22) describes the Chooz experiment for reactor $\bar{\nu}_e$'s. Due to its negative result, $P(\bar{\nu}_e \to \bar{\nu}_e)_{\rm Chooz} \sim 1$, there appears the experimental constraint for s_{25}^2 [9]:

$$(1+c_{25}^2)s_{25}^2 \equiv \sin^2 2\theta_{\rm Chooz} \lesssim 0.1 \ \text{ if }\ \Delta m_{25}^2 \equiv \Delta m_{\rm Chooz}^2 \gtrsim 0.1\,{\rm eV}^2 . \qquad (29)$$

This implies for the LSND effect (in our texture) the Chooz upper bound

$$\sin^2 2\theta_{\rm LSND} \equiv \tfrac{1}{2}s_{25}^4 \lesssim 1.3\times 10^{-3} \qquad (30)$$

if $\Delta m_{25}^2 \gg \Delta m_{32}^2 \sim 3\times 10^{-3}\ {\rm eV}^2$, what is consistent with $\Delta m_{25}^2 \gtrsim 0.1\ {\rm eV}^2$ and gives $(x_{25})_{\rm Chooz} \gg (x_{32})_{\rm Chooz} \simeq (x_{32})_{\rm atm} = O(1)$ as $(x_{ji})_{\rm Chooz} \simeq (x_{ji})_{\rm atm}$ numerically. Then,

$$\sin^2(x_{25})_{\rm Chooz} \simeq \tfrac{1}{2} \qquad (31)$$

in the fourth formula (22). When combined with Eq. (27), the Chooz bound (30) leads to the lower limit for $\overset{0}{m}$:

$$\overset{0}{m} \gtrsim 1.0\times 10^{-2}\ {\rm eV} . \qquad (32)$$

This gives in turn the lower limits

$$\Delta m^2_{\mathrm{LSND}} \equiv \Delta m^2_{25} \gtrsim 1.1 \times 10^{-4}\ \mathrm{eV}^2 \tag{33}$$

and

$$\sin^2 2\theta_{\mathrm{sol}} \equiv c^2_{25} \gtrsim 0.95\,, \quad \sin^2 2\theta_{\mathrm{atm}} \equiv \frac{1}{2}(1 + c^2_{25})c^2_{36} \gtrsim 0.061 \tag{34}$$

due to Eqs. (17) and (28), respectively. Evidently, this lower limit for $\sin^2 2\theta_{\mathrm{atm}}$ is not reached experimentally. If *e.g.* $\overset{0}{m} \sim 1$ eV corresponding to $\sin^2 2\theta_{\mathrm{atm}} \sim 0.998$, then $\sin^2 2\theta_{\mathrm{LSND}} \sim 1.4 \times 10^{-11}$, $\Delta m^2_{\mathrm{LSND}} \sim 1\ \mathrm{eV}^2$ and $\sin^2 2\theta_{\mathrm{sol}} \sim 1$.

The effective weighted sum of Majorana neutrino masses contributing to the neutrinoless double β decay $\langle m_e \rangle \equiv |\sum_i U^2_{\alpha i} m_i|$ is in our texture equal to $\overset{0}{m}$. Thus, the experimental upper limit for $\langle m_e \rangle$ gives $\overset{0}{m} = \langle m_e \rangle <$ 0.4 (0.2) eV – 1 (0.6) eV (*cf.* Baudis 99B in Ref. [5]). If *e.g.* $\overset{0}{m} \sim 0.2$ eV corresponding to $\sin^2 2\theta_{\mathrm{atm}} \sim 0.96$, then $\sin^2 2\theta_{\mathrm{LSND}} \sim 8.8 \times 10^{-9}$, $\Delta m^2_{\mathrm{LSND}} \sim 4.0 \times 10^{-2}\ \mathrm{eV}^2$ and $\sin^2 2\theta_{\mathrm{sol}} \sim 1$.

Very recently, a possible positive evidence of the neutrinoless double β decay has been reported for the first time [10]. The proposed estimation is $0.05\ \mathrm{eV} \leq \langle m_e \rangle \leq 0.84\ \mathrm{eV}$ with the best fit $\langle m_e \rangle \sim 0.39$ eV. Then, in our texture, for $\overset{0}{m} = \langle m_e \rangle \sim (0.05 - 0.39 - 0.84)$ eV corresponding to $\sin^2 2\theta_{\mathrm{atm}} \sim 0.63 - 0.99 - 0.998$ one obtains $\sin^2 2\theta_{\mathrm{LSND}} \sim 2.2 \times 10^{-6} - 6.0 \times 10^{-10} - 2.8 \times 10^{-11}$, $\Delta m^2_{\mathrm{LSND}} \sim (2.5 \times 10^{-3} - 1.5 \times 10^{-1} - 7.1 \times 10^{-1})\ \mathrm{eV}^2$ and $\sin^2 2\theta_{\mathrm{sol}} \sim 0.998 - 1 - 1$. If this evidence is confirmed, we will be sure that ν_e is a Majorana neutrino and, moreover, we will gain the first experimental estimate of its mass scale. In the case such as in our texture, where neutrino masses m_1, m_2, m_3 are nearly degenerate, this scale shall be also the mass scale of Majorana neutrinos ν_μ and ν_τ. The case of near degeneracy of m_1, m_2, m_3 is here supported by the considerably large best fit of the mass-squared scale, $\langle m_e \rangle^2 \sim (0.39)^2\ \mathrm{eV}^2 = 0.15\ \mathrm{eV}^2$, distinctly larger than the mass-squared differences $\Delta m^2_{21} \ll \Delta m^2_{32} \sim 3 \times 10^{-3}\ \mathrm{eV}^2$.

4. Conclusions

We presented in this note an effective texture for six Majorana neutrinos, three active and three (conventional) sterile, based on the 6×6 mass matrix defined in Eqs. (3) and (5), and leading to the mixing matrix given in Eqs. (6) and (7), as well as to the mass spectrum (9) or (10). We conjectured that the Dirac 3×3 component of such a neutrino mass matrix (when the bimaximal mixing, specific for neutrinos, is transformed out unitarily) gets a *fermion* *universal form* (15) similar to the 3×3 mass matrix for charged leptons and 3×3 mass matrices for up and down quarks, constructed previously with a considerable success [2,3].

This texture *predicts* reasonably oscillations of solar ν_e's in a form not inconsistent with LMA solar solution, if the SuperKamiokande value of the mass-squared scale for atmospheric ν_μ's is taken as an input. In both cases, neutrino oscillations are practically maximal. The proposed texture also *predicts* very small, perhaps unobservable, LSND effect with the oscillation amplitude of the order $O[10^{-11}\,(\mathrm{eV}/\overset{0}{m})^4]$ and the mass-squared scale of the order $O(\overset{0}{m}{}^2)+O(10^{-5}\,\mathrm{eV}^2)$. If *e.g.* $\overset{0}{m}=O(10^{-1}\,\mathrm{eV})-O(1\mathrm{eV})$ corresponding to $\sin^2 2\theta_{\mathrm{atm}}=O(0.9)-O(1)$, then $\sin^2 2\theta_{\mathrm{LSND}}=O(10^{-7})-O(10^{-11})$, $\Delta m^2_{\mathrm{LSND}}=O(10^{-2}\,\mathrm{eV}^2)-O(1\,\mathrm{eV}^2)$ and $\sin^2 2\theta_{\mathrm{sol}}=O(1)$.

The negative result of Chooz experiment imposes on the oscillation amplitude of LSND effect (in our texture) an upper bound of the order $O(10^{-3})$ which corresponds for $\overset{0}{m}$ to a lower limit of the order $O(10^{-2}\,\mathrm{eV})$ and for $\Delta m^2_{\mathrm{LSND}}$ to a lower limit of the order $O(10^{-4}\,\mathrm{eV}^2)$. Notice that the estimations following from the original LSND experiment [8] are *e.g.* $\sin^2 2\theta_{\mathrm{LSND}}=O(10^{-2})$ and $\Delta m^2_{\mathrm{LSND}}=O(1\,\mathrm{eV}^2)$. The new miniBooNE experiment may confirm or revise the original LSND results.

As far as the neutrino mass spectrum is concerned, our model of neutrino texture is of 3 + 3 type, in contrast to the models of 3 + 1 or 2 + 2 types [11] discussed in the case when, beside three active neutrinos ν_e, ν_μ, ν_τ, there is one *extra* sterile neutrino ν_{s}. In those models, three Majorana *conventional* sterile neutrinos $\nu_{e_{\mathrm{s}}}$, $\nu_{\mu_{\mathrm{s}}}$, $\nu_{\tau_{\mathrm{s}}}$ are decoupled through the familiar seesaw mechanism, as being practically identical with three very heavy neutrino mass states ν_4, ν_5, ν_6 (of the GUT mass scale). In our model, on the contrary, $\nu_{e_{\mathrm{s}}}$, $\nu_{\mu_{\mathrm{s}}}$, $\nu_{\tau_{\mathrm{s}}}$ are practically identical with three mass states ν_4, ν_5, ν_6 that this time are constructed to be massless.

In this paper, the most crucial may be the pertinent question, what is the physical (Higgs?) origin of the Dirac component $M^{(\mathrm{D})}$, Eq. (5), of the neutrino mass matrix M, where its bimaximal-mixing-free unitary transform $\overset{0}{M}{}^{(\mathrm{D})}$, Eq. (13), is conjectured to be of the fermion universal form (15) (with α/μ negligible in the case of neutrinos). A somewhat different question arises also about the physical (explicit or effective?) origin of the lefthanded and righthanded components $M^{(\mathrm{L})}$ and $M^{(\mathrm{R})}$, Eqs. (5), of M.

The reader can find three Appendices added at the end of this paper. In Appendix A, an alternative, effective 6×6 neutrino texture is sketched, where due to a specific degeneracy of the mass matrix there are no oscillations of the (conventional) sterile neutrinos and, therefore, no LSND effect can arise. Appendix B contains a proposal of the explanation, why in nature there are three and only three generations of leptons and quarks, and also an argument for the particular form of the Dirac-type 3×3 mass matrix used in this paper for neutrinos (and in Refs. [2] and [3] for charged leptons and quarks, respectively). Finally, Appendix C deals briefly with the problem of new boson hierarchy, appearing as an unavoidable by-product of explaining the observed fermion hierarchy in the way presented in Appendix B.

Appendix A

An alternative 6×6 *texture without the LSND effect*

In this Appendix, we report on another effective texture for three active and three (conventional) sterile neutrinos, where there are *no oscillations* of the latter neutrinos due to a specific degeneracy of the mass matrix. Thus, they are decoupled from the former neutrinos, evidently in a different way than through the familiar seesaw mechanism.

In such a texture, the 3×3 components of the neutrino 6×6 mass matrix (3) get the form

$$\begin{aligned} M^{(\mathrm{L})} &= \overset{0}{m}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -M^{(\mathrm{R})}\,, \\ M^{(\mathrm{D})} &= \overset{0}{m}\begin{pmatrix} \frac{\tan 2\theta_{14}}{\sqrt{2}} & \frac{\tan 2\theta_{25}}{\sqrt{2}} & 0 \\ -\frac{\tan 2\theta_{14}}{2} & \frac{\tan 2\theta_{25}}{2} & \frac{\tan 2\theta_{36}}{\sqrt{2}} \\ \frac{\tan 2\theta_{14}}{2} & -\frac{\tan 2\theta_{25}}{2} & \frac{\tan 2\theta_{36}}{\sqrt{2}} \end{pmatrix} \end{aligned} \tag{A.1}$$

with $\overset{0}{m} > 0$ being a mass scale and $\tan 2\theta_{ij}$ $(ij = 14, 25, 36)$ denoting three dimensionless parameters. Its unitary diagonalizing matrix is given as before in Eqs. (6) and (7), but now the relations

$$c_{ij}^2 - s_{ij}^2 = \cos 2\theta_{ij} = \frac{1}{\sqrt{1+\tan^2 2\theta_{ij}}} \tag{A.2}$$

work and the neutrino mass spectrum becomes

$$\begin{aligned} m_{1,4} &= \pm\overset{0}{m}\sqrt{1+\tan^2 2\theta_{14}}\,, \\ m_{2,5} &= \pm\overset{0}{m}\sqrt{1+\tan^2 2\theta_{25}}\,, \\ m_{3,6} &= \pm\overset{0}{m}\sqrt{1+\tan^2 2\theta_{36}}\,, \end{aligned} \tag{A.3}$$

satisfying the equalities

$$\left(c_{14}^2 - s_{14}^2\right) m_{1,4} = \left(c_{25}^2 - s_{25}^2\right) m_{2,5} = \left(c_{36}^2 - s_{36}^2\right) m_{3,6} = \pm\overset{0}{m}\,. \tag{A.4}$$

This can be seen by applying the formula $M_{\alpha\beta} = \sum_i U_{\alpha i} m_i U^*_{\beta i}$ with the use of mass spectrum described in Eqs. (A.3) or (A.4).

For the new texture, the neutrino oscillation formulae (20) lead to the relations

$$\begin{aligned} P(\nu_e \to \nu_e)_{\rm sol} &= 1 - \sin^2(x_{21})_{\rm sol}\,, \\ P(\nu_\mu \to \nu_\mu)_{\rm atm} &= 1 - \frac{1}{4}\sin^2(x_{21})_{\rm atm} - \frac{1}{2}\left[\sin^2(x_{31})_{\rm atm} + \sin^2(x_{32})_{\rm atm}\right] \\ &\simeq 1 - \sin^2(x_{32})_{\rm atm}\,, \\ P(\nu_\mu \to \nu_e)_{\rm LSND} &= \frac{1}{2}\sin^2(x_{21})_{\rm LSND} \simeq 0\,, \\ P(\bar\nu_e \to \bar\nu_e)_{\rm Chooz} &= 1 - \sin^2(x_{21})_{\rm Chooz} \simeq 1\,, \end{aligned} \tag{A.5}$$

where $0 \simeq (x_{21})_{\rm atm} \ll (x_{31})_{\rm atm} \simeq (x_{32})_{\rm atm}$, $(x_{21})_{\rm LSND} \simeq 0$ and $(x_{21})_{\rm Chooz} \simeq (x_{21})_{\rm atm} \simeq 0$. Note that the formulae (A.5) describe oscillations having the same form as those in the case of the simple bimaximal texture of three active neutrinos [12], but now with the specific mass spectrum (A.3). On the other hand, oscillations of three (conventional) sterile neutrinos vanish in the new texture, $P(\nu_\alpha \to \nu_{\beta_{\rm s}}) = 0$ and $P(\nu_{\alpha_{\rm s}} \to \nu_{\beta_{\rm s}}) = \delta_{\beta_{\rm s}\,\alpha_{\rm s}}$ $(\alpha, \beta = e, \mu, \tau)$, in consequence of the degeneracy $\Delta m^2_{41} = \Delta m^2_{52} = \Delta m^2_{63} = 0$ following from the equalities $m_1 = -m_4$, $m_2 = -m_5$, $m_3 = -m_6$.

The oscillation formulae (A.5) imply *bimaximal mixing* for solar ν_e's and atmospheric ν_μ's, *negative result* for Chooz reactor $\bar\nu_e$'s and *no* LSND *effect* for accelerator ν_μ's ($\bar\nu_\mu$'s).

In the case of the conjecture (15) with $\overset{0}{M}{}^{(\rm D)} = U^{(3)\dagger} M^{(\rm D)}$, the new texture gives

$$\begin{aligned} \overset{0}{m} \tan 2\theta_{14} &= \frac{\mu}{29}\varepsilon \to 0\,, \\ \overset{0}{m} \tan 2\theta_{25} &= \frac{\mu}{29}\frac{4\times 80}{9} = 1.23\mu\,, \\ \overset{0}{m} \tan 2\theta_{36} &= \frac{\mu}{29}\frac{24\times 624}{25} = 20.7\mu\,, \end{aligned} \tag{A.6}$$

and then, from Eq. (A.3)

$$m_{1,4} = \pm\overset{0}{m}\,, \quad m_{2,5} = \pm\sqrt{\overset{0}{m}{}^2 + 1.50\mu^2}\,, \quad m_{3,6} = \pm\sqrt{\overset{0}{m}{}^2 + 427\mu^2}\,. \tag{A.7}$$

Hence,

$$\Delta m^2_{21} = 1.50\,\mu^2\,, \quad \Delta m^2_{32} = 425\,\mu^2\,, \quad \Delta m^2_{31} = 427\,\mu^2\,. \tag{A.8}$$

Thus, using the SuperKamiokande result $\Delta m_{32}^2 \sim 3 \times 10^{-3}\ \mathrm{eV}^2$ for atmospheric ν_μ's described by the second formula (A.5), we obtain from Eq. (A.8)

$$\mu^2 \sim 7.1 \times 10^{-6}\ \mathrm{eV}^2 \text{ or } \mu \sim 2.7 \times 10^{-3}\ \mathrm{eV} \tag{A.9}$$

in place of the estimate (24). Then, from Eq. (A.8) we *predict*

$$\Delta m_{21}^2 \sim 1.1 \times 10^{-5}\ \mathrm{eV}^2 \tag{A.10}$$

for solar ν_e's presented in the first formula (A.5). So, the solar mass-squared scale Δm_{21}^2 turns out to be the same as estimated before in Eq. (26), being not inconsistent with the LMA solar solution.

Appendix B

Foundations for the fermion hierarchy

The form of Dirac mass matrix

$$M^{(f)} = \frac{1}{29}\begin{pmatrix} \mu^{(f)}\varepsilon^{(f)} & 2\alpha^{(f)} & 0 \\ 2\alpha^{(f)} & 4\mu^{(f)}(80+\varepsilon^{(f)})/9 & 8\sqrt{3}\,\alpha^{(f)} \\ 0 & 8\sqrt{3}\,\alpha^{(f)} & 24\mu^{(f)}(624+\varepsilon^{(f)})/25 \end{pmatrix}, \tag{B.1}$$

explored previously for charged leptons ($f = e$) [2] as well as for up and down quarks ($f = u,\, d$) [3] with a considerable success, is applied in the present paper [Eq. (15)] to neutrinos ($f = \nu$), namely to the bimaximal-mixing-free unitary transform $\overset{0}{M}{}^{(\mathrm{D})}$ of Dirac component of their 6×6 mass matrix M (*cf.* also Ref. [4]). In this case, $\varepsilon^{(\nu)} \to 0$ and $\alpha^{(\nu)}/\mu^{(\nu)}$ is negligible. In consequence, $\Delta m_{\mathrm{sol}}^2 = \Delta m_{21}^2$ is predicted just a little bit below the range suggested by the LMA solar solution, if the SuperKamiokande result for $\Delta m_{\mathrm{atm}}^2 = \Delta m_{32}^2$ is used. Notice that in the quark case ($f = u,\, d$) the parameter $\varepsilon^{(f)}$ must be replaced in the matrix element $M_{33}^{(f)}$ by $\varepsilon^{(f)} + C^{(f)}$, where $C^{(f)} > 0$ is large.

In this Appendix, we argue, first of all, for there being three and only three generations of leptons and quarks, and then, for the particular form (B.1) of the Dirac-type mass matrix. This argumentation is based on two assumptions:

(i) the conjecture that all kinds of matter's fundamental particles existing in nature can be deduced from Dirac square-root procedure $\sqrt{p^2} \to \Gamma \cdot p$, but constrained by an intrinsic Pauli principle, and

(ii) a simple specific ansatz for the shape of Dirac mass matrix, formulated on the ground of the first assumption.

The conjecture (i) turns out to be sufficient to explain the puzzling existence of *three and only three* generations of leptons and quarks. Then, the ansatz (ii) reproduces the specific form (B.1) of the Dirac mass matrix. At the end of this Appendix, we speculate on the physical origin of the ansatz (ii).

It is not difficult to see that, in the interaction-free case, Dirac's square-root procedure implies generically the sequence $N = 1, 2, 3, \ldots$ of generalized Dirac equations [13,2]:

$$\left(\Gamma^{(N)} \cdot p - M^{(N)}\right) \psi^{(N)}(x) = 0\,, \tag{B.2}$$

where for any N the Dirac algebra

$$\left\{\Gamma^{(N)}_\mu,\, \Gamma^{(N)}_\nu\right\} = 2g_{\mu\nu} \tag{B.3}$$

holds, constructed by means of a Clifford algebra:

$$\Gamma^{(N)}_\mu \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma^{(N)}_{i\mu}\,, \quad \left\{\gamma^{(N)}_{i\mu},\, \gamma^{(N)}_{j\nu}\right\} = 2\delta_{ij} g_{\mu\nu} \tag{B.4}$$

with $i,\, j = 1, 2, \ldots, N$ and $\mu,\, \nu = 0, 1, 2, 3$. The mass $M^{(N)}$ is independent of $\Gamma^{(N)}_\mu$. In general, the mass $M^{(N)}$ should be replaced by a mass matrix of elements $M^{(N,N')}$ which would couple $\psi^{(N)}(x)$ with all appropriate $\psi^{(N')}(x)$, and it might be natural to assume for $N \neq N'$ that $\gamma^{(N)}_{i\mu}$ and $\gamma^{(N')}_{j\nu}$ commute, and so do $\Gamma^{(N)}_\mu$ and $\Gamma^{(N')}_\nu$.

For $N = 1$, Eq. (B.2) is evidently the usual Dirac equation and for $N = 2$ it is known as the Dirac form [14] of Kähler equation [15], while for $N \geq 3$ Eq. (B.2) give us *new* Dirac-type equations [13,2]. They describe some spin-halfinteger or spin-integer particles for N odd or N even, respectively.

The Dirac-type matrices $\Gamma^{(N)}_\mu$ for any N can be embedded into the new Clifford algebra

$$\left\{\Gamma^{(N)}_{i\mu}\,,\, \Gamma^{(N)}_{j\nu}\right\} = 2\delta_{ij} g_{\mu\nu}\,, \tag{B.5}$$

isomorphic with the Clifford algebra of $\gamma^{(N)}_{i\mu}$, if $\Gamma^{(N)}_{i\mu}$ are defined by the properly normalized Jacobi linear combinations of $\gamma^{(N)}_{i\mu}$:

$$\begin{aligned} \Gamma^{(N)}_{1\mu} &\equiv \Gamma^{(N)}_\mu \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma^{(N)}_{i\mu}\,, \\ \Gamma^{(N)}_{i\mu} &\equiv \frac{1}{\sqrt{i(i-1)}} \left[\gamma^{(N)}_{1\mu} + \ldots + \gamma^{(N)}_{i-1\,\mu} - (i-1)\gamma^{(N)}_{i\mu}\right] \end{aligned} \tag{B.6}$$

for $i = 1$ and $i = 2, \ldots, N$, respectively. So, $\Gamma^{(N)}_{1\mu}$ and $\Gamma^{(N)}_{2\mu}, \ldots, \Gamma^{(N)}_{N\mu}$, respectively, present the "centre-of-mass" and "relative" Dirac-type matrices. Note that the Dirac-type equation (B.2) for any N does not involve the "relative" Dirac-type matrices $\Gamma^{(N)}_{2\mu}, \ldots, \Gamma^{(N)}_{N\mu}$, including solely the "centre-of-mass" Dirac-type matrix $\Gamma^{(N)}_{1\mu} \equiv \Gamma^{(N)}_{\mu}$. Since $\Gamma^{(N)}_{i\mu} = \sum_{j=1}^{N} O_{ij}\gamma^{(N)}_{j\mu}$, where $O = (O_{ij})$ is an orthogonal $N \times N$ matrix ($O^T = O^{-1}$), we obtain for the total spin tensor the equality

$$\sum_{i=1}^{N} \sigma^{(N)}_{i\mu\nu} = \sum_{i=1}^{N} \Sigma^{(N)}_{i\mu\nu} \,, \tag{B.7}$$

where

$$\sigma^{(N)}_{j\mu\nu} \equiv \frac{i}{2}\left[\gamma^{(N)}_{j\mu}, \gamma^{(N)}_{j\nu}\right], \quad \Sigma^{(N)}_{j\mu\nu} \equiv \frac{i}{2}\left[\Gamma^{(N)}_{j\mu}, \Gamma^{(N)}_{j\nu}\right]. \tag{B.8}$$

The total spin tensor (B.7) is the generator of Lorentz transformations for $\psi^{(N)}(x)$.

In place of the chiral representations for individual $\gamma^{(N)}_{j} = \left(\gamma^{(N)}_{j\mu}\right)$, where

$$\gamma^{(N)}_{j5} \equiv i\gamma^{(N)}_{j0}\gamma^{(N)}_{j1}\gamma^{(N)}_{j2}\gamma^{(N)}_{j3}\,, \quad \sigma^{(N)}_{j3} \equiv \sigma^{(N)}_{j12} \tag{B.9}$$

are diagonal, it is convenient to use for any N the chiral representations of Jacobi $\Gamma^{(N)}_{j} = \left(\Gamma^{(N)}_{j\mu}\right)$, where now

$$\Gamma^{(N)}_{j5} \equiv i\Gamma^{(N)}_{j0}\Gamma^{(N)}_{j1}\Gamma^{(N)}_{j2}\Gamma^{(N)}_{j3}\,, \quad \Sigma^{(N)}_{j3} \equiv \Sigma^{(N)}_{j12} \tag{B.10}$$

are diagonal (all matrices (B.9) and similarly (B.10) commute simultaneously, both with equal and different j).

When using the Jacobi chiral representations, the "centre-of-mass" Dirac-type matrices $\Gamma^{(N)}_{1\mu} \equiv \Gamma^{(N)}_{\mu}$ and $\Gamma^{(N)}_{15} \equiv \Gamma^{(N)}_{5} \equiv i\Gamma^{(N)}_{0}\Gamma^{(N)}_{1}\Gamma^{(N)}_{2}\Gamma^{(N)}_{3}$ can be taken in the reduced forms

$$\Gamma^{(N)}_{\mu} = \gamma_\mu \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{N-1 \text{ times}}, \quad \Gamma^{(N)}_{5} = \gamma_5 \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{N-1 \text{ times}}, \tag{B.11}$$

where γ_μ, $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3$ and $\mathbf{1}$ are the usual 4×4 Dirac matrices.

Then, the Dirac-type equation (B.2) for any N can be rewritten in the reduced form

$$\left(\gamma \cdot p - M^{(N)}\right)_{\alpha_1\beta_1} \psi^{(N)}_{\beta_1\alpha_2\ldots\alpha_N}(x) = 0\,, \tag{B.12}$$

where α_1 and $\alpha_2, \ldots, \alpha_N$ are the "centre-of-mass" and "relative" Dirac bispinor indices, respectively ($\alpha_i = 1, 2, 3, 4$ for any $i = 1, 2, \ldots, N$). Note that in the Dirac-type equation (B.12) for any $N > 1$ there appear the "relative" Dirac indices $\alpha_2, \ldots, \alpha_N$ which are free from any coupling, but still are subjects of Lorentz transformations.

The Standard Model gauge interactions can be introduced to the Dirac-type equations (B.12) by means of the minimal substitution $p \to p - gA(x)$, where p plays the role of the "centre-of-mass" four-momentum, and so, x — the "centre-of-mass" four-position. Then,

$$\left\{\gamma \cdot [p - gA(x)] - M^{(N)}\right\}_{\alpha_1\beta_1} \psi^{(N)}_{\beta_1\alpha_2\ldots\alpha_N}(x) = 0\,, \tag{B.13}$$

where $g\gamma \cdot A(x)$ symbolizes the Standard Model gauge coupling that involves within $A(x)$ the familiar weak-isospin and color matrices as well as the usual Dirac chiral matrix γ_5. The last arises from the "centre-of-mass" Dirac-type chiral matrix $\Gamma^{(N)}_5$, when a generic $g\Gamma^{(N)} \cdot A(x)$ is reduced to $g\gamma \cdot A(x)$ in Eq. (B.13) [see Eq. (B.11)].

In Eq. (B.13) the Standard Model gauge fields interact only with the "centre-of-mass" index α_1 that, therefore, is distinguished from the physically unobserved "relative" indices $\alpha_2, \ldots, \alpha_N$. This was the reason, why some time ago we conjectured that the "relative" Dirac bispinor indices $\alpha_2, \ldots, \alpha_N$ are all indistinguishable physical objects obeying Fermi statistics along with the Pauli principle requiring the full antisymmetry of wave function $\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N}(x)$ with respect to $\alpha_2, \ldots, \alpha_N$ [13,2]. Hence, due to this "intrinsic Pauli principle", only five values of N satisfying the condition $N - 1 \leq 4$ are allowed, namely $N = 1, 3, 5$ for N odd and $N = 2, 4$ for N even. Then, from the postulate of relativity and the probabilistic interpretation of $\psi^{(N)}(x) = \left(\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N}(x)\right)$ we were able to infer that these N odd and N even correspond to states with total spin $1/2$ and total spin 0, respectively [13,2].

Thus, the Dirac-type equation (B.13), jointly with the intrinsic Pauli principle, if considered on a fundamental level, justifies the existence in nature of *three and only three* generations of spin-$1/2$ fundamental fermions coupled to the Standard Model gauge bosons (they are identified with leptons and quarks). In addition, there should exist *two and only two* generations of spin-0 fundamental bosons also coupled to the Standard Model gauge bosons (they are not identified yet).

The wave functions or fields of spin-$1/2$ fundamental fermions (leptons and quarks) of three generations $N = 1, 3, 5$ can be presented in terms of $\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N}(x)$ as follows:

$$\begin{aligned}
\psi^{(f_1)}_{\alpha_1}(x) &= \psi^{(1)}_{\alpha_1}(x)\,,\\
\psi^{(f_3)}_{\alpha_1}(x) &= \tfrac{1}{4}\left(C^{-1}\gamma_5\right)_{\alpha_2\alpha_3} \psi^{(3)}_{\alpha_1\alpha_2\alpha_3}(x) = \psi^{(3)}_{\alpha_1 12}(x) = \psi^{(3)}_{\alpha_1 34}(x)\,,
\end{aligned}$$

$$\psi^{(f_5)}_{\alpha_1}(x) \;=\; \tfrac{1}{24}\varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5}\psi^{(5)}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}(x) = \psi^{(5)}_{\alpha_1 1234}(x)\,, \tag{B.14}$$

where $\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N}(x)$ carries also the Standard Model (composite) label, suppressed in our notation, and C denotes the usual 4×4 charge-conjugation matrix. Here, writing explicitly, $f_1 = \nu_e, e^-, u, d, f_3 = \nu_\mu, \mu^-, c, s$ and $f_5 = \nu_\tau, \tau^-, t, b$, thus each f_N corresponds to the same suppressed Standard Model (composite) label. We can see that, due to the full antisymmetry in α_i indices for $i \geq 2$, the wave functions or fields $N = 1, 3$ and 5 appear (up to the sign) with the multiplicities 1, 4 and 24, respectively. Thus, for them, there is defined the weighting matrix

$$\rho^{1/2} = \frac{1}{\sqrt{29}}\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{24} \end{array}\right)\,, \tag{B.15}$$

where Tr $\rho = 1$.

Concluding the first part of this Appendix, we would like to point out that our algebraic construction of *three and only three* generations of leptons and quarks may be interpreted *either* as ingenuously algebraic (much like the famous Dirac's algebraic discovery of spin-1/2), *or* as a summit of an iceberg of really composite states of N spatial partons with spin-1/2 whose Dirac bispinor indices manifest themselves as our Dirac bispinor indices α_1, α_2, ..., α_N ($N = 1, 3, 5$) which thus may be called "algebraic partons", as being algebraic building blocks for leptons and quarks. Among all N "algebraic partons" in any generation N of leptons and quarks, there are one "centre-of-mass algebraic parton" (α_1) and $N-1$ "relative algebraic partons" (α_2, ..., α_N), the latter undistinguishable from each other and so, obeying our intrinsic Pauli principle.

In the second part of this Appendix we introduce a simple specific ansatz for the shape of Dirac mass matrix by putting [13,2]

$$M^{(f)} = \rho^{1/2} h^{(f)} \rho^{1/2}\,, \tag{B.16}$$

where $\rho^{1/2}$ is given in Eq. (B.15) and

$$h^{(f)} = \mu^{(f)}\left[N^2 - (1-\varepsilon^{(f)})N^{-2}\right] + \alpha^{(f)}(a + a^\dagger) \tag{B.17}$$

with $\mu^{(f)} > 0$ and $\varepsilon^{(f)} > 0$ being parameters, while $f = \nu$, e, u, d refers to neutrinos, charged leptons, up quarks and down quarks, respectively. Here, the matrix

$$N = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{array}\right) = 1 + 2n \tag{B.18}$$

describes the number of all α_i indices with $i = 1, 2, \ldots, N$ (all "algebraic partons"), appearing in any of three fermion generations $N = 1, 3, 5$, while

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad a^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{B.19}$$

play the role of "truncated" annihilation and creation matrices for pairs of "relative" indices $\alpha_i \alpha_j$ with $(i, j) = (2, 3), \ldots, (N-1, N)$ (pairs of "relative algebraic partons"):

$$[a\,, n] = a\,, \, [a^\dagger\,, n] = -a^\dagger\,, \quad n = a^\dagger a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{B.20}$$

where the "truncation" condition $a^3 = 0 = a^{\dagger\,3}$ is satisfied.

It is not difficult to show that the formulae (B.16) and (B.17) lead explicitly to the particular form (B.1) of Dirac-type mass matrix.

Finally, a few words about a possible physical origin of the ansatz (B.17). In the kernel (B.17) of the Dirac mass matrix (B.16), the first term $\mu^{(f)} N^2$ may be intuitively interpreted as coming from an interaction of all N "algebraic partons" treated on equal footing, while the second term $-\mu^{(f)}(1-\varepsilon^{(f)})N^{-2}$ may be considered as being a subtraction term caused by the fact that there is one "centre-of-mass algebraic parton" distinguished (due to its external coupling to the Standard Model gauge fields) among all N "algebraic partons" of which $N-1$ are "relative algebraic partons", indistinguishable from each other. This distinguished "algebraic parton" appears, therefore, with the probability $[N!/(N-1)!]^{-1} = N^{-1}$ that, when squared, leads to the additional term $\mu^{(f)}(1-\varepsilon^{(f)})N^{-2}$ [with a coefficient $\mu^{(f)}(1-\varepsilon^{(f)})$] which should be subtracted in the kernel (B.17) from the former term in order to obtain the mass matrix element $M_{11}^{(f)} = \mu^{(f)}\varepsilon^{(f)}/29$ tending to zero if $\varepsilon^{(f)} \to 0$. Eventually, the third term $\alpha^{(f)}(a + a^\dagger)$ in the kernel (B.17) annihilates and creates pairs of "relative algebraic partons" and so, is responsible in a natural way for mixing of three fermion generations in the Dirac mass matrix $M^{(f)}$.

Appendix C

Problem of new boson hierarchy

The way of explanation of the observed fermion hierarchy (especially, of the existence of three and only three generations of leptons and quarks), as is described in Appendix B, suggests also the existence of a new boson hierarchy, consisting of *two and only two* generations of spin-0 fundamental bosons. These boson generations correspond to the numbers $N = 2, 4$ of the Dirac bispinor indices $\alpha_1, \alpha_2, \ldots, \alpha_N$, among which there are one "centre-of-mass" index α_1 and $N-1 = 1, 3$ "relative" indices α_2 or $\alpha_2, \alpha_3, \alpha_4$, respectively. Only the "centre-of-mass" index α_1 is coupled to the Standard Model gauge bosons.

The wave functions or fields of spin-0 fundamental bosons of two generations $N = 2, 4$ can be written down in terms of $\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N}(x)$ as follows:

$$\begin{aligned}
\psi^{(b_2)}(x) &= \tfrac{1}{4}(C^{-1}\gamma_5)_{\alpha_1\alpha_2}\psi^{(2)}_{\alpha_1\alpha_2}(x) \\
&= \psi^{(2)}_{12}(x) = -\psi^{(2)}_{21}(x) = \psi^{(2)}_{34}(x) = -\psi^{(2)}_{43}(x)\,, \\
\psi^{(b_4)}(x) &= \tfrac{1}{24}\varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\psi^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4}(x) \\
&= \psi^{(4)}_{1234}(x) = -\psi^{(4)}_{2134}(x) = \psi^{(4)}_{3412}(x) = -\psi^{(4)}_{4312}(x)\,, \qquad \text{(C.1)}
\end{aligned}$$

where the wave function or field $\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N}(x)$ carries the suppressed Standard Model (composite) label. In consequence, there are four sorts of fundamental scalars carrying the same Standard Model signature as four sorts of fundamental fermions, namely as neutrinos ($f = \nu$), charged leptons ($f = e$), up quarks ($f = u$) and down quarks ($f = d$). These fermions, however, are realized in three generations $N = 1, 3, 5$, while the fundamental scalars are predicted in two generations $N = 2, 4$. So, one cannot hope here for a construction of the full supersymmetry (at most, there might appear a partial supersymmetry: two to two).

Two lepton-like scalar doublets (corresponding, as far as the Standard Model signature is concerned, to three lepton doublets) might play the role of two generations of Higgs doublets [16]. On the other hand, two quark-like scalar doublets (corresponding to three quark doublets) should lead to a lot of new (colorless) hadrons, composed dynamically of these colored scalars from two generations and (also colored) quarks from three generations [17]. Most of them should be highly unstable, but perhaps not all, allowing then for some new observations.

In the rest of this Appendix, we discuss some structural aspects of the problem of new boson hierarchy, comparing it with the fermion hierarchy.

It is not difficult to derive the second-order differential equation for $\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N}(x)$ arising from the Dirac-type equation (B.13) through its matrix multiplication from the left by $\gamma\cdot[p - gA(-\gamma_5)] + M^{(N)}$, where the sign of γ_5 within $A_\mu(-\gamma_5) \equiv A_\mu(x, -\gamma_5)$ is reversed in comparison with Eq. (B.13). Such a Klein–Gordon-type equation gets the form

$$\left\{[p-gA(\gamma_5)]^2-M^{(N)\,2}-\tfrac{1}{2}g\sigma^{\mu\nu}F_{\mu\nu}(\gamma_5)-gM^{(N)}\gamma\cdot[A(\gamma_5)-A(-\gamma_5)]\right\}_{\alpha_1\beta_1}$$
$$\times\psi^{(N)}_{\beta_1\alpha_2\ldots\alpha_N}=0\,, \qquad \text{(C.2)}$$

with

$$F_{\mu\nu}(\gamma_5)=\partial_\mu A_\nu(\gamma_5)-\partial_\nu A_\mu(\gamma_5)+ig\,[A_\mu(\gamma_5),\,A_\nu(\gamma_5)] \qquad \text{(C.3)}$$

denoting the Standard Model gauge forces. Here, $\gamma\cdot A(-\gamma_5)=A(\gamma_5)\cdot\gamma$ since $A_\mu(\gamma_5)\equiv A_\mu(0)+A'_\mu(0)\gamma_5$. Thus, $A_\mu(\gamma_5)-A_\mu(-\gamma_5)\equiv 2A'_\mu(0)\gamma_5$.

For N even, when only the scalar wave functions or fields $\psi^{(b_N)}(x)$ $(N=2,4)$ given in Eqs. (C.1) appear, the Klein–Gordon-type equation (C.2) can be reduced to the usual Klein–Gordon equation

$$\left\{[p-gA(0)]^2+g^2A'(0)^2-M^{(N)\,2}\right\}\psi^{(b_N)}=0\,, \qquad \text{(C.4)}$$

because the 4×4 Dirac matrices γ_5, $\sigma^{\mu\nu}$, $\sigma^{\mu\nu}\gamma_5$ and $\gamma^\mu\gamma_5$ appearing in Eq. (C.2) are traceless and so, are reduced to zero for the scalars $\psi^{(b_N)}(x)$ $(N=2,4)$, when there are no other boson wave functions or fields with $N=2,4$. The Klein–Gordon equation (C.4) implies in turn the existence of the relativistic covariant *conserved* current of the usual form

$$j^{(b_N)}_{\mu\mathrm{KG}}\equiv\lambda^{-1}_{b_N}\psi^{(b_N)*}\left[i\overleftrightarrow{\partial}_\mu-gA_\mu(0)\right]\psi^{(b_N)}\,, \qquad \text{(C.5)}$$

where $p_\mu=i\partial_\mu$ and $\overleftrightarrow{\partial}_\mu\equiv\frac{1}{2}(\partial_\mu-\overleftarrow{\partial}_\mu)$ with $f(x)\overleftarrow{\partial}_\mu\equiv\partial_\mu f(x)$. In fact, $\partial^\mu j^{(b_N)}_{\mu\mathrm{KG}}=0$, because $A^\dagger_\mu(0)=A_\mu(0)$ and $\partial^\mu A_\mu(0)=0$. Here, $\lambda_{b_N}>0$ is a normalization mass scale such that $\phi^{(b_N)}(x)\equiv\lambda^{-1/2}_{b_N}\psi^{(b_N)}(x)$ is normalized as a Klein–Gordon wave function or field *i.e.*, $\int d^3x j^{(b_N)}_{0\mathrm{KG}}(x)$ is equal to 1 (for positive energies) or to the operator of b_N-boson number, respectively.

For N even, the Dirac-type equation (B.13) does not give any $j^{(b_N)}_{\mu\mathrm{D}}(x)$, of course.

The remainder of this Appendix is devoted to other consequences of the Klein–Gordon-type equation (C.2) and to its comparison with the Dirac-type equation (B.13).

For N odd, where only the bispinor wave functions or fields $\psi^{(f_N)}_{\alpha_1}(x)$ $(N=1,3,5)$ defined in Eq. (B.14) exist, the Dirac-type equation (B.13) can be reduced to the usual Dirac equation

$$\left\{\gamma\cdot[p-gA(\gamma_5)]-M^{(N)}\right\}_{\alpha_1\beta_1}\psi^{(f_N)}_{\beta_1}=0\,. \qquad \text{(C.6)}$$

This gives in turn the relativistic covariant *conserved* current of the usual form

$$j^{(f_N)}_{\mu\mathrm{D}} \equiv \psi^{(f_N)*}_{\alpha_1} (\gamma_0\gamma_\mu)_{\alpha_1\beta_1} \psi^{(f_N)}_{\beta_1} , \tag{C.7}$$

valid for N odd ($N = 1, 3, 5$). In fact, $\partial^\mu j^{(f_N)}_{\mu\mathrm{D}}(x) = 0$ since $A^\dagger_\mu(\gamma_5) = A_\mu(\gamma_5)$.

For N odd, the Klein–Gordon-type equation (C.2) is reducible to the form

$$\left\{[p-gA(\gamma_5)]^2 - M^{(N)\,2} - \frac{1}{2}g\sigma^{\mu\nu}F_{\mu\nu}(\gamma_5) - gM^{(N)}\gamma\cdot[A(\gamma_5)-A(-\gamma_5)]\right\}_{\alpha_1\beta_1} \times\psi^{(f_N)}_{\beta_1} = 0 \,. \tag{C.8}$$

This implies in turn the following divergence relationship:

$$\partial^\mu j^{(f_N)}_{\mu\mathrm{KG}} = -2gA'^{\,\mu}(0) j^{(f_N)}_{\mu\,5\mathrm{KG}} , \tag{C.9}$$

where

$$j^{(f_N)}_{\mu\mathrm{KG}} \equiv \lambda^{-1}_{f_N}\psi^{(f_N)*}_{\alpha_1}(\gamma_0)_{\alpha_1\beta_1}\left[i \overleftrightarrow{\partial}_\mu - gA_\mu(0)\right]\psi^{(f_N)}_{\beta_1} \tag{C.10}$$

and

$$j^{(f_N)}_{\mu\,5\mathrm{KG}} \equiv \lambda^{-1}_{f_N}\psi^{(f_N)*}_{\alpha_1}(i\gamma_0\gamma_5)_{\alpha_1\beta_1}\left[i \overleftrightarrow{\partial}_\mu - gA_\mu(0)\right]\psi^{(f_N)}_{\beta_1} \tag{C.11}$$

are respectively the vector and axial-vector Klein–Gordon currents of f_N bispinor fermions. Here, the form $A_\mu(\gamma_5) \equiv A_\mu(0) + \gamma_5 A'_\mu(0)$ is used, and $\lambda_{f_N} > 0$ denotes a normalization mass scale. Due to the chiral character of electroweak interactions (where $A'_\mu(0) \neq 0$), the current $j^{(f_N)}_{\mu\mathrm{KG}}(x)$ is generically *nonconserved* locally, in contrast to the current $j^{(f_N)}_{\mu\mathrm{D}}(x)$.

In Eq. (C.7) as well as in Eqs. (C.10) and (C.11), and also (C.5), there appears the operation $* =$ (complex conjugation)$\times$(transposition) = (hermitian conjugation), where the transposition pertains to the Standard Model (composite) label, suppressed in this notation, and to the quantum-field degrees of freedom, the latter if $\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N}(x)$ is a field rather than a wave function. For pure Dirac degrees of freedom $*$ is equivalent to the complex conjugation, in particular, when Dirac bispinor indices are written down explicitly as *e.g.* in Eq. (C.7).

At this point, we would like to emphasize that the Standard Model (composite) label, suppressed in $\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N}(x)$, *is summed up* in the discussed currents *within any generation* $N = 1,3,5$ and $N = 2,4$, separately. As was mentioned already in Appendix B, in order to mix generations $N = 1,3,5$ or $N = 2,4$, separately for any Standard Model signature, we should introduce nondiagonal mass matrix of elements $M^{(N,N')}$ with $N, N' = 1,3,5$ or

$N, N' = 2, 4$, dependent on the suppressed Standard Model signature. We should also stress that in the Dirac-type equation (B.13) as well as in the Klein–Gordon-type equation (C.2) *there are no source terms* which could change the global number of all f fermions and all b bosons, respectively. Of course, the numbers of gauge bosons and possible Higgs bosons do change (the latter, due to their definition, do get nonzero f-fermion and b-boson sources).

Now, some more formal remarks are due. The relativistic covariant Dirac-type current and Klein–Gordon-type current must have the forms

$$j^{(N)}_{\mu\mathrm{D}} \equiv \psi^{(N)*}_{\alpha_1\alpha_2\ldots\alpha_N} \xi^{(N)} \left(\Gamma^{(N)}_{10}\Gamma^{(N)}_{20}\ldots\Gamma^{(N)}_{N0}\Gamma^{(N)}_{1\mu}\right)_{\alpha_1\alpha_2\ldots\alpha_N,\beta_1\beta_2\ldots\beta_N} \psi^{(N)}_{\beta_1\beta_2\ldots\beta_N} \tag{C.12}$$

and

$$j^{(N)}_{\mu\mathrm{KG}} \equiv \lambda_N^{-1}\psi^{(N)*}_{\alpha_1\alpha_2\ldots\alpha_N} \eta^{(N)} \left(\Gamma^{(N)}_{10}\Gamma^{(N)}_{10}\ldots\Gamma^{(N)}_{N0}\right)_{\alpha_1\alpha_2\ldots\alpha_N,\beta_1\beta_2\ldots\beta_N} \times \left[i \overset{\leftrightarrow}{\partial}_\mu - gA_\mu(0)\right] \psi^{(N)}_{\beta_1\beta_2\ldots\beta_N}\,, \tag{C.13}$$

respectively, where $\Gamma^{(N)}_{i\mu}$ are the Dirac-type matrices in their Jacobi version, introduced in Eqs. (B.6), while $\xi^{(N)}$ and $\eta^{(N)}$ are phase factors making Hermitian the $N \times N$ bispinor matrices appearing in these currents and $\lambda_N > 0$ denote normalization mass scales.

The Dirac-type matrices $\Gamma^{(N)}_{i\mu}$ $(i = 1, 2, \ldots, N)$, satisfying the anticommutation relations of Clifford algebra (B.5), can be represented in terms of the usual 4×4 Dirac matrices as follows:

$$\begin{aligned}
\Gamma^{(N)}_{1\mu} &= \gamma_\mu \otimes \underbrace{\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\ldots\otimes\mathbf{1}\otimes\mathbf{1}}_{N-1 \text{ times}},\\
\Gamma^{(N)}_{2\mu} &= \gamma_5 \otimes i\gamma_\mu\gamma_5 \otimes \mathbf{1}\otimes\mathbf{1}\otimes\ldots\otimes\mathbf{1}\otimes\mathbf{1}\,,\\
\Gamma^{(N)}_{3\mu} &= \gamma_5 \otimes \gamma_5 \otimes \gamma_\mu \otimes \mathbf{1}\otimes\ldots\otimes\mathbf{1}\otimes\mathbf{1}\,,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
\Gamma^{(N)}_{N\mu} &= \underbrace{\gamma_5\otimes\gamma_5\otimes\gamma_5\otimes\gamma_5\otimes\ldots\otimes\gamma_5}_{N-1 \text{ times}} \otimes \begin{cases}\gamma_\mu & \text{for } N \text{ odd}\\ i\gamma_\mu\gamma_5 & \text{for } N \text{ even}\end{cases}\,.
\end{aligned} \tag{C.14}$$

Then, forming their product for $\mu = 0$,

$$\Gamma^{(N)}_{10}\Gamma^{(N)}_{20}\ldots\Gamma^{(N)}_{N0} = \begin{cases} i^{\frac{N-1}{2}}\ \gamma_0 \otimes \gamma_0 \otimes\ldots\otimes \gamma_0 & \text{for } N \text{ odd}\\ (-i)^{\frac{N}{2}} i\gamma_0\gamma_5 \otimes i\gamma_0\gamma_5 \otimes\ldots\otimes i\gamma_0\gamma_5 & \text{for } N \text{ even}\end{cases}\,, \tag{C.15}$$

and multiplying from the right by $\Gamma^{(N)}_{1\mu}$, we obtain

$$\Gamma^{(N)}_{10}\Gamma^{(N)}_{20}\ldots\Gamma^{(N)}_{N0}\Gamma^{(N)}_{1\mu}=\left\{\begin{array}{ll} i^{\frac{N-1}{2}}\gamma_0\,\gamma_\mu\,\otimes\,\gamma_0\,\otimes\ldots\otimes\,\gamma_0 & \text{for}\,N\,\text{odd}\\ (-i)^{\frac{N-2}{2}}\gamma_0\gamma_5\gamma_\mu\otimes i\gamma_0\gamma_5\otimes\ldots\otimes i\gamma_0\gamma_5 & \text{for}\,N\,\text{even}\end{array}\right. . \tag{C.16}$$

Hence, we can define the phase factors in Eqs. (C.12) and (C.13) as follows:

$$\xi^{(N)} = \left\{\begin{array}{ll} (-i)^{\frac{N-1}{2}} = 1, -i, -1 & \text{for } N=1,3,5\\ (-i)^{\frac{N-2}{2}} = 1, -i & \text{for } N=2,4\end{array}\right. ,$$

$$\eta^{(N)} = \left\{\begin{array}{ll} (-i)^{\frac{N-1}{2}} = 1, -i, -1 & \text{for } N=1,3,5\\ (-i)^{\frac{N}{2}} = -i, -1 & \text{for } N=2,4\end{array}\right. . \tag{C.17}$$

Making use of Eqs. (C.16) and (C.15) with (C.17) we can represent the currents (C.12) and (C.13) in the following forms:

$$j^{(N)}_{\mu\mathrm{D}} = \psi^{(N)*}_{\alpha_1\alpha_2\ldots\alpha_N}\left\{\begin{array}{cccc} (\gamma_0\,\gamma_\mu)_{\alpha_1\beta_1} & (\gamma_0)_{\alpha_2\beta_2} & \cdots & (\gamma_0)_{\alpha_N\beta_N}\\ (-1)^{\frac{N-2}{2}}(\gamma_0\gamma_5\gamma_\mu)_{\alpha_1\beta_1} & (i\gamma_0\gamma_5)_{\alpha_2\beta_2} & \cdots & (i\gamma_0\gamma_5)_{\alpha_N\beta_N}\end{array}\right\}$$
$$\times\psi^{(N)}_{\beta_1\beta_2\ldots\beta_N} \tag{C.18}$$

and

$$j^{(N)}_{\mu\mathrm{KG}} = \lambda_N^{-1}\psi^{(N)*}_{\alpha_1\alpha_2\ldots\alpha_N}\left\{\begin{array}{cccc} (\gamma_0)_{\alpha_1\beta_1} & (\gamma_0)_{\alpha_2\beta_2} & \cdots & (\gamma_0)_{\alpha_N\beta_N}\\ (-1)^{\frac{N}{2}}(i\gamma_0\gamma_5)_{\alpha_1\beta_1} & (i\gamma_0\gamma_5)_{\alpha_2\beta_2} & \cdots & (i\gamma_0\gamma_5)_{\alpha_N\beta_N}\end{array}\right\}$$
$$\times\left[i\overset{\leftrightarrow}{\partial}_\mu - gA_\mu(0)\right]\psi^{(N)}_{\beta_1\beta_2\ldots\beta_N}\,. \tag{C.19}$$

Here, the alternative $\left\{\begin{array}{c}\cdot\\ \cdot\end{array}\right\}$ is valid for $\left\{\begin{array}{c}N\,\text{odd}\\ N\,\text{even}\end{array}\right\}$.

For $N = 1,3,5$, the current $j^{(N)}_{\mu\mathrm{D}}(x)$ can be reduced to the *conserved* current $j^{(f_N)}_{\mu\mathrm{D}}(x)$ defined in Eq. (C.7). For $N = 2,4$, on the other hand, the current $j^{(N)}_{\mu\mathrm{D}}(x)$ as well as the current $j^{(b_N)}_{\mu\mathrm{D}}(x)$ do not exist.

For $N = 1,3,5$, the current $j^{(N)}_{\mu\mathrm{KG}}(x)$ is reducible to the generically *non-conserved* current $j^{(f_N)}_{\mu\mathrm{KG}}(x)$ introduced in Eq. (C.10). For $N = 2,4$, however, the current $j^{(N)}_{\mu\mathrm{KG}}(x)$ can be reduced to the *conserved* current $j^{(b_N)}_{\mu\mathrm{KG}}(x)$ defined in Eq. (C.5). For an illustration we will perform the last reduction in the case of $N = 2$.

In fact, for $N = 2$, due to the definition (C.1) of the scalar $\psi^{(b_2)}(x) = \frac{1}{4}\,\mathrm{Tr}\left[C^{-1}\gamma_5\psi^{(2)\,\mathrm{T}}(x)\right]$, where $\psi^{(2)}(x) = \left(\psi^{(2)}_{\alpha_1\alpha_2}(x)\right)$ is a 4×4 formal matrix, we can infer that $\psi^{(2)}(x) = (\gamma_5 C)^{\rm T}\, \psi^{(b_2)}(x) + R(x)$ with $\mathrm{Tr}[C^{-1}\gamma_5 R^{\rm T}(x)] = 0$. Here, T denotes the transposition with respect to Dirac bispinor indices. Expanding the matrix $\psi^{(2)\,\rm T}(x)$ in terms of the Dirac relativistic covariants S, P, V, A, T, where S $= \mathbf{1}$, we can write

$$\begin{aligned}\psi^{(2)\,\rm T} &= \left\{\mathbf{1}\psi^{(\rm S)} + \gamma_5\psi^{(\rm P)} + \gamma_\rho\psi^{(\rm V)\,\rho} + \gamma_\rho\gamma_5\psi^{(\rm A)\,\rho} + \frac{i}{2}\left[\gamma_\rho,\, \gamma_\sigma\right]\psi^{(\rm T)\,\rho\sigma}\right\}\gamma_5 C \\ &= \gamma_5 C\psi^{(\rm S)} + R^{\rm T} \qquad \text{(C.20)}\end{aligned}$$

with

$$R^{\rm T} = C\psi^{(\rm P)} + \gamma_\rho\gamma_5 C\psi^{(\rm V)\,\rho} + \gamma_\rho C\psi^{(\rm A)\,\rho} + \frac{i}{2}\left[\gamma_\rho,\, \gamma_\sigma\right]\gamma_5 C\psi^{(\rm T)\,\rho\sigma}\,. \qquad \text{(C.21)}$$

Hence, $\psi^{(b_2)}(x) = \frac{1}{4}(\mathrm{Tr}\,\mathbf{1})\psi^{(\rm S)}(x) = \psi^{(\rm S)}(x)$. When for $N = 2$ only the scalar $\psi^{(b_2)}(x) = \psi^{(\rm S)}(x)$ appears, we get the truncation $R(x) = 0$ and so,

$$\psi^{(2)} = \psi^{(2)}_{\rm S} \equiv (\gamma_5 C)^{\rm T}\psi^{(\rm S)}\,. \qquad \text{(C.22)}$$

For a justification of the absence of $\psi^{(\rm P)}(x)$, $\psi^{(\rm V)\rho}(x)$, $\psi^{(\rm A)\rho}(x)$ and $\psi^{(\rm T)\rho\sigma}(x)$ from the expansion (C.20) see Eqs. (C.30) and (C.31) later on. Now, after a simple calculation, we show that the form (C.22) of $\psi^{(2)}(x)$ leads through Eq. (C.19) with $N = 2$ to the expression

$$\begin{aligned}\frac{1}{4} j^{(2)}_{\mu\rm KG} &\equiv \frac{1}{4}\lambda_2^{-1}\psi^{(2)*}_{\alpha_1\alpha_2}(\gamma_0\gamma_5)_{\alpha_1\beta_1}(\gamma_0\gamma_5)_{\alpha_2\beta_2}\left[i\overleftrightarrow{\partial}_\mu - gA_\mu(0)\right]\psi^{(2)}_{\beta_1\beta_2} \\ &= \lambda_2^{-1}\psi^{(\rm S)*}\left[i\overleftrightarrow{\partial}_\mu - gA_\mu(0)\right]\psi^{(\rm S)} \qquad \text{(C.23)}\end{aligned}$$

that with $\lambda_2 = \lambda_{b_2}$ is equal to the Klein–Gordon current $j^{(b_2)}_{\mu\rm KG}(x)$ of b_2 scalar bosons, introduced in Eq. (C.5). The factor $\frac{1}{4}$ in Eq. (C.23) is caused by reduction of the number of wave-function components due to the relations $\psi^{(2)}_{\alpha_1\alpha_2}(x) = -\psi^{(2)}_{\alpha_2\alpha_1}(x)$ and $\psi^{(2)}_{12}(x) = \psi^{(2)}_{34}(x)$ [see the definition (C.1)]. These relations follow explicitly from the form of

$$\psi^{(2)} \equiv \left(\psi^{(2)}_{\alpha_1\alpha_2}\right) = -\gamma_5 C\psi^{(\rm S)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\psi^{(\rm S)}\,, \qquad \text{(C.24)}$$

valid in the chiral representation, where

$$\gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -i\sigma_2^{(\mathrm{P})} & 0 \\ 0 & i\sigma_2^{(\mathrm{P})} \end{pmatrix} . \tag{C.25}$$

Here, $(\gamma_5 C)^{\mathrm{T}} = -\gamma_5 C$.

In order to justify the truncated form (C.22) of $\psi^{(2)}(x)$, let us introduce for $N = 2$ the matrix of total internal parity:

$$\eta^{(2)} \mathit{\Gamma}_{10}^{(2)} \mathit{\Gamma}_{20}^{(2)} = \gamma_0\gamma_5 \otimes \gamma_0\gamma_5 = ((\gamma_0\gamma_5)_{\alpha_1\beta_1}(\gamma_0\gamma_5)_{\alpha_2\beta_2}) \,, \tag{C.26}$$

where $\eta^{(2)} = -i$, and the matrix of total chirality:

$$\mathit{\Gamma}_{15}^{(2)} \mathit{\Gamma}_{25}^{(2)} = \gamma_5 \otimes \gamma_5 = ((\gamma_5)_{\alpha_1\beta_1}(\gamma_5)_{\alpha_2\beta_2}) \tag{C.27}$$

[see Eqs. (C.14) and (C.17)]. With these matrices acting on $\psi_{\mathrm{S}}^{(2)}(x) \equiv (\gamma_5 C)^{\mathrm{T}} \psi^{(\mathrm{S})}(x) = \left(\psi_{\mathrm{S}\beta_1\beta_2}^{(2)}(x)\right)$ [see Eq. (C.22)] we can show after a simple calculation that

$$\eta^{(2)} \mathit{\Gamma}_{10}^{(2)} \mathit{\Gamma}_{20}^{(2)} \psi_{\mathrm{S}}^{(2)} = \psi_{\mathrm{S}}^{(2)} ,\ \mathit{\Gamma}_{15}^{(2)} \mathit{\Gamma}_{25}^{(2)} \psi_{\mathrm{S}}^{(2)} = \psi_{\mathrm{S}}^{(2)} . \tag{C.28}$$

In addition, of course, $\frac{1}{2}(\Sigma_{1\mu\nu}^{(2)} + \Sigma_{2\mu\nu}^{(2)})\psi_{\mathrm{S}}^{(2)} = 0$. This means that our spin-0 wave function or field $\psi_{\mathrm{S}}^{(2)}(x) = \left(\psi_{\mathrm{S}\alpha_1\alpha_2}^{(2)}(x)\right)$ gets the eigenvalue of total internal parity equal to $+1$ and the eigenvalue of total chirality equal also to $+1$. Here, evidently, the matrices $\eta^{(2)} \mathit{\Gamma}_{10}^{(2)} \mathit{\Gamma}_{20}^{(2)}$ and $\mathit{\Gamma}_{15}^{(2)} \mathit{\Gamma}_{25}^{(2)}$ commute, and also commute with $\frac{1}{2}(\Sigma_{1\mu\nu}^{(2)} + \Sigma_{2\mu\nu}^{(2)})$ for $\mu, \nu = 1, 2, 3$.

After a calculation, it turns out that, among the terms

$$\begin{aligned} \psi_{\mathrm{P}}^{(2)} &\equiv C^{\mathrm{T}} \psi^{(\mathrm{P})} ,\ \psi_{\mathrm{V}}^{(2)} \equiv (\gamma_\rho\gamma_5 C)^{\mathrm{T}} \psi^{(\mathrm{V})\rho} ,\ \psi_{\mathrm{A}}^{(2)} \equiv (\gamma_\rho C)^{\mathrm{T}} \psi^{(\mathrm{A})\rho} , \\ \psi_{\mathrm{T}}^{(2)} &\equiv \left(\frac{i}{2}[\gamma_\rho,\, \gamma_\sigma]\gamma_5 C\right)^{\mathrm{T}} \psi^{(\mathrm{T})\rho\sigma} \end{aligned} \tag{C.29}$$

whose sum gives the matrix $R(x)$ introduced through Eq. (C.21), only $\psi_{\mathrm{P}}^{(2)}(x)$ is an eigenstate of the total internal parity $\eta^{(2)} \mathit{\Gamma}_{10}^{(2)} \mathit{\Gamma}_{20}^{(2)}$ (though, this time, with the eigenvalue equal to -1), while three others are not such eigenstates. In fact,

$$\begin{aligned} \eta^{(2)} \mathit{\Gamma}_{10}^{(2)} \mathit{\Gamma}_{20}^{(2)} \psi_{\mathrm{P}}^{(2)} &= -\psi_{\mathrm{P}}^{(2)} , \\ \eta^{(2)} \mathit{\Gamma}_{10}^{(2)} \mathit{\Gamma}_{20}^{(2)} \psi_{\mathrm{V}}^{(2)} &= -\gamma_0 \psi_{\mathrm{V}}^{(2)} \gamma_0^{\mathrm{T}} , \\ \eta^{(2)} \mathit{\Gamma}_{10}^{(2)} \mathit{\Gamma}_{20}^{(2)} \psi_{\mathrm{A}}^{(2)} &= +\gamma_0 \psi_{\mathrm{A}}^{(2)} \gamma_0^{\mathrm{T}} , \\ \eta^{(2)} \mathit{\Gamma}_{10}^{(2)} \mathit{\Gamma}_{20}^{(2)} \psi_{\mathrm{T}}^{(2)} &= -\gamma_0 \psi_{\mathrm{T}}^{(2)} \gamma_0^{\mathrm{T}} . \end{aligned} \tag{C.30}$$

Now, in view of Eqs. (C.30), we make tentatively the assumption that only terms with *definite* total intrinsic parity can contribute to the expansion of the state $\psi^{(2)}(x)$, described in Eq. (C.20). Such a constraint imposed on $\psi^{(2)}(x)$ excludes $\psi^{(2)}_{\rm V}(x)$, $\psi^{(2)}_{\rm A}(x)$, $\psi^{(2)}_{\rm T}(x)$ from the set (C.29), when Eqs. (C.30) are referred to. Then, the Klein–Gordon-type current (C.19) for $N=2$ may be evaluated as follows:

$$\begin{aligned}\frac{\lambda_2}{4} j^{(2)}_{\mu{\rm KG}} \equiv \psi^{({\rm S})*}[\;]_\mu \psi^{({\rm S})} - \psi^{({\rm P})*}[\;]_\mu \psi^{({\rm P})} - \psi^{({\rm V})*}_\rho[\;]_\mu \psi^{({\rm V})\rho} \\ -\psi^{({\rm A})*}_\rho[\;]_\mu \psi^{({\rm A})\rho} - 2\psi^{({\rm T})*}_{\rho\sigma}[\;]_\mu \psi^{({\rm T})\rho\sigma} \\ = \psi^{({\rm S})*}[\;]_\mu \psi^{({\rm S})} - \psi^{({\rm P})*}[\;]_\mu \psi^{({\rm P})}\,,\end{aligned} \tag{C.31}$$

where $[\;]_\mu \equiv \Big[i\overset{\leftrightarrow}{\partial}_\mu - gA_\mu(0)\Big]$. But, the second term in Eq. (C.31) can spoil the positive-definiteness of $\psi^{(2)}(x)$ (for positive energies), expressed by the requirement of $j^{(2)}_{0{\rm KG}}(x) > 0$ (for positive energies). This excludes $\psi^{(2)}_{\rm P}(x)$ from the set (C.29). Then, from Eq. (C.31) the Klein–Gordon current (C.23) follows.

In general, for any N even, if the total internal parity is diagonal for $\psi^{(N)}(x) = \Big(\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N}(x)\Big)$ with the eigenvalue equal to $+1$:

$$\eta^{(N)}\Gamma^{(N)}_{10}\Gamma^{(N)}_{20}\ldots\Gamma^{(N)}_{N0}\psi^{(N)} = \psi^{(N)}\,, \tag{C.32}$$

then the *conserved* Klein–Gordon-type current (C.13) or (C.19) takes the simplified form:

$$j^{(N)}_{\mu{\rm KG}} = \lambda_N^{-1}\psi^{(N)*}_{\alpha_1\alpha_2\ldots\alpha_N}\Big[i\overset{\leftrightarrow}{\partial}_\mu - gA_\mu(0)\Big]\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N}\,. \tag{C.33}$$

This form is relativistic covariant *only* for states satisfying the constraint (C.32) (which is *not* explicitly covariant). The form (C.33) implies the positive-definiteness of $\psi^{(N)}(x)$ (for positive energies), since in the case of Klein–Gordon-type wave function (of positive energies)

$$j^{(N)}_{0{\rm KG}} = \lambda_N^{-1}\psi^{(N)*}_{\alpha_1\alpha_2\ldots\alpha_N}\Big[i\overset{\leftrightarrow}{\partial}_0 - gA_0(0)\Big]\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N} > 0\,. \tag{C.34}$$

Note that the total internal parity $\eta^{(N)}\Gamma^{(N)}_{10}\Gamma^{(N)}_{20}\ldots\Gamma^{(N)}_{N0}$ is a constant of motion, because $\partial^\mu j^{(N)}_{\mu{\rm KG}}(x) = 0$ for N even. So, the constraint (C.32) is stationary.

As was seen in Eq. (C.28), the constraint (C.32) is satisfied for $N = 2$ with $\psi^{(2)}_{\alpha_1\alpha_2}(x) = (\gamma_5 C)_{\alpha_2\alpha_1}\psi^{(b_2)}(x)$. Notice that also for $N = 4$ with

$\psi^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4}(x) = \varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\psi^{(b_4)}(x)$ the constraint (C.32) holds. Here also, the total chrality $\Gamma^{(4)}_{15}\Gamma^{(4)}_{25}\Gamma^{(4)}_{35}\Gamma^{(4)}_{45}$ gets its eigenvalue $+1$.

For any N odd, an analogical role is played by the total "relative" internal parity which, if diagonal for $\psi^{(N)}(x) = \left(\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N}(x)\right)$ with the eigenvalue equal to $+1$:

$$\xi^{(N)}\Gamma^{(N)}_{20}\ldots\Gamma^{(N)}_{N0}\psi^{(N)} = \psi^{(N)}\,, \tag{C.35}$$

simplifies the form of *conserved* Dirac-type current (C.12) or (C.18):

$$j^{(N)}_{\mu\mathrm{D}} = \psi^{(N)*}_{\alpha_1\alpha_2\ldots\alpha_N}\left(\gamma_0\gamma_\mu\right)_{\alpha_1\beta_1}\psi^{(N)}_{\beta_1\alpha_2\ldots\alpha_N}\,. \tag{C.36}$$

Similarly as before, this form is relativistic covariant *only* for states fulfilling the constraint (C.35) (that is *not* explicitly covariant in the world of "relative" Dirac degrees of freedom). The form (C.36) leads to the positive-definiteness of $\psi^{(N)}(x)$ (for all energies), because in the case of Dirac wave function (of all energies)

$$j^{(N)}_{0\mathrm{D}} = \psi^{(N)*}_{\alpha_1\alpha_2\ldots\alpha_N}\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N} > 0\,. \tag{C.37}$$

Note that, this time, the total "relative" internal parity $\xi^{(N)}\Gamma^{(N)}_{20}\ldots\Gamma^{(N)}_{N0}$ is a constant of motion, since $\partial^\mu j^{(N)}_{\mu\mathrm{D}}(x) = 0$ for N odd. Thus, the constraint (C.35) is stationary. For an illustration, we will show that the constraint (C.35) is satisfied for the form of $\psi^{(3)}_{\alpha_1\alpha_2\alpha_3}(x)$ corresponding to the bispinor $\psi^{(f_3)}_{\alpha_1}(x)$ defined in Eq. (B.14).

In fact, for $N = 3$, due to the definition (B.14) of the bispinor

$$\psi^{(f_3)}_{\alpha_1}(x) = \tfrac{1}{4}(C^{-1}\gamma_5)_{\alpha_2\alpha_3}\psi^{(3)}_{\alpha_1\alpha_2\alpha_3}(x),$$

we can conclude that $\psi^{(3)}_{\alpha_1\alpha_2\alpha_3}(x) = (\gamma_5 C)_{\alpha_3\alpha_2}\psi^{(f_3)}_{\alpha_1}(x)$, as for $N = 3$ only the bispinor $\psi^{(f_3)}_{\alpha_1}(x)$ exists. Since

$$\xi^{(3)}\Gamma^{(3)}_{20}\Gamma^{(3)}_{30} = \mathbf{1}\otimes\gamma_0\otimes\gamma_0 = \left(\delta_{\alpha_1\beta_1}(\gamma_0)_{\alpha_2\beta_2}(\gamma_0)_{\alpha_3\beta_3}\right)\,, \tag{C.38}$$

where $\xi^{(3)} = -i$ [see Eqs. (C.14) and (C.17)], the above form of $\psi^{(3)}_{\alpha_1\alpha_2\alpha_3}(x)$ leads after a simple calculation to the equality

$$\xi^{(3)}\left(\Gamma^{(3)}_{20}\Gamma^{(3)}_{30}\right)_{\alpha_1\alpha_2\alpha_3,\beta_1\beta_2\beta_3}\psi^{(3)}_{\beta_1\beta_2\beta_3} = \psi^{(3)}_{\alpha_1\alpha_2\alpha_3} \tag{C.39}$$

which is the constraint (C.35) for $N = 3$. Note that also for $N = 5$, where $\psi^{(f_5)}_{\alpha_1}(x) = \frac{1}{24}\varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5}\psi^{(5)}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}(x)$ and $\psi^{(5)}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}(x) = \varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5}\psi^{(f_5)}_{\alpha_1}(x)$ (as for $N = 5$ only the bispinor $\psi^{(f_5)}_{\alpha_1}(x)$ exists), we get the eigenvalue equal to $+1$ for the total "relative" intrinsic parity $\xi^{(5)}\Gamma^{(5)}_{20}\Gamma^{(5)}_{30}\Gamma^{(5)}_{40}\Gamma^{(5)}_{50}$, where $\xi^{(5)} = -1$, and so, the constraint (C.35) for $N = 5$ holds.

REFERENCES

[1] For a theoretical summary *cf.* J. Ellis, *Nucl. Phys. Proc. Suppl.* **91**, 503 (2001); and references therein.

[2] W. Królikowski, in *Spinors, Twistors, Clifford Algebras and Quantum Deformations*, Proc. 2nd Max Born Symposium 1992, eds. Z. Oziewicz *et al.*, Kluwer Acad. Press, 1993; *Acta Phys. Pol.* **B27**, 2121 (1996); and references therein.

[3] W. Królikowski, *Acta Phys. Pol.* **B30**, 2631 (1999); *Acta Phys.Pol.* **B32**, 2961 (2001); and references therein.

[4] W. Królikowski, *Acta Phys. Pol.* **B33**, 641 (2002).

[5] The Particle Data Group, *Eur. Phys. J.* **C15**, 1 (2000).

[6] T. Kajita, Y. Totsuka, *Rev. Mod. Phys.* **73**, 85 (2001); T. Toshito, `hep-ex/0105023`.

[7] For a recent analysis *cf.* M.V. Garzelli, C. Giunti, *J. High Energy Phys.* **0112**, 017 (2001); `hep-ph/0111254`; *cf.* also V. Barger, D. Marfatia, K. Whisnant, *Phys. Rev. Lett.* **88**, 011302 (2002).

[8] G. Mills, *Nucl. Phys. Proc. Suppl.* **91**, 198 (2001); R.L. Imlay, Talk at ICHEP 2000 at Osaka; and references therein.

[9] M. Appolonio *et al.*, *Phys. Lett.* **B420**, 397 (1998); **B466**, 415 (1999).

[10] H.V. Klapdor-Kleingrothaus *et al.*, *Mod. Phys. Lett.* **A16**, 2409 (2002).

[11] *Cf. e.g.* V. Barger, B. Kayser, J. Learned, T. Weiler, K. Whisnant, *Phys. Lett.* **B489**, 345 (2000); M.C. Gonzalez-Garcia, M. Maltoni, C. Peña-Garay, `hep-ph/0108073`; and references therein; *cf.* also W. Królikowski, `hep-ph/0106350`; O. Yasuda, `hep-ph/0109067`.

[12] *Cf. e.g.* W. Królikowski, `hep-ph/0007255`.

[13] W. Królikowski, *Acta Phys. Pol.* **B21**, 871 (1990); *Phys. Rev.* **D45**, 3222 (1992); *Acta Phys. Pol.* **B24**, 1149 (1993); *cf.* also Appendix in *Acta Phys. Pol.* **B32**, 2961 (2001) (the second Ref. [3]).

[14] T. Banks, Y. Dothan, D. Horn, *Phys. Lett.* **B117**, 413 (1982).

[15] E. Kähler, *Rendiconti di Matematica* **21**, 425 (1962); *cf.* also D. Ivanenko, L. Landau, *Z. Phys.* **48**, 341 (1928).

[16] W. Królikowski, *Phys. Rev.* **D46**, 5188 (1992).

[17] W. Królikowski, *Acta Phys. Pol.* **B24**, 1149 (1993); **B26**, 1217 (1995); *Nuovo Cim.* **107A**, 69 (1994).