PARTON DISTRIBUTIONS FUNCTIONS OF PION, KAON AND ETA PSEUDOSCALAR MESONS IN THE NJL MODEL

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Parton distributions of pseudoscalar π, K and η mesons obtained within the NJL model using the Pauli–Villars regularization method are analyzed in terms of LO and NLO evolution, and the valence sea quark and gluon parton distributions for the pion are obtained at $Q^2 = 4\text{GeV}^2$ and compared to existing parametrizations at that scale. Surprisingly, the NLO order effects turn out to be small compared to the LO ones. The valence distributions are in good agreement with experimental analyses, but the gluon and sea distributions come out to be softer in the high-x region and harder in the low-x region than the experimental analyses suggest.

1. Introduction

The study of structure functions of hadrons in the Bjorken limit and high enough Q^2, as probed in inclusive Deep Inelastic Scattering (DIS), is traditionally considered the domain of perturbative QCD since the running coupling constant, $\alpha(Q^2)$, becomes small [1]. Present day QCD Leading Order (LO) and Next to Leading Order (NLO) phenomenological calculations can relate leading twist contributions to structure functions among

different momentum scales through the well known linear integro-differential Gribov–Lipatov–Altarelli–Parisi (GLAP) equations [2]. This makes sense if Q^2 is high enough so that only leading twist logarithmic corrections contribute and higher twist power-like corrections are negligible. To start with, some theoretical or experimental nonperturbative profile function is needed as initial condition for the GLAP equations. In the nucleon case, QCD scaling violations have been confirmed by relating experimental partonic distributions at many Q^2 values, and many phenomenological parametrizations have been proposed [3–6]. Naturally, these parametrizations are under continuous update to incorporate increasing information obtained from current experiments. The net result is that uncertainties in the parton distribution functions of the nucleon include not only a large body of experimental data but also theoretical NNLO or higher twist error estimates which provide a, perhaps inaccurate but undoubtedly systematic description within a large region in the x, Q^2 plane (see, $e.g.$, the talks in Ref. [7]).

In contrast with the nucleon case, our present knowledge of parton distribution functions for other hadrons is rather poor. As suggested long ago [8], it is possible to estimate distribution functions using constituent quark models to evaluate the low energy initial condition under the assumption that the gluon and sea content of hadrons vanish at the corresponding low energy resolution scale, and dynamically generate them by QCD evolution to higher Q^2 scales. These estimates can then be used to test the sensitivity of various experiments to the distributions of interest. Recently, this approach has also been applied to generalized parton distributions (GPD's) [9–14], which are generalizations of the usual parton distributions considered in this work and are related via a sum rule to the elastic form factors. Unlike the usual parton distributions, the GPD's are not directly measurable as observables are always expressed in terms of them via a convolution formula. This has lead some [15] to conclude that, at present, model calculations of the GPD's at a low scale are needed to assess the sensitivity of various observables to the GPD's. In practice, this requires an evolution of the GPD's calculated at the low energy scale to the scale relevant to the experiment. As α is rather large at the low energy scale, one must worry about the use of perturbative evolution to connect the low energy model with the high energy data. Thus, it seems prudent to test this procedure in a situation where data are available to compare with theory, $i.e.$, the usual parton distributions. To test the validity of this approach, it is necessary to compare the LO and NLO results not only for the valence distributions but also for the sea and gluon distributions.

From a theoretical viewpoint, pseudoscalar mesons and specifically π and K mesons are particularly distinguished hadrons since most of their low energy properties follow the patterns dictated by chiral symmetry. Actually, we do not expect to understand the properties of any hadron better than the pion, as Chiral Perturbation Theory suggests. By extension, one might think that the parton structure of a pion is the simplest one to consider provided chiral symmetry constraints, *i.e.*, spontaneous and explicit chiral symmetry breaking, are properly incorporated. The recent work [16] clarifies this point regarding explicit chiral symmetry breaking; ChPT allows one to systematically compute chiral corrections to the moments of structure functions, but says nothing about the soft pion limit. Each moment corresponds to a undetermined low energy parameter which renormalizes a local operator. On the other hand, improved QCD sum rules have also been employed [17] to determine the quark distribution functions of the pion in the intermediate x region, $0.15 < x < 0.7$ at $Q^2 = 2 \, \text{GeV}^2$ where the model is applicable. The absolute normalization becomes a problem since some ansatz for the distributions must be made outside this x-range.

Among the quark models where spontaneous breaking of chiral symmetry plays a dominant role, the Nambu–Jona-Lasinio (NJL) model provides a particular example of a chiral quark model where a unified picture of vacuum, mesons and nucleons is achieved [18]. Pseudoscalar mesons appear as quark–antiquark excitations of the spontaneously broken vacuum. Several calculations of the pion structure functions within chiral quark loop models have been made and many different results for the initial conditions have been obtained. One important and tricky reason for the discrepancies lies in the use of different regularization procedures. As the bosonized version of the NJL model is similar to other quark-loop models of the pion (the πqq coupling is γ_5-like), we think it of interest to briefly review them and comment on the main differences. The use of different regularizations might be regarded as an objection to the NJL model itself. However, not every regularization scheme can be considered acceptable. Actually, some of the quark-loop calculations violate some necessary conditions on the regularization. We argue in the following that in some cases one should blame the regularization scheme instead of the model. At a formal level, the process of going from the hadronic to the distribution function can be done by using the so-called quark-target scattering formula [19]. A large body of quark loop model calculations have been done making use of these ideas [20–31]. The problem of proceeding in that way is that the distribution function may turn out to be non-normalizable.

Unfortunately, no first principle calculations of structure functions for pseudoscalar mesons are yet available with the exception of some standard lattice calculations of the lowest moments [32–35], albeit in the quenched

approximation and subjected to well-known problems with chiral extrapola-
tions. In addition, the reconstruction of the x dependent structure function
via some inverse moments method is strongly biased in the intermediate and
low x regions. The transverse lattice approach employed in Refs. [36,37] of-
fers the possibility of directly computing structure functions in x-space. In
any case, as one might expect from the quenched approximation, the lattice
results provide a larger momentum fraction of valence quarks than those
suggested by phenomenological analyses [38–40].

Although not as well determined as the nucleon, the parton structure
of the pion has been analyzed on a phenomenological level [38] and a sim-
ple parametrization at $Q^2 = 4\,\mathrm{GeV}^2$ has been given. The valence quark
distributions extracted in this work [38] from Drell–Yan experiments [41]
seem well determined, whereas the gluon distribution as obtained from the
analysis of prompt-photon emission data [42] is less well determined. On
a phenomenological level, the constituent model proposed in Ref. [43] for
the valence distributions of the pion has been further extended to the sea
and gluon distributions [39] and the K/π valence up-quark ratio. In these
calculations and in the recent update [40] the required total valence mo-
mentum fraction in the pion at $Q^2 = 4\,\mathrm{GeV}^2$ is taken to be the same as
for the nucleon, $\langle xV_\pi \rangle = \langle xV_N \rangle = 0.40$, a bit below the value $\langle xV_\pi \rangle = 0.47$
of Ref. [38]. As different data sets have been fitted and different nucleon
parton distributions have been used in the different analyses, it is not clear
what to make of the differences. In addition, although Ref. [38] includes
error estimates, the model analysis of Refs. [39,40] does not include them,
and therefore it is not possible to know if the differences are significant. Let
us note that the E615 experiment [41] suggests the valence density of the
pion may be enhanced by about 20% compared to the proton, and a recent
analysis [44] of the ZEUS di-jet data seem to favor the gluon distributions
of Ref. [38]. Thus, in determining the low-energy scale of our model, we use
the valence momentum fraction found in Ref. [38]. Finally, we also compare
with the K^-/π^- structure function ratio at $Q^2 = 20\,\mathrm{GeV}^2$, which has been
measured using the Drell–Yan process [45].

2. Remarks on pion parton distribution functions
in quark loop models

In a previous work [22], we found the structure function of the pion to
be a constant function of x in the NJL model in the chiral limit and in the
leading order of a large N_c expansion. To get this result the use of a suit-
able regularization method was needed. A thorough study of regularization
methods in the NJL model may be found in Ref. [46] and we refer to that
work for a more detailed description. By suitable we mean several desirable
properties that should be incorporated, namely:

- The connection between the forward Compton amplitude and the quark-target scattering amplitude is valid only for gauge invariant, finite amplitudes. For this reason, some gauge invariant regularization must be imposed on the Compton amplitude. Naive sharp cut-offs are not acceptable from this viewpoint. In addition, this way of proceeding represents a further advantage, since in the NJL model it is only known how to regularize closed quark loops. The quark target scattering amplitude corresponds to an open quark line.

- The regularization must produce exact scaling in the Bjorken limit. The main reason is that this is the only way we know how to extract the leading, and eventually higher, twist contributions for which QCD evolution is known. This eliminates proper-time regularization, since it produces unrealistic scaling violations.

- The regularization must also be able to work away from the chiral limit, but without spoiling the QCD anomaly. The former condition precludes a single Pauli–Villars subtraction.

- The regularization should allow calculations in both Minkowski and Euclidean space, *i.e.*, dispersion relations must be fulfilled. This turns out to be very convenient for DIS calculations, since cutting rules may be used.

- The resulting distributions should satisfy the normalization condition and the momentum sum rule.

We found in Ref. [22] that the Pauli–Villars with two subtractions fulfills the desired requirements. In addition, the Pauli–Villars scheme does not spoil the good description of other low energy hadronic properties found in the NJL model [18,47,48], fulfills dispersion relations [49], and allows one to regularize the Dirac sea of the chiral soliton away from the chiral limit [26]. Taking π^+ for definiteness, one gets in the chiral limit

$$u_{\pi^+}(x, Q_0^2) = \bar{d}_{\pi^+}(x, Q_0^2) = \theta(x)\theta(1 - x) \ . \tag{1}$$

The results for $m_\pi \neq 0$ are displayed for completeness in the Appendix. By construction, Eq. (1) is consistent with chiral symmetry. The result was obtained by several means within the NJL model either using Pauli–Villars regularization [22,26] on the virtual Compton amplitude or imposing a transverse cut-off [25] upon the quark-target amplitude. This result has been recently re-derived [31] in a chiral quark model solving chiral Ward identities by using the so-called gauge technique [50]. The easiest way to understand Eq. (1) is perhaps in terms of phase space arguments and

point couplings (*i.e.*, constant matrix elements) [51]. For a massless pion this is justified since intermediate states in the quark-target amplitude have $p_n^+ = m_\pi (1 - x) \to 0$ and the low momentum components of $\pi \bar{q} q$ matrix element dominate. Let us mention that Eq. (1) disagrees with other NJL calculations, due to the use of different regularizations. If the virtual Compton amplitude is used with a four-dimensional cut-off [20] or the quark-target amplitude is used with Lepage–Brodsky regularization [25], different shapes for the quark distributions are obtained. The null-plane [21] NJL model with sharp cut-off [20], Light-Cone (LC) quantized NJL model [27] and spectator model [24] calculations also produce different results. In all cases, the use of momentum dependent form factors or non-gauge invariant regularizations make the connection between Compton amplitude and quark-target amplitude doubtful and, furthermore, spoil normalization. The results based on a quark loop with momentum dependent quark masses [28–30] seem to produce a non-constant distribution. Recent calculations on the transverse lattice reveal [36] either an almost flat structure very much resembling Eq. (1) at a scale $Q^2 = 1\,\mathrm{GeV}^2$ or a more bumped form [37]. The reason for the discrepancy between these two transverse-lattice calculations is not obvious to us.

In this paper we study within LO and NLO the parton content of pseudoscalar mesons, namely π, K and η including valence, gluon and sea distributions, thus extending our previous work [22] where only the initial conditions were presented and the LO evolution for the valence distributions. There, we analyzed the LO valence contribution and impressive agreement with SMRS [38] parametrization at $Q^2 = 4\,\mathrm{GeV}^2$ was obtained. Encouraged by this success we extend our analysis to the sea and gluon distributions both in LO and NLO evolution.

3. Numerical results

3.1. Momentum fraction analysis

To perform the evolution, one must determine the scale Q_0 of the model. We determine this scale by fitting the valence quark momentum fraction at 4 GeV². For definiteness, we take the running strong coupling constant at the Z mass, $M_Z = 91.12\,\mathrm{GeV}$, to be $\alpha(M_Z^2) = 0.116$ and evolve it down by exactly solving the differential equation

$$\frac{d\alpha}{dt} = \beta(\alpha) = -\alpha \left[\beta_0 \left(\frac{\alpha}{4\pi} \right) + \beta_1 \left(\frac{\alpha}{4\pi} \right)^2 + \cdots \right], \qquad (2)$$

where $t = \ln(Q^2/Q_0^2)$ and $\alpha = g^2/(4\pi)$. We take the number of active flavors to diminish by one unit each time a quark threshold is crossed, *i.e.*,

$N_{\rm F}(Q^2) = \sum_{i=u,ds,c,b,t} \theta(Q^2 - m_i^2)$, with $m_b = 4.5\,{\rm GeV}$ and $m_c = 2.0\,{\rm GeV}$. For $N_{\rm F} = 3$ one has $\beta_0 = 9$, $\beta_1 = 64$. This yields the value $\alpha(4{\rm GeV}^2) = 0.284$. Below that scale we fix the number of flavors equal to three, since we consider evolution below charm threshold. The previous formula, Eq. (2), is used to transform the variable t into the variable α, by exactly[1] solving the differential equation. Since we numerically perform the NLO evolution of the sea and gluon distributions, it is convenient to specify the initial α_i at t_i and numerically integrate Eq. (2) to t_f to obtain α_f. We note, however, that it is also possible to find an implicit solution for α_f in terms of α_i, t_i and t_f. Specifically, we find

$$
\begin{aligned}
\frac{1}{\alpha_f} + \frac{\beta_1}{4\pi\beta_0}\ln\left(\frac{\alpha_f}{\alpha_i}\right) - \frac{\beta_1}{4\pi\beta_0}\ln\left(\frac{1 + \frac{\beta_1}{4\pi\beta_0}\alpha_f}{1 + \frac{\beta_1}{4\pi\beta_0}\alpha_i}\right) &= \frac{\beta_0}{4\pi}(t_f - t_i) + \frac{1}{\alpha_i} \\
&= \frac{\beta_0}{4\pi}\ln\left(\frac{Q_f^2}{Q_i^2}\right) + \frac{1}{\alpha_i}\,. \quad (3)
\end{aligned}
$$

Although we are not aware of an analytic solution for α_f in terms of the other parameters, this equation may be solved numerically very quickly and accurately using Newton's method. For example, taking $Q_i = m_c$ and $\alpha_i = 0.284$, and using this α_i as the initial seed for α_f, one obtains at least eight-significant digit accuracy for α_f after at most ten iterations all the way down to Q_f of 0.4 GeV. This form also enables one to determine at what scale α_f diverges. For $\alpha_f \to \infty$, we obtain

$$
\frac{2\pi\beta_0}{\beta_1\alpha_f^2} \approx \frac{1}{\alpha_i} + \frac{\beta_0}{2\pi}\ln\left(\frac{Q_f}{Q_i}\right) - \frac{\beta_1}{4\pi\beta_0}\ln\left(1 + \frac{4\pi\beta_0}{\beta_1\alpha_i}\right)\,. \quad (4)
$$

Evidently, α_f diverges when the right-hand side of the above equation vanishes, which happens at a scale of $Q_f \approx 0.365$ GeV.

The non-singlet momentum fraction satisfies the differential equation

$$
\beta(\alpha)\frac{dV_2(\alpha)}{d\alpha} = \gamma_{2,{\rm NS}}(\alpha)V_2(\alpha)\,, \quad (5)
$$

where $V_2(\alpha)$ can be any non-singlet quark distribution. Up to two loops one obtains the expansion [2]

$$
\gamma_{2,{\rm NS}}(\alpha) = \gamma_{2,{\rm NS}}^{(0)}\left(\frac{\alpha}{4\pi}\right) + \gamma_{2,{\rm NS}}^{(1)}\left(\frac{\alpha}{4\pi}\right)^2 + \cdots\,. \quad (6)
$$

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[1] By 'exact' we mean solving the differential equation with a numerical accuracy much greater than the experimental uncertainty on α.

To proceed further, we use the results from Ref. [38] where it was found that at $Q^2 = 4\,\mathrm{GeV}^2$ valence quarks carry 47% of the total momentum fraction in the pion, *e.g.*, for π^+,

$$\langle x\left(u_\pi - \bar{u}_\pi + \bar{d}_\pi - d_\pi\right)\rangle = 0.47 \qquad \text{at} \qquad Q^2 = 4\,\mathrm{GeV}^2\,. \tag{7}$$

Evolving downwards, we get that for $\alpha_0 = 1.89(1.487)$ valence quarks carry 100% of the total momentum in the pion in LO (NLO).

3.2. Pion structure functions

Having determined Q_0 of our model, we evolve the structure functions to $Q^2 = 4$ GeV^2 using the scheme presented in Ref. [52], which requires an analytical formula for the moments of the distribution function. In the chiral limit, $m_\pi = 0$, the moments may be trivially computed. Away from it, $m_\pi \neq 0$, they can be expressed in terms of hypergeometric functions $_2F_1$, but it is more convenient, and just as accurate, to make a polynomial approximation in x-space and then compute analytically the moments. For completeness, the result of such a fit for both $u(x)$ and $\bar{d}(x)$ is presented in the Appendix. For the η, it is more convenient to make an expansion in terms of $x(1-x)$, as is discussed in the Appendix. In this work, we take $m_\pi = 139.6$ MeV, $m_k = 494$ MeV, $f_\pi = 93.3$ MeV and $M_u = M_d = 280$ MeV, resulting in $M_s = 527$ MeV, $\Lambda = 870$ MeV and $m_\eta = 501$ MeV (exp. 549 MeV).

Our LO and NLO valence, sea and gluon distribution functions evolved from the quark model point, Q_0^2, where the valence quarks carry all the momentum, to the point $Q^2 = 4\,\mathrm{GeV}^2$ where gluon and sea distributions are dynamically generated, are shown in Fig. 1. They are compared to the phenomenological analysis of Refs. [38] and [40]. The remaining distributions trivially fulfill

$$\begin{aligned}
\bar{d}_{\pi^+}(x, Q^2) &= u_{\pi^+}(x, Q^2)\,, \\
\bar{u}_{\pi^+}(x, Q^2) &= d_{\pi^+}(x, Q^2)\,, \\
\bar{s}_{\pi^+}(x, Q^2) &= s_{\pi^+}(x, Q^2)
\end{aligned} \tag{8}$$

as a consequence of our initial condition and properties of evolution. The LO valence result was already presented in our previous work [22]. We see here that NLO evolution does not make a big difference, providing some confidence in perturbative evolution, even though the quark model point corresponds to α's larger than unity. Actually, it has been suggested that the natural expansion parameter for DIS is α/π, which in our case is about a half, $\alpha(Q_0)/\pi \sim 0.5$.
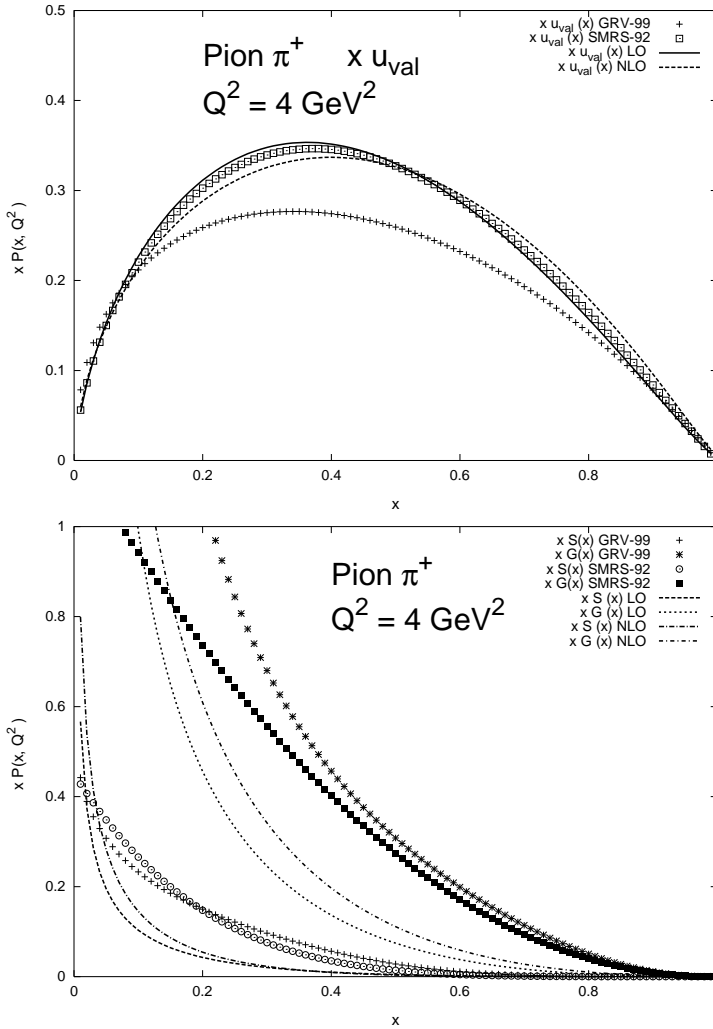
Fig. 1. Valence (upper panel), gluon and sea (lower panel) distributions in the pion, π^+, at $Q^2 = 4\,\mathrm{GeV}^2$ in the NJL model compared with phenomenological analysis for the pion SMRS92 [38] and GRS99 [40]. We take the valence momentum fraction $\langle xV \rangle_\pi = 0.47$ at $Q^2 = 4\mathrm{GeV}^2$.

As can be seen in Fig. 4, the finite pion mass effects turn out to be rather small because chiral corrections to the initial condition are small within the model at the one loop level. While it is conceivable that pion loop effects could provide, as is frequently the case, some logarithmic enhancement to chiral corrections, it is a feature of GLAP evolution equations that upward evolution tends to wash out the differences in the initial condition.
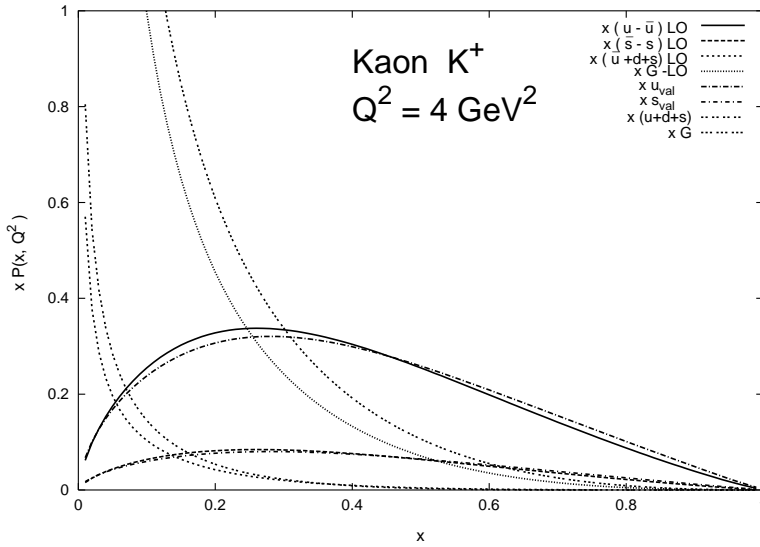
Fig. 2. Valence, gluon and sea distributions in the kaon, K^+, at $Q^2 = 4\,\mathrm{GeV}^2$ in the NJL model. We take the total valence momentum fraction $\langle xV \rangle_\pi = 0.47$ at $Q^2 = 4\,\mathrm{GeV}^2$.
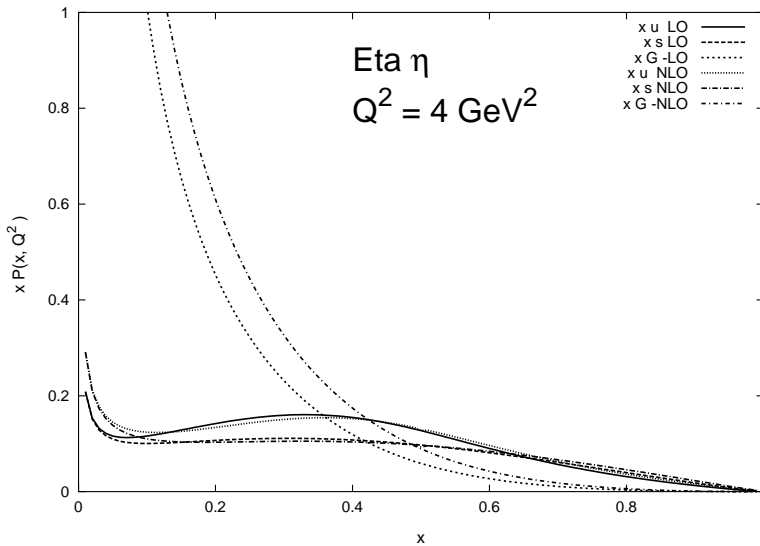

Fig. 3. Up and strange quark and gluon LO and NLO distribution functions in the η-meson, at $Q^2 = 4\,\mathrm{GeV}^2$ in the NJL model.
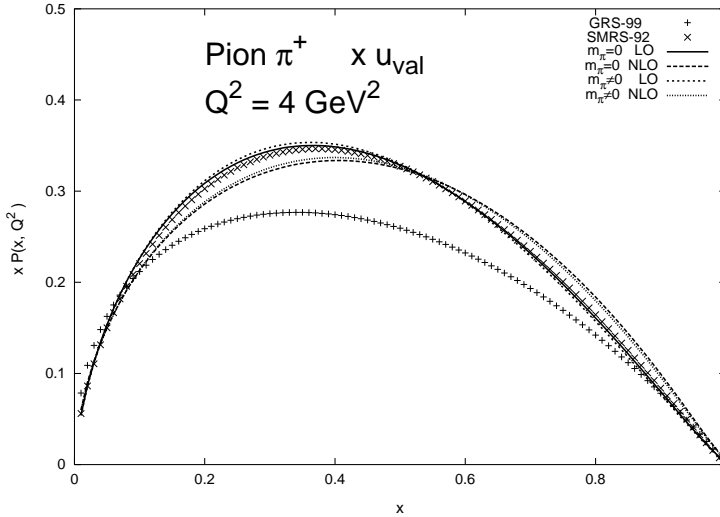
Fig. 4. Chiral corrections to the u-quark valence LO an NLO distribution functions at $Q^2 = 4\,\mathrm{GeV}^2$ compared with phenomenological analysis for the pion SMRS92 [38] and GRS99 [40]. As suggested in Ref. [38] we take $\langle xV \rangle_\pi = 0.47$ at $Q^2 = 4\,\mathrm{GeV}^2$.

We finish our discussion on the pion parton distribution by comparing in Fig. 5 the results obtained by taking either $\langle xV \rangle_\pi = 0.47$ as suggested by the SMRS92 analysis [38] or $\langle xV \rangle_\pi = 0.40$ as implied by the GRS99 parametrization [40]. The sea and gluon distributions are not shown because their dependence on the momentum fraction is rather small. As can be deduced from the figure, the shape of the valence distribution is much better described if, as determined in Ref. [38], the valence quarks carry 47% of the total pion momentum at $Q^2 = 4\,\mathrm{GeV}^2$. Note that, as one might expect, Fig. 5 also illustrates the fact that reproducing the momentum fraction is not sufficient to accurately determine the full shape of the distribution functions. From this point of view the agreement of the NJL evolved valence quark distribution with the SMRS92 parametrization [38] is not entirely trivial.

For comparison, let us also mention that early lattice calculations of Ref. [32, 33] provided $\langle xV_\pi \rangle = 0.64 \pm 0.10$ scale $Q^2 \approx 4.84 \pm 2.2\,\mathrm{GeV}^2$. A recent and more accurate lattice QCD calculation [34] extrapolated to the chiral limit yields the number $\langle xV_\pi \rangle = 0.56 \pm 0.02$ at the scale $Q^2 \approx 5.8\mathrm{GeV}^2$, a larger value than suggested by phenomenology [38,40] and expected from a quenched approximation. The transverse lattice calculation of Ref. [36] gives $\langle xV_\pi \rangle = 0.86 \pm 0.02$ at $Q^2 \approx 1\,\mathrm{GeV}^2$, whereas that of Ref. [37] provides, still at very low scales $Q^2 \approx 0.4\,\mathrm{GeV}^2$, a form for the distribution amplitude surprisingly close to the asymptotic value, $6x(1-x)$. From their parton distribution function one gets $\langle xV_\pi \rangle \approx 0.76$.
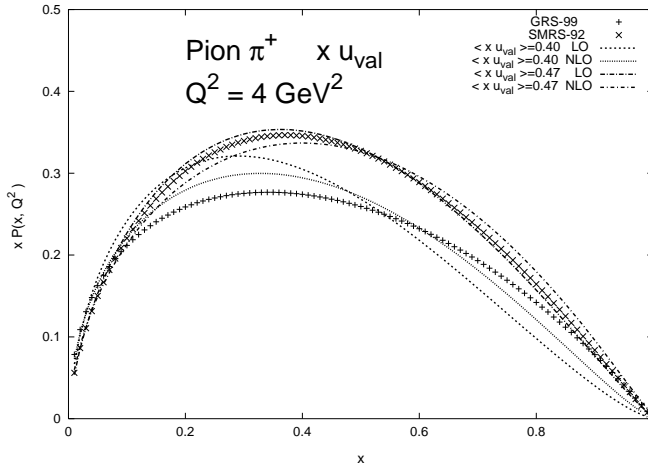
Fig. 5. Dependence of the u-quark valence LO an NLO distribution functions at $Q^2 = 4\,\mathrm{GeV}^2$ on the momentum fraction at that scale, compared with phenomenological analysis for the pion SMRS92 [38] which takes $\langle xV \rangle_\pi = 0.47$ and GRS99 [40] where $\langle xV \rangle_\pi = 0.40$ is used.

3.3. Kaon and eta structure functions

For the kaon and eta, we assume the same Q_0 as for the pion. For the K^+, this immediately leads to

$$\langle x \left(u_K - \bar{u}_K + \bar{s}_K - s_K \right) \rangle = 0.47 \qquad \text{at} \qquad Q^2 = 4\,\mathrm{GeV}^2 \ . \tag{9}$$

Our LO and NLO evolved results for the K^+ parton distributions are shown in Fig. 2. A practical parametrization of the corresponding initial condition may be found in the Appendix. Similar to the pion case, there are only small differences between LO and NLO evolution. The only known information regarding K structure functions is the ratio between the valence up quark distribution in the kaon and the pion, which was originally reported in Ref. [45] and has been reanalyzed in Ref. [39]. In Fig. 6 we show the NJL results, together with the data obtained from Ref. [45]. Besides the LO result, already shown in our previous work [22], we provide the NLO ratio, which does not differ much from the former and is in fair agreement with the experimental data. For the K^+ meson the momentum fraction for the up and strange valence quarks turn out to be

$$\langle x \left(u_K - \bar{u}_K \right) \rangle = 0.20 \,, \quad \langle x \left(\bar{s}_K - s_K \right) \rangle = 0.27 \quad \text{at} \quad Q^2 = 4\,\mathrm{GeV}^2 \,. \tag{10}$$

As could be anticipated from Fig. 2, the difference for these momentum fractions between LO and NLO evolution are small and do not show up within the presented accuracy.
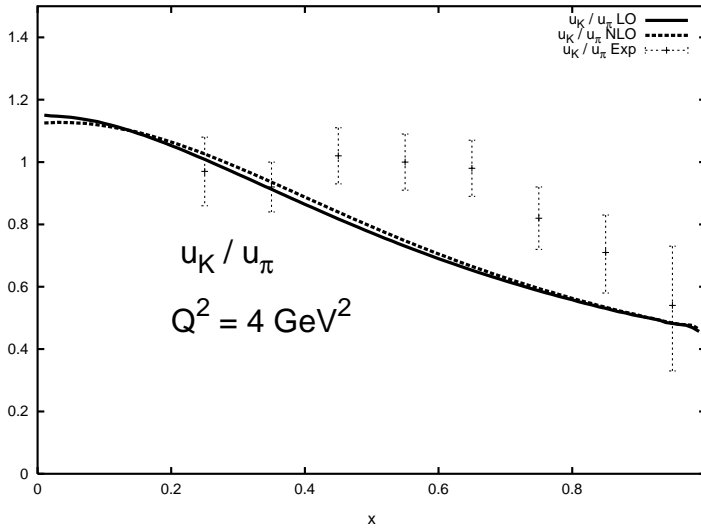
Fig. 6. Valence u-quark kaon/pion ratio of LO and NLO distribution functions in the NJL model at $Q^2 = 4\,\mathrm{GeV}^2$ compared with phenomenological analysis. We take a total valence momentum fraction $\langle xV \rangle_\pi = \langle xV \rangle_K = 0.47$ at $Q^2 = 4\,\mathrm{GeV}^2$. Experimental data from Ref. [45].

Although a phenomenological analysis of the η partonic distributions seems unlikely, for the sake of completeness we show in Fig. 3 our results for the η meson. We do this by evolving from the scale where $\alpha = 1.89\,(1.49)$ at LO (NLO) to $Q^2 = 4\,\mathrm{GeV}^2$ the NJL distributions conveniently parametrized in the Appendix. As explained in our previous work [22], our description relies on a very particular ansatz which provides flavor mixing without quark mass mixing. For the momentum fractions, we obtain

$$\langle xu_\eta \rangle = \langle xd_\eta \rangle = 0.10\,, \quad \langle xs_\eta \rangle = 0.08 \quad \text{at} \quad Q^2 = 4\,\mathrm{GeV}^2\,. \tag{11}$$

As we have noted, the differences in parton distribution functions for massless and massive pions are tiny. In fact, even for K and η, many of the distributions are close to those of the massless pion. By comparing Fig. 1, Fig. 2 and Fig. 3 we point out the strong similarities in the gluon parton distributions between the π,K and η mesons. Likewise, we also find very similar shapes for the sea distributions in the π and K mesons, see Fig. 1 and Fig. 2, as well as in the total valence distributions, see Fig. 7. This is in agreement with having identical total valence momentum fractions for the pion and the kaon at $Q^2 = 4\,\mathrm{GeV}^2$.
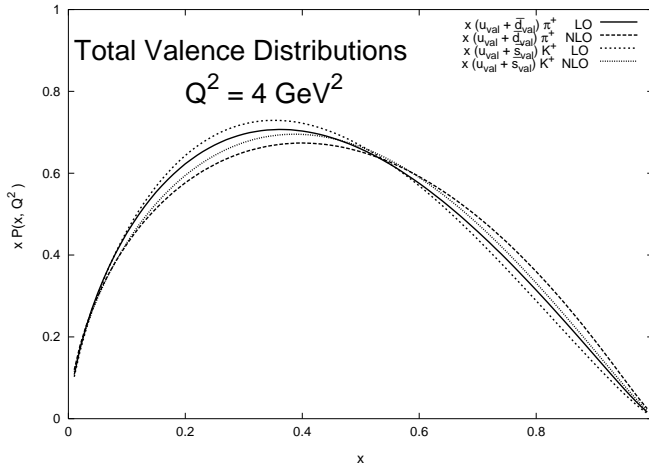
Fig. 7. Total valence π^+ and K^+ LO and NLO distribution functions in the NJL model at $Q^2 = 4\,\mathrm{GeV}^2$. We take $\langle xV \rangle_\pi = \langle xV \rangle_K = 0.47$ at $Q^2 = 4\,\mathrm{GeV}^2$. For π^+ we define $V = u - \bar{u} + \bar{d} - d$ and for K^+ we have $V = u - \bar{u} + \bar{s} - s$.

4. Conclusions

In the present work, we have computed the parton distribution functions of the lowest pseudoscalar mesons, namely π, K and η. To this end we have used the Nambu–Jona-Lasinio distribution functions at the low resolution scale found in our previous work. In common with state of the art calculations, the corresponding sea and gluon distribution functions vanish at that scale, and are dynamically generated through standard GLAP evolution to higher Q^2-values at LO and NLO approximations. For both π and K we have assumed that the valence quarks carry 47% of the total momentum fraction at 4 GeV^2. Despite the fact that $\alpha(Q_0)/\pi \approx 0.5$, the differences between LO and NLO evolution are small. The agreement between the u-quark valence distribution in the pion in the NJL model at LO and the phenomenological analyses is not spoiled at NLO. In addition, we have confirmed at NLO the successful description at LO of the ratio of the valence up quark content in the kaon with respect that of the pion. This provides one with some confidence in the validity of this approach to the study of structure functions, or GPD's in general.

We have also presented LO and NLO sea and gluon distributions of the pseudoscalar mesons. For the pion, we find disagreement with the phenomenological expectations; the gluon and sea distributions come out to be softer in the high-x region and harder in the low-x region than the experimental analysis suggests. We have also provided results for the η meson, which interest seems only theoretical, given the lack of experimental data.

Our analysis, however, reveals some clear trends: all gluon distributions look strikingly similar, and the total valence π and K distributions do not differ much. We hope these observations to be useful to get further insight and guidance into the theoretical description of the poorly known meson structure functions.

Appendix

The π, K and η structure functions found in Ref. [22] may be conveniently written as

$$u_\pi(x) = \bar{d}_\pi(1-x) \;=\; 4N_c g^2_{\pi uu} \frac{d}{dm^2_\pi} \left[m^2_\pi F_{uu}(m^2_\pi, x) \right] \;, \quad (12)$$

$$u_K(x) = \bar{s}_K(1-x) \;=\; 4N_c g^2_{\pi us} \frac{d}{dm^2_K} \left[m^2_K F_{us}(m^2_K, x) \right] , (13)$$

$$u_\eta(x) = \bar{u}_\eta(x) = d_\eta(x) = \bar{d}_\eta(x) \;=\; 4N_c \left(\frac{1}{g^2_{\eta uu}} + \frac{2}{g^2_{\eta ss}} \right)^{-1}$$
$$\times \frac{d}{dm^2_\eta} \left[m^2_\eta F_{uu}(m^2_\eta, x) \right] \;, \quad (14)$$

$$s_\eta(x) = \bar{s}_\eta(x) \;=\; 8N_c \left(\frac{1}{g^2_{\eta uu}} + \frac{2}{g^2_{\eta ss}} \right)^{-1}$$
$$\times \frac{d}{dm^2_\eta} \left[m^2_\eta F_{ss}(m^2_\eta, x) \right] \quad (15)$$

in the interval $0 < x < 1$. The Pauli–Villars regularized one-loop integrals are defined,

$$F_{\alpha\beta}(p^2, x) = -\frac{1}{16\pi^2} \sum_i c_i \log \left[-x(1-x)p^2 + (1-x)M^2_\alpha + xM^2_\beta + \Lambda^2_i \right] \;,$$
$$(16)$$

where $\sum_i c_i f(\Lambda^2_i) = f(0) - f(\Lambda^2) + \Lambda^2 f'(\Lambda^2)$. All other distribution functions are exactly zero, since we do not have gluons or sea quarks in the model. The meson-quark–quark couplings are defined in terms of the residues of the poles in the $q - \bar{q}$ scattering amplitude, and have the precise form needed to ensure the normalization conditions

$$\begin{aligned}
\langle u_\pi(x) \rangle = \langle \bar{d}_\pi(x) \rangle \;&=\; 1\,, \\
\langle u_K(x) \rangle = \langle \bar{s}_K(x) \rangle \;&=\; 1\,, \\
\langle u_\eta + d_\eta + s_\eta \rangle \;&=\; 1\,, \\
\langle \bar{u}_\eta + \bar{d}_\eta + \bar{s}_\eta \rangle \;&=\; 1\,.
\end{aligned} \quad (17)$$

The function $F_{\alpha\beta}$ satisfies the symmetry relation $F_{\alpha\beta}(p^2, x) = F_{\beta\alpha}(p^2, 1-x)$. This feature, along with the normalization condition, ensures the momentum sum rule. For the kaon, for example, one obtains

$$
\begin{aligned}
\langle xu_K(x) + x\bar{s}_K(x)\rangle &= \langle xu_K(x) + xu_K(1-x)\rangle \\
&= \langle xu_K(x) + (1-x)u_K(x)\rangle = \langle u_K(x)\rangle = 1 \,. \quad (18)
\end{aligned}
$$

To apply the evolution method employed in Ref. [52] some analytical formula for the moments is needed. To obtain an approximate analytic formula for the moments, we note that for $0 < x < 1$, a convergent Taylor expansion of x dependence of Eq. (16) exists. Thus, the distribution functions may be accurately approximated by an nth degree polynomial, and the accuracy may be increased by keeping higher order terms. The pion and kaon distribution functions at Q_0^2 are accurately represented by

$$
\begin{aligned}
u_{\pi^+}(x, Q_0^2) = \bar{d}_{\pi^+}(x, Q_0^2) &= 0.9535 + 0.2664x - 0.2074x^2 - 0.1046x^3 \\
&\quad + 0.0190x^4 + 0.0400x^5 - 0.0133x^6 \,, \quad (19) \\
u_{K^+}(x, Q_0^2) &= 1.1039 + 1.8071x - 1.0739x^2 - 16.2227x^3 \\
&\quad + 33.1781x^4 - 25.5372x^5 + 7.1872x^6 \,, \quad (20) \\
\bar{s}_{K^+}(x, Q_0^2) &= 0.4425 + 0.8593x + 1.7623x^2 - 4.8611x^3 \\
&\quad + 13.2997x^4 - 17.5858x^5 + 7.1872x^6 \,. \quad (21)
\end{aligned}
$$

In each case, the remaining quark and gluon distribution functions are assumed to be zero. For the η meson, the expansion does not converge rapidly because one of the expansion parameters is

$$
\frac{M_\eta^2 x(1-x)}{M_u^2} \leq \frac{M_\eta^2}{4M_u^2} \approx 0.8 \,. \quad (22)
$$

The convergent series for $u_\eta(x)$ we find to be given by

$$
\begin{aligned}
u_\eta(x) = {}& A_u \left[\ln\left(\frac{M_u^2 + \Lambda^2}{M_u^2}\right) - \frac{\Lambda^2}{M_u^2 + \Lambda^2} \right. \\
&+ \left. \sum_{n=1}^\infty [x(1-x)]^n (n+1)\left(\frac{\alpha^n}{n} - \frac{\beta^n}{n} - \frac{\Lambda^2}{M_u^2 + \Lambda^2}\beta^n\right)\right] \,, \quad (23)
\end{aligned}
$$

where $\alpha = M_\eta^2/M_u^2$, $\beta = M_\eta^2/(M_u^2 + \Lambda^2)$ and $A_u = 0.09077$. The same expression holds for $s_\eta(x)$ with the replacements $M_u \to M_s$ and $A_u \to A_s = 0.36307$. Although this series could be rearranged into a polynomial in x, it is easier to express the moments in terms of Euler complex Beta functions. In practice, 30 terms in the expansion are kept, providing a reasonable 0.1% accuracy.

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