D-BRANES AND NON-COMMUTATIVE GEOMETRY*

Jacek Pawełczyk

Institute of Theoretical Physics, Warsaw University
Hoża 69, 00-681 Warsaw, Poland

(Received June 3, 2002)

Dedicated to Stefan Pokorski on his 60th birthday

An algebraic description of (untwisted) D-branes on compact group manifolds G using quantum algebras related to $U_q(\mathfrak{g})$ is discussed. It reproduces the known characteristics of stable branes in the WZW models. A toy model of NCG based on a quiver diagram for branes on orbifold is also presented.

PACS numbers: 11.25.Hf, 11.25.Mj, 02.40.Gh, 02.20.Uw

1. Introduction

Recently the physics of D-branes has been extensively studied with help of NCG tools. Flat branes in a constant B background [1] leads to quantum spaces with a Moyal–Weyl star product. A rather different situation is given by D-branes on compact Lie groups G, which carry a (NSNS) B field which is not closed. It has been shown, using CFT [2] and DBI (Dirac–Born–Infeld) [3] descriptions that stable branes can wrap certain conjugacy classes in the group manifold. On the other hand, the matrix model [4] and CFT calculations [5] led to a beautiful picture where, in a special limit, the macroscopic branes are formed as a bound state of $D0$-branes. Attempting to unify these various approaches, we proposed in recent papers [6, 7] a (quantum) matrix description of D-branes on group manifold G. This led to a quantum algebra based on quantum group symmetries, which reproduced all static properties of stable D-branes on G. The first part of the paper is devoted to this subject.

NCG language can also be useful in description of branes on orbifolds [8, 9]. Here I present some preliminary results concerning a toy model of

* Work supported in part by the Polish State Committee for Scientific Research (KBN) under contract no 5 P03B 150 20 (2001–2002).

NCG which exhibits the field theory of branes on orbifold as a certain Yang–Mills theory. The approach bears some resemblance with the deconstruction ideas [10]. These results were obtained in collaboration with S. Pokorski and A. Sitarz.

2. CFT of untwisted D-branes

The WZW model is specified by a group G and a level k [11,12]. We shall consider only simple, compact groups (G will be $\mathrm{SU}(N)$ mainly), so that the level k must be a positive integer. The WZW branes can be described by boundary states $|B\rangle\rangle \in \mathcal{H}^{\mathrm{closed}}$ respecting a set of boundary conditions. A large class of boundary conditions (so called untwisted branes) is of the form

$$\left(J_n + \tilde{J}_{-n}\right)|B\rangle\rangle = 0 \qquad n \in \boldsymbol{Z}\,. \tag{1}$$

Here J_n are the modes of the left-moving currents and $\tilde{J}_n$ are the modes of the right-moving currents. The boundary condition (1) breaks half of the symmetries of the WZW model $\widehat{\mathfrak{g}}_{\mathrm{L}} \times \widehat{\mathfrak{g}}_{\mathrm{R}}$ down to the vector part $\widehat{\mathfrak{g}}_V$. The chiral symmetry (the isometry of G) $\mathfrak{g}_{\mathrm{L}} \times \mathfrak{g}_{\mathrm{R}}$ acts on (1) rotating branes. The untwisted branes are labelled by $\lambda \in P_k^+$ corresponding to integrable irreps of $\widehat{\mathfrak{g}}$, which are precisely the weights in the "fundamental alcove" (A.2). The CFT description yields also an important formula for the energy of the brane λ,

$$E_\lambda = \prod_{\alpha>0} \frac{\sin\left(\pi \frac{\alpha\cdot(\lambda+\rho)}{k+g^\vee}\right)}{\sin\left(\pi\frac{\alpha\cdot\rho}{k+g^\vee}\right)}\,. \tag{2}$$

For $k \gg N$, one can expand the denominator in (2) to obtain a formula which compared with results obtained from DBI action [3, 13] shows that the leading k-dependence fits perfectly with the interpretation of a brane wrapping once a conjugacy class given by an element t_λ of the maximal torus of G.

The CFT provides hints towards the description of branes as quantum manifolds. It is known that the dynamics of D-branes is given by open string excitations. The relevant operators, entering as building blocks the string operators, are the primary fields of the BCFT. The number of lowest conformal weight primaries is finite for any compact WZW model (in general for any RCFT). In the $k \to \infty$ limit, the primaries can be interpreted as corresponding to a (finite dimensional) algebra of functions on the brane (see [14, 15]). For finite k, the interpretation is not that clear because the candidate algebra as given in [14] is not associative. However, the algebra becomes associative [6,14] after "twisting" (resulting in a modification of the product of the primary fields), so that it can be considered as an algebra of functions of a quantum manifold. Then the primaries become modules of the quantum group $U_q(\mathfrak{g}_V)$ instead of $\widehat{\mathfrak{g}}_V$. The CFT considerations make us to expect that the relations defining the algebra of functions on the quantum manifold to be invariant under the chiral counterpart of the chiral algebra, *i.e.* under $U_q(\mathfrak{g}_{\mathrm{L}} \times \mathfrak{g}_{\mathrm{R}})_{\mathcal{R}}$.

3. Branes on G by quantum algebras

We expect that the relevant quantum spaces are described by quantum algebras $\mathcal{M}$ which transform appropriately under a quantum symmetry. To find $\mathcal{M}$ we shall make an "educated guess" based on considerations contained in the previous section, and justify it by comparing its predictions with the results listed above. Thus first we postulate the form of the relations between generators of the quantum algebra. We expect the relations to be at most quadratic in generators, and to have appropriate covariance under the action of a quantum group. Moreover, we require the central terms of the algebra to be invariant under the "vector" subalgebra of this quantum symmetry. Thus our constructions mimic the symmetry pattern and its breaking by the D-branes in CFT.

Let $\mathcal{M}$ be generated by elements M^i_j with indices i, j in the defining representation V_N of G, subject to some commutation relations and constraints. With hindsight, we claim that these relations are given by the so-called reflection equation (RE) [16], which in a short notation reads

$$R_{21} M_1 R_{12} M_2 = M_2 R_{21} M_1 R_{12} \,. \tag{3}$$

Here R is the $\mathcal{R}$ matrix of $U_q(\mathfrak{g})$ in the defining representation. Displaying the indices explicitly we have

$$(\mathrm{RE})\, {}^{i}_{j}\,{}^{k}_{l} : \qquad R^k{}_a{}^i{}_b \;\, M^b_c \;\, R^c{}_j{}^a{}_d \;\, M^d_l = M^k_a \;\, R^a{}_b{}^i{}_c \;\, M^c_d \;\, R^d{}_j{}^b{}_l \;. \tag{4}$$

The indices $\{i, j\}$, $\{k, l\}$ correspond to the first (1) and the second (2) vectors space V_N in (3). An example of the algebra generated by RE relations is presented in Sec. 3.4. Because $\mathcal{M}$ should describe branes embedded in G, we need to impose constraints which ensure it. In the case $G = \mathrm{SU}(N)$, these are $\det_q(M) = 1$ where $\det_q$ is the so-called quantum determinant (8) and some reality conditions imposed on M^i_j. One can show that RE and the above constraints are invariant under the Hopf algebra $U_q(\mathfrak{g}_{\mathrm{L}} \times \mathfrak{g}_{\mathrm{R}})_{\mathcal{R}}$ which can be characterized by the fact that its Hopf subalgebra is $U_q(\mathfrak{g}_V)$.

M^i_j's can also be thought of as some matrices (as in Myers model [4]) out of which we can form an action invariant under the relevant quantum groups. The action has the structure $S = \mathrm{Tr}_q(1+\ldots)$, where dots represent some expressions in the M's (the quantum trace is defined in (12)). The point of [6] was that for some equations of motion, the "dots"-terms vanish on classical configurations. We postulate that the equations of motion for M are given by RE (3). If so, then their energy is equal to

$$E = \mathrm{Tr}_q(1)\,. \tag{5}$$

As we shall see this energy is not just a constant (as might be suggested by the notation), but it depends on the representations of the algebra, where it becomes the quantum dimension (12).

3.1. Central elements of RE

Below we discuss some general properties of the algebra defined by (3). We need to find the central elements, which are expected to characterize its irreps. This problem was solved in [17]. The (generic) central elements of the algebra (3) are

$$c_n = \mathrm{Tr}_q(M^n) \equiv \mathrm{Tr}_{V_N}(M^n\ v)\ \in \mathcal{M}\,, \tag{6}$$

where the trace is taken over the defining representation V_N, and

$$v = \pi\left(q^{-2H_\rho}\right) \tag{7}$$

is a numerical matrix which satisfies $S^2(r) = v^{-1}rv$ for the generator r of $\mathcal{G}_V$. These elements c_n are independent for $n = 1, 2, \ldots, \mathrm{rank}(G)$ and the c_n's for $n = 1, \ldots \mathrm{rank}(G) - 1$ fix the position of the brane configuration on the group manifold.

There is another central term known as the quantum determinant and denoted by $\det_q(M)$. While it can be expressed as a polynomial in c_n's ($n = 1, \ldots, \mathrm{rank}(G)$), $\det_q(M)$ is invariant under the full chiral quantum algebra. We need it to impose the constraint [1]

$$1 = \det_q(M)\,. \tag{8}$$

For $q = 1$, RE reduces to $[M^i_j, M^k_l] = 0$, thus any polynomial in M^i_j is central and we recover ordinary group manifold. It is worth to mention that the number of central elements is finite thus providing kind of foliation of the group manifold defined by $\mathcal{M}$.

[1] For other groups such as SO(N) and SP(N), additional constraints (which are also invariant under the full chiral quantum algebra) must be imposed.

3.2. Representations of $\mathcal{M}$ and quantum D-branes

By construction, M^i_j's can be considered as quantized coordinate functions on G providing some kind of quantization of the manifold G. However, we are interested here in the quantization of the orbits $\mathcal{C}(t_\lambda)$, which are submanifolds of G. We claim that they are described by irreps (fixed by the set of Casimirs) $\pi_V : \mathcal{M} \to \mathrm{Mat}(V, C)$ of $\mathcal{M}$. Indeed, the map π_V can be considered as the dual of the embedding map $\mathcal{C}(t_\lambda) \to G$. This will allow us to make statements on the location of the branes in G.

Consider an irreducible representation of $\mathcal{M}$. The Casimirs c_n (6) then take distinct values which can be calculated. Moreover, they are invariant under (vector) rotations. In view of their form (6), this suggests that an irrep of $\mathcal{M}$ should be considered as quantization of (the algebra of functions on) some conjugacy class $\mathcal{C}(t_\lambda)$, the position of which is determined by the values of the Casimirs c_n.

The representations of the algebra $\mathcal{M}$ coincide with those of $U_q(\mathfrak{g})$, which are largely understood, although quite complicated at roots of unity. The fact relevant for us is that representations V_λ of $U_q(\mathfrak{g})$ with $\lambda \in P_k^+$ have the following properties:

- they are unitary,
- their quantum-dimension $\dim_q(V_\lambda) = \mathrm{Tr}_{V_\lambda}(q^{2H_\rho})$ given in (12) is positive [18],
- λ corresponds precisely to the integrable modules of the affine Lie algebra $\widehat{\mathfrak{g}}$ which governs the CFT.

The representations belonging to the boundary of P_k^+ will correspond to the degenerate branes.

Having characterized the admissible representations V_λ, we propose that **the representation of $\mathcal{M}$ on V_λ for $\lambda \in P_k^+$ is a quantized or "fuzzy" *D*-brane, denoted by D_λ**. It is an algebra of maps from V_λ to V_λ which transforms under the quantum adjoint action of $U_q(\mathfrak{g})$. For "small" weights λ, this algebra coincides with $\mathrm{Mat}(V_\lambda)$. There are some modifications for "large" weights λ because q is a root of unity. The reason is that $\mathrm{Mat}(V_\lambda)$ then contains unphysical degrees of freedom which should be truncated.

A first justification is that there is indeed a one-to-one correspondence between the (untwisted) branes in string theory and these quantum branes, since both are labeled by $\lambda \in P_k^+$. To give a more detailed comparison, we calculate the traces (6) and show that the energy (2) of the branes in string theory will be recovered precisely in terms of the quantum dimension.

3.3. Value of the central terms

The values of the Casimirs c_n on D_λ are:

$$\begin{aligned} c_0 &= \mathrm{Tr}_{V_N}\,(q^{-2H_\rho}) = \dim_q(V_N)\,, & (9)\\ c_1(\lambda) &= \mathrm{Tr}_{V_N}\,(q^{2(H_\rho+H_\lambda)})\,, & (10)\\ c_n(\lambda) &= \sum_{\nu\in V_N;\,\lambda+\nu\in P_k^+} q^{2n((\lambda+\rho)\cdot\nu-\lambda_N\cdot\rho)}\,\frac{\dim_q(V_{\lambda+\nu})}{\dim_q(V_\lambda)}, \quad n\geq 1\,. & (11)\end{aligned}$$

Here λ_N is the highest weight of the defining representation V_N, and the sum in (11) goes over all $\nu\in V_N$ such that $\lambda+\nu$ lies in P_k^+. c_0 is λ-independent uninteresting number.

It is worth emphasizing here the agreement of the values of c_n with their classical counterparts.

The positions and the "size" of the branes essentially agree with the results from string theory [13]. In particular, their size shrinks to zero if λ approaches a corner of P_k^+, as can be seen easily in the SU(2) case [6]: as λ goes from 0 to k, the branes start at the identity e, grow up to the equator, and then shrink again around $-e$. We will see that the algebra of functions on D_λ precisely reflects this behavior; however this is more subtle and will be discussed below. All of this is fundamentally tied to the fact that q is a root of unity.

Furthermore, the quantum dimension of the representation space V_λ is

$$\dim_q(V_\lambda) = \mathrm{Tr}_q(1) = \mathrm{Tr}_{V_\lambda}(q^{2H_\rho}) = \prod_{\alpha>0}\frac{\sin(\pi\frac{\alpha\cdot(\lambda+\rho))}{k+g^\vee})}{\sin(\pi\frac{\alpha\cdot\rho}{k+g^\vee})}\,. \qquad (12)$$

The last equality above follows from Weyl's character formula. According to the interpretation (5) it should be the energy of the D-brane, and this is indeed the case (see (2)).

3.4. $G = SU(2)$ model

In this section we shall show how one can recover the results of [6] from the general formalism we discussed so far. The representation of the RE given by generators of $U_q(\mathrm{su}(2))$ (see appendix A) is

$$M = \begin{pmatrix} q^H & q^{-\frac{1}{2}}\lambda q^{H/2}X_- \\ q^{-\frac{1}{2}}\lambda X_+\, q^{H/2} & q^{-H}+q^{-1}\lambda^2 X_+X_- \end{pmatrix}\,. \qquad (13)$$

Let us parameterize the M matrix as ($[2] \equiv [2]_q = q + q^{-1}$)

$$M = \begin{pmatrix} M^4 - iM^0 & -iq^{-3/2}\sqrt{[2]}M^+ \\ iq^{-1/2}\sqrt{[2]}M^- & M^4 + iq^{-2}M^0 \end{pmatrix}, \tag{14}$$

then RE is equivalent to

$$[M^4, M^l] = 0, \quad \varepsilon^l_{ij}\ M^i M^j = i(q - q^{-1})M^4 M^l\,. \tag{15}$$

In order to calculate the central terms we need $v = \pi(q^{-2H_\rho}) = \pi(q^{-H}) = \mathrm{diag}(q^{-1}, q)$ so that (using (6))

$$\begin{aligned} c_1 &= \mathrm{Tr}_q(M) = q^{-1}a + qd = [2]\, M^4\,, & (16)\\ c_2 &= \mathrm{Tr}_q(M^2) = [2]\ ((M^4)^2 - q^{-2} g_{ij} M^i M^j)\,, & (17)\\ \det_q(M) &= (M^4)^2 + (M^0)^2 - q^{-1}M^+M^- - q\, M^-M^+ & \\ &= (M^4)^2 + g_{ij}M^iM^j\,. & (18) \end{aligned}$$

Only $\det_q(M)$ is invariant under $U_q(\mathfrak{g}_\mathrm{L} \times \mathfrak{g}_\mathrm{R})_\mathcal{R}$. The explicit value of $M^4 = c_1/[2]$ is obtained from

$$M^4 = \frac{1}{[2]}\left(q^{-1}a + q\, d\right) = \frac{1}{[2]}\left(q^{H-1} + q^{-(H-1)} + \lambda^2 X_+ X_-\right) \tag{19}$$

which is proportional to the standard Casimir of $U_q(\mathrm{su}(2))$. On the n-th brane D_n, H takes the value $-n$ on the lowest weight vector, thus $M^4 = \cos(\frac{(n+1)\pi}{k+2})/\cos(\frac{\pi}{k+2})$. If the square of radius of the quantum S^3 is chosen to be $\det_q(M) = k$ (which is the value given by the supergravity solution for the background), $g_{ij}M^iM^j$ leads to the correct formulae for the square of the radius of the n-th branes.

Comparison of the results. Here we shall show that all the results obtained from the quantum matrix model gives in a limit either DBI results [3] or matrix model results [5]. The limits of k and n for those models are: DBI — $1 \ll n \leq k,\ k \gg 1$, the matrix model — $n \ll k,\ k \to \infty$.

q-matrix		DBI	matrix
$r_n^2 = k\frac{\sin(\frac{n\pi}{k+2})\sin(\frac{(n+2)\pi}{k+2})}{\cos^2(\frac{\pi}{k+2})}$	$\to$	$k\sin^2(\frac{n\pi}{k})$	$\pi^2\frac{(n+1)^2-1}{k}$
$E_n = T_0\frac{\sin(\frac{(n+1)\pi}{k+2})}{\sin(\frac{\pi}{k+2})}$	$\to$	$k\sin(\frac{\pi n}{k})$	$T_0(n+1)(1+\frac{\pi^2}{6}\frac{(n+1)^2-1}{k^2})$

Comparing with [3, 5] on can conclude that the quantum matrix model results are in perfect agreement with those obtained from the DBI and the ordinary matrix model. Let us also make the approximation of small spheres $S^2_{q,n}$ and substitute $M^4 = \sqrt{k} + O(g_{ij}M^iM^j)$. Then $x^l = iM^l/((q-q^{-1})\sqrt{k})$ respects q-fuzzy sphere equation as written in [19],

$$i\epsilon_{ijl}\, x^i x^j = x^l\,. \tag{20}$$

The above is the ordinary fuzzy sphere equation obtained in this context in *e.g.* [5].

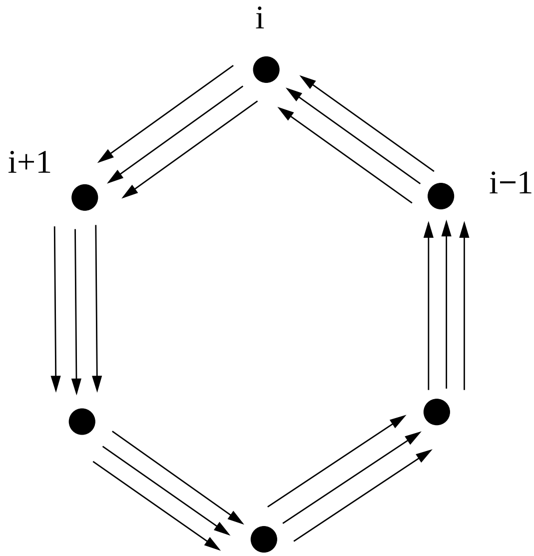
4. D-branes on orbifolds and NCG

Consider $k\ N$ D3 branes sitting at the origin of C^3. One can get string states of the brane on C^3/Γ starting from the above case by making certain projection[2] [20]. Let $\Gamma = Z_N$ act on C^3: $(z_1, z_2, z_3) \to \mathrm{e}^{-2\pi i/N}(z_1, z_2, z_3)$. The flat branes states are the gauge fields for the group SU($k\ N$) and 6 scalars (Φ^a, $a = 1, 2, 3$) in the adjoint of SU($k\ N$) which we grouped in a triplet of SU(3)⊂ SU(4) (here SU(4) is the R-group of the $\mathcal{N} = 4$ theory). Orbifolding identifies this SU(3) with the C^3 rotations. This sets the action of Z_N on Φ^a. Moreover the orbifold group acts on the adjoint indices of the gauge group. The indices of the fundamental of U($k\ N$) (thus also branes) are split into N equal sets (indexed by $i, j, \ldots = 1, \ldots N$) on which Z_N acts naturally. The adjoint (gauge bosons) under SU($k\ N$) belonging to the $(i, \bar{j})$ set transform as

$$A_{\bar{i},j} \to \mathrm{e}^{2\pi(j-i)/N} A_{i,\bar{j}}\,. \tag{21}$$

Then $\Phi^a_{\bar{i},j} \to \mathrm{e}^{2\pi(j-i-1)\ i/N}\,\Phi^a_{\bar{i},j}$. The appropriate orbifold projection cuts all Z_N-non-invariant fields breaking $\mathrm{SU}(k\ N) \to \mathrm{SU}(k)_1 \times \ldots \mathrm{SU}(k)_N$, where we have indicated the index of the gauge group. The spectrum of the massless fields consist of $\mathrm{SU}(k)_i$ gauge fields and 3 complex scalars in the bifundamental $(\bar{k}_i, k_{i+1})$ of $\mathrm{SU}(k)_i \times \mathrm{SU}(k)_{i+1}$. This can be depicted on the so-called quiver diagram (for $\Gamma = Z_6$). The Lagrangian for the interaction between fields one gets form $\mathcal{N} = 4$ theory projecting out all terms with the non-invariant fields.

[2] We limit consideration to massless bosonic fields.



4.1. YM theory on set of points

Here we show that one can construct the above theory (with the correct interaction) out of YM theory on a NC space times ordinary (commutative) Minkowski space.

Thus we define NCG for set of N points and arrows as in the quiver. The arrows will represent the differentials. Explicitly the differential of a function $f = (f_1, \dots f_N)$ is

$$df_i = (f_{i+1} - f_i) \sum_a \chi^a + (f_i - f_{i-1}) \sum_a \chi^{*a} \,, \tag{22}$$

where $*$ is a conjugation and χ^a are differentials assign to different arrows connecting the same nodes of the quiver. The conjugation for function is denoted with "bar". It is clear that the differential is Z_N covariant. All functions are commutative but the they do not commute with differentials. One can check that $d(fg) = df\ g + f\ dg$ if $\chi f_i = f_{i+1}\chi, \chi^* f_{i+1} = f_i \chi^*$. With above rule we also have $(df)^* = d(\bar{f})$. Imposing $d^2 = 0$ we get $(\chi^a)^2 = (\chi^{*a})^2 = 0$ and *e.g.*:

$$\chi^a \chi^{*b} = -\chi^{*b}\chi^a, \quad d\chi^a = d\chi^{*a} = 0 \,. \tag{23}$$

Let us assume that over each point i we have a vector $\mathrm{U}(k)_i$-bundle *i.e.* the space-time gauge group is as in the orbifold case $\mathrm{SU}(k)_1 \times \dots \mathrm{SU}(k)_N$. The differential constructed above allows to define the connection on a the bundle (we display its components along set of N points only):

$$\nabla_i = d + \sum_a \Phi_i^a \chi_a - \sum \Phi_{i-1}^{*a} \chi_a^* \,, \tag{24}$$

where Φ_i^a are the gauge fields corresponding to χ_a. One can check that the connection is Hermitian and Φ_i^a is in bifundamental $(\bar{k}_i, k_{i+1})$. Thus we get the proper field content of the theory and also proper coupling between scalars and gauge fields because it is completely determined by the gauge group properties.

The curvature $\nabla_i^2 = F_i$ of the connection depends on the commutations between χ_a's.

$$F_i = \nabla_i^2 = (\Phi_i^a \Phi_i^{*b} - \Phi_{i-1}^{*b} \Phi_{i-1}^a)\chi^a \chi^{*b} + (\Phi_i^a \Phi_{i+1}^b - \Phi_i^b \Phi_{i+1}^a)\chi^a \chi^b + \text{c.c}\,. \tag{25}$$

One can check that the square of the above curvature yields the same self-interaction of the scalars as for the orbifold model.

I would like to thank my teacher S. Pokorski for his past, present and future collaboration and kind interest in my various projects.

Appendix A

Some properties of $\mathfrak{g}$ and $U_q(\mathfrak{g})$

We collect some notations used throughout this paper. $\mathfrak{g}$ denotes the (simple, finite-dimensional) Lie algebra of G, with Cartan matrix $A_{ij} = 2\frac{\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j}$. Here "$\cdot$" is the Killing form which is defined for arbitrary weights, and α_i are the simple roots. The set of dominant integral weights is denoted by

$$P^+ = \left\{ \sum n_i \Lambda_i ; \;\; n_i \in Z_{\geq 0} \right\} , \tag{A.1}$$

where the fundamental weights Λ_i satisfy $\alpha_i \cdot \Lambda_j = d_{\alpha_i} \delta_{ij}$, and the length of a root α is $d_\alpha = \frac{\alpha \cdot \alpha}{2}$. The Weyl vector is the sum over all positive roots, $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$. For a positive integer k, one defines the "fundamental alcove" in weight space as

$$P_k^+ = \{\lambda \in P^+ ; \; \lambda \cdot \theta \leq k\} \,, \tag{A.2}$$

where θ is the highest (maximal) root. It is a finite set of dominant integral weights. For $G = \mathrm{SU}(N)$, this is explicitly $P_k^+ = \{\sum n_i \Lambda_i ; \; \sum_i n_i \leq k\}$. We shall normalize the Killing form such that $d_\theta = 1$, so that the dual Coxeter number is given by $g^\vee = (\rho + \frac{1}{2}\theta) \cdot \theta$, which is N for $\mathrm{SU}(N)$.

For any weight λ, we define $H_\lambda \in \mathfrak{g}$ to be the Cartan element which takes the value $H_\lambda v_\mu = (\lambda \cdot \mu)\; v_\mu$ on vectors v_μ with weight μ in some representation. We shall consider only finite-dimensional representations (=modules) of $\mathfrak{g}$. V_λ denotes the irreducible highest-weight module of G with highest weight $\lambda \in P^+$, and V_{λ^+} is the conjugate (=dual) module of V_λ. The defining representation of the classical matrix groups SU(N), SO(N), and Sp(N) will be denoted by V_N, being N-dimensional.

The generators $X_i^\pm, H_i$ of $U_q(\mathfrak{g})$ satisfy the relations

$$[H_i, H_j] = 0, \quad [H_i, X_j^\pm] = \pm A_{ji} X_j^\pm \,, \tag{A.3}$$

$$[X_i^+, X_j^-] = \delta_{i,j} \frac{q^{d_i H_i} - q^{-d_i H_i}}{q^{d_i} - q^{-d_i}} = \delta_{i,j} [H_i]_{q_i} \,, \tag{A.4}$$

where $q_i = q^{d_i}$ and $[a]_q \equiv \frac{q^a - q^{-a}}{q - q^{-1}}$. Comultiplication and antipode are defined by

$$\begin{aligned} \Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i, \qquad \Delta(X_i^\pm) = X_i^\pm \otimes q^{d_i H_i/2} + q^{-d_i H_i/2} \otimes X_i^\pm \,, \\ S(H_i) &= -H_i, \qquad S(X_i^\pm) = -q^{\mp d_i} X_i^\pm \,. \end{aligned} \tag{A.5}$$

The coproduct is conveniently written in Sweedler-notation as $\Delta(u) = u_1 \otimes u_2$, for $u \in U_q(\mathfrak{g})$, where a summation is implied. It is easy to verify that $S^2(u) = q^{2H_\rho} u q^{-2H_\rho}$ for all $u \in U_q(\mathfrak{g})$, where $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$ is the Weyl vector. This is used in the definition of the quantum traces (6), (12).

REFERENCES

[1] M.R. Douglas, C. Hull, *J. High Energy Phys.* **9802**, 008 (1998); A. Connes, M.R. Douglas, A. Schwarz, *J. High Energy Phys.* **9802**, 003 (1998); V. Schomerus, *J. High Energy Phys.* **9906**, 030 (1999); N. Seiberg, E. Witten, *J. High Energy Phys.* **9909**, 032 (1999).

[2] A.Yu. Alekseev, V. Schomerus, *Phys. Rev.* **D60**, 061901 (1999).

[3] C. Bachas, M. Douglas, C. Schweigert, *J. High Energy Phys.* **0005**, 048 (2000); J. Pawełczyk, *J. High Energy Phys.* **08**, 006 (2000).

[4] R. C. Myers, *J. High Energy Phys.* **9912**, 022 (1999); D. Kabat, W. Taylor, *Adv. Theor. Math. Phys.* **2**, 181 (1998); S.-J. Rey, `hep-th/9711081`.

[5] A.Yu. Alekseev, A. Recknagel, V. Schomerus, *J. High Energy Phys.* **0005**, 010 (2000).

[6] J. Pawełczyk, H. Steinacker, *J. High Energy Phys.* **0112**, 018 (2001).

[7] J. Pawełczyk, H. Steinacker, `hep-th/0203110`.

[8] N.A. Nekrasov, `hep-th/0203112`.

[9] M. Alishahiha, *Phys. Lett.* **B517**, 406 (2001).

[10] N. Arkani-Hamed, A.G. Cohen, H. Georgi, *Phys. Rev. Lett.* **86**, 4757 (2001); C.T. Hill, S. Pokorski, J. Wang, *Phys. Rev.* **D64**, 105005 (2001).

[11] J. Fuchs, *Affine Lie Algebras and Quantum Groups*, Cambridge University Press, 1992.

[12] P. Di Francesco, P. Mathieu, D. Senechal, *Conformal Field Theory*, Springer-Verlag, 1997.

[13] G. Moore, J. Maldacena, N. Seiberg, *J.High Energy Phys.* **0111**, 062 (2001).

[14] A.Yu. Alekseev, A. Recknagel, V. Schomerus, *J. High Energy Phys.* **9909**, 023 (1999).

[15] G. Felder, J. Fröhlich, J. Fuchs, C. Schweigert, *J. Geom. Phys.* **34**, 162 (2000).

[16] E. Sklyanin, *J. Phys. A* **21**, 2375 (1988); L. Mezinescu, R. Nepomeniche, *Int. J. Mod. Phys.* **A6**, 5231 (1991).

[17] P.P. Kulish, E.K. Sklyanin, *J. Phys. A* **25**, 5963 (1992); P.P. Kulish, R. Sasaki, *Prog. Theor. Phys.* **89**, 741 (1993).

[18] V. Chari, A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, 1994.

[19] H. Grosse, J. Madore, H. Steinacker, *J. Geom. Phys.* **38**, 308 (2001); `hep-th/0005273`; H. Steinacker, *Mod. Phys. Lett.* **A16**, 361 (2001).

[20] M.R. Douglas, G.W. Moore, `hep-th/9603167`; A.E. Lawrence, N. Nekrasov, C. Vafa, *Nucl. Phys.* **B533**, 199 (1998).