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FUN WITH GAUGE THEORIES IN 5 DIMENSIONS

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Dedicated to Stefan Pokorski on his 60th birthday

We discuss gauge symmetry breaking with Wilson loops in 5 dimensions. We present a simple example with the fifth dimension compactified on an S^1/Z_2 orbifold. The Wilson loop in this SO(3) example replaces the adjoint Higgs scalar (needed to break SO(3) to U(1)) in the well-known Georgi–Glashow model. We then show that gauge symmetry breaking with a Wilson loop on this S^1/Z_2 orbifold is gauge equivalent to gauge symmetry breaking on a particular $S^1/(Z_2 \times Z_2')$ orbifold. The latter orbifold has been used in many recent constructions with gauge symmetry breaking in five dimensional supersymmetric and non-supersymmetric models. Finally we explicitly construct a magnetic monopole string solution; the analog of the 't Hooft–Polyakov monopole. The monopole string has finite energy, and length equal to the size of the extra dimension.

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1. Introduction

Recently there has been quite a bit of interest in non-Abelian gauge field theories in 4 + d dimensions with d extra dimensions compactified on an orbifold [1–5]. The extra dimensions can have inverse radii of order a few TeV, of order the GUT scale or anything in between. In these recent studies, symmetry breaking via orbifold boundary conditions has replaced the traditional method using the vacuum expectation values of Higgs scalars. In an illustrative and simple example in one extra dimension, the orbifolds S^1/Z_2 and $S^1/(Z_2 \times Z_2')$ have been used to break the GUT groups $\mathrm{SU(5)} \to \mathrm{SU(3)} \times \mathrm{SU(2)} \times \mathrm{U(1)}$ [1,2], $\mathrm{SO(10)} \to \mathrm{SU(4)} \times \mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ [3], the left-right gauge symmetry $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R} \times \mathrm{U(1)_{B-L}} \to \mathrm{SU(2)_L} \times \mathrm{U(1)_R} \times \mathrm{U(1)_{B-L}}$ [4] or the electroweak unified group $\mathrm{SU(3)} \to \mathrm{SU(2)} \times \mathrm{U(1)}$ [5].

In this letter we argue that magnetic monopoles are generic consequences of gauge symmetry breaking with Wilson loops on S^1/Z_2 orbifolds. We consider the simple example of an SO(3) gauge theory defined on the orbifold $M \times S^1/Z_2$ with a background gauge field. In an attempt to define notation and set up some of the ideas we introduce the concept of Wilson loop symmetry breaking in the simple example of the circle S^1. We then generalize this discussion to the orbifold S^1/Z_2 and also elucidate the equivalence of gauge symmetry breaking with Wilson loops on S^1/Z_2 and gauge symmetry breaking on the orbifold $S^1/(Z_2 \times Z_2')$. Finally, we explicitly construct the monopole string solution and discuss some of its properties. This letter is based on the recent paper [6] and the earlier work [7–13]. For a recent discussion of Wilson loops on orbifolds see [14].

2. SO(3) gauge theory on $M \times S^1$

Consider a general gauge theory with symmetry group G in five dimensional spacetime. The Lagrangian is given by

$$\mathcal{L}_5 = -\frac{1}{4e_5^2 k}\mathrm{Tr}\,(F_{MN}F^{MN}) \tag{1}$$

where $F_{MN} \equiv \sum_a F^a_{MN}T^a$, T^a are generators in some finite dimensional representation of G normalized such that $\mathrm{Tr}\,(T^aT^b) = k\delta^{ab}$ and $M, N = \{0,1,2,3,5\}$:

$$F_{MN} = \partial_M A_N - \partial_N A_M + i[A_M, A_N]. \tag{2}$$

(For the adjoint representation of SO(3) we use the standard normalization of the generators with $k = 2$.) The gauge transformation of the gauge field $A_M(x_\mu, y) \equiv \sum_a A^a_M T^a(x_\mu, y)$ (greek indices correspond to 4-dimensional Minkowski spacetime and $y \equiv x_5$) is given by

$$A_M(x_\mu, y) \to UA_M(x_\mu, y)U^\dagger - iU\partial_M U^\dagger, \tag{3}$$

where

$$U = \exp(i\theta^a(x_\mu, y)T^a). \tag{4}$$

In our notation, Eq. (1), the gauge fields have mass dimensions [1], and the charge e_5 has dimension [-1/2]. We can also define the effective four dimensional, dimensionless, gauge coupling e by rescaling e_5 in Eq. (1) via the expression $e_5 = \sqrt{2\pi R}\; e$. Note, if $\partial_5 A_\mu = 0$, then $F_{\mu 5}$ reduces to the covariant derivative of the 5th component of the gauge field A_5. In this case we can conveniently define $\Phi \equiv A_5/e_5 = \widetilde{\Phi}/\sqrt{2\pi R}$, where the scalars Φ and

$\widetilde{\Phi}$ have dimension [3/2] and [1]. The Lagrangian (1) can then be rewritten in the suggestive form:

$$\mathcal{L}_5 = \frac{1}{2\pi R}\left[-\frac{1}{4e^2 k}\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2k}\mathrm{Tr}\,(D_\mu \widetilde{\Phi} D^\mu \widetilde{\Phi})\right]. \tag{5}$$

This resembles the Georgi–Glashow model [9] of an SO(3) gauge theory interacting with an isovector Higgs field. There are two differences, however. First, there is no potential $V(\widetilde{\Phi}) = \lambda(\widetilde{\Phi}^a\widetilde{\Phi}^a - V^2)^2$ for the Higgs field which would break the gauge symmetry down to U(1) and second, the Higgs field depends on the 5th coordinate. Although this analysis is limited to gauge fields satisfying $\partial_5 A_\mu = 0$, it nevertheless inspires the following discussion of symmetry breaking via Wilson loops and the further consideration of monopoles with Wilson loops. In general, however, $\partial_5 A_\mu \neq 0$ and we need to keep the full $\mathrm{Tr}\,(F_{\mu 5}^2)$ term.

2.1. Wilson loop gauge symmetry breaking on $M \times S^1$

Assume the 5th dimension is compactified on a circle S^1 parametrized by $y \in [0, 2\pi R]$. The gauge symmetry can then be broken by the presence of a background gauge field A_5. This symmetry breaking mechanism is known as Hosotani or Wilson loop symmetry breaking [11]. Consider the constant background to be along the third isospin direction,

$$A_5(y) = A_5^3 T^3. \tag{6}$$

Using the single valued gauge transformation (periodic under $y \to y + 2\pi R$) given by Eqs. (3), (4) with $\theta(x_\mu, y) = -ny/R$, $n \in \mathbb{Z}$:

$$U(y) = \exp\left(-inT^3 \frac{y}{R}\right), \tag{7}$$

we obtain the transformation of A_5^3:

$$A_5^3 \to A_5^3 + \frac{n}{R}. \tag{8}$$

Therefore the gauge non-equivalent values of A_5^3 can be chosen to lie between 0 and $1/R$. The holonomy due to this constant background gauge field is given by

$$T = \exp\left(i\oint A_5 dy\right) = \exp\left(i\alpha T^3\right). \tag{9}$$

with the arbitrary parameter $\alpha \equiv 2\pi R A_5^3$. Note the set of possible holonomies $\{\mathbb{1}, T^{\pm 1}, T^{\pm 2}, \cdots\}$ provides a mapping of the gauge group into the discrete group $\mathbb{Z}$. This non-trivial holonomy affects the spectrum of the theory. A massless periodic scalar field ϕ (satisfying $\phi(y + 2\pi R) = \phi(y)$) with isospin eigenvalue I_3 can be decomposed into Kaluza–Klein modes

$$\phi_{(n)}(x_\mu) \exp\left(\frac{iny}{R}\right). \tag{10}$$

The 5-dimensional wave equation $D^M D_M \phi = 0$ splits into an infinite set of 4-dimensional wave equations for Kaluza–Klein modes $\phi_{(n)}$ with masses given by

$$\begin{aligned} m_{(n)}^2 \phi_{(n)} \exp\left(\frac{iny}{R}\right) &= -\left(\partial_y + iA_5^3 T^3\right)^2 \phi_{(n)} \exp\left(\frac{iny}{R}\right) \\ &= \left(\frac{n}{R} + A_5^3 I_3\right)^2 \phi_{(n)} \exp\left(\frac{iny}{R}\right). \end{aligned} \tag{11}$$

It is now easy to obtain the spectrum of gauge fields [1]. The gauge field $A_\mu^3(y)$ has $I_3 = 0$ and therefore its KK modes are not affected by the holonomy. The zero mode of this field corresponds to the gauge field of the unbroken U(1). On the other hand, the masses of the KK modes of the $W^\pm$ gauge bosons, with $I_3 = \pm 1$, are given by $m_{(n)} = |\frac{n}{R} \pm A_5^3|$. If $A_5^3 \neq \frac{k}{R}$, where $k \in \mathbb{Z}$, the gauge bosons $W^\pm$ are all massive. Clearly the SO(3) symmetry is broken to U(1). Note, the symmetry breaking scale satisfies $0 \leq A_5^3 < 1/R$, but is otherwise unconstrained.

2.2. Gauge picture with vanishing background

A constant background gauge field A_5^3 may be gauged away with the non-periodic gauge transformation

$$U(y) = \exp\left(iyA_5^3 T^3\right). \tag{12}$$

In this gauge the covariant derivative in Eq. (11) is trivial, *i.e.* $D_5 = \partial_5$. Nevertheless it is easy to see that, as expected, the physics is unchanged.

This gauge transformation is not single valued and thus the periodicity condition $\phi(y + 2\pi R) = \phi(y)$ becomes

$$\phi(y + 2\pi R) = \exp\left(i\alpha T^3\right) \phi(y). \tag{13}$$

[1] Consider a background field gauge with $A_M = B_M + a_M$ where B_M is the background value of the gauge field and a_M are the small fluctuations. The background covariant derivative is given by $D_M \equiv \partial_M + i[B_M, \ ]$. If we use the covariant gauge fixing condition $D^M a_M \equiv 0$, then the gauge field equations of motion are given by $D^M D_M a_N + 3i[B_{NM}, a^M] = 0$. Note, for a constant background gauge field $B_{NM} \equiv 0$.

Now the mode expansions are of the form

$$\phi_{(n)}(x_\mu)\exp\left[i\left(\frac{n}{R}+A_5^3 I_3\right)y\right] \tag{14}$$

resulting in the identical spectrum as before.

3. SO(3) gauge theory on S^1/Z_2

3.1. The S^1/Z_2 orbifold

The S^1/Z_2 orbifold is a circle S^1 modded out by a Z_2 parity symmetry: $y \to -y$. The 5th dimension is now a line segment $y \in [0, \pi R]$. This orbifold has two fixed points at $y = 0$ and πR. The Lagrangian (1) is invariant under the parity transformation

$$A_\mu(-y) \;=\; A_\mu(y)\,, \tag{15}$$
$$A_5(-y) \;=\; -A_5(y)\,. \tag{16}$$

As in the case of compactification on a circle we consider a constant background for A_5^3 (Eq. (6)). Clearly such a background is not consistent with the parity operation, Eq. (16). However, following [14] we define a generalized parity by combining the parity transformation (16) with the gauge transformation (8), for $n = 1$, $A_5^3 \to A_5^3 + 1/R$. We then look for a consistent solution with constant A_5^3. There are now only two possible values for A_5^3. The possibility $A_5^3 = 0$ is obviously allowed, but in this case the gauge symmetry is unbroken. The only nontrivial choice corresponds to $A_5^3(y) = \frac{1}{2R}$ which changes sign under the "naive" parity, $A_5^3(-y) = -\frac{1}{2R}$, but is gauge equivalent to its original value. Therefore, instead of (15) – (16) we define the fields for negative y, in the region $-\pi R < y < 0$, in terms of the fields defined for positive y in the fundamental domain, $0 < y < \pi R$, via the generalized parity transformation (*i.e.* a combined "naive" parity transformation (16) and a gauge transformation) such that, in general:

$$A_\mu(-y) = U(-y)A_\mu(y)U^\dagger(-y) - iU(-y)\partial_\mu U^\dagger(-y), \tag{17}$$

$$A_5(-y) = -U(-y)A_5(y)U^\dagger(-y) - iU(-y)\partial_{-y} U^\dagger(-y), \tag{18}$$

with

$$U(y) = \exp\left(-i\frac{y}{R}T^3\right). \tag{19}$$

It is useful to define new fields, $W^\pm$, in a usual way from A^1 and A^2:

$$W^\pm = \frac{1}{\sqrt{2}}\left(A^1 \mp iA^2\right), \quad T^\pm = \frac{1}{\sqrt{2}}\left(T^1 \pm iT^2\right). \tag{20}$$

With this definition we have $A^1T^1+A^2T^2 = W^+T^+ + W^-T^-$ and $[T^3, T^\pm] = \pm T^\pm$. Using the identity

$$\exp\left(i\frac{y}{R}T^3\right) T^\pm \exp\left(-i\frac{y}{R}T^3\right) = \exp\left(\pm i\frac{y}{R}\right) T^\pm \tag{21}$$

it is easy to show that the generalized parity tranformation acts on gauge fields as follows:

$$\begin{aligned} W_\mu^\pm(-y) &= \exp\left(\pm i\frac{y}{R}\right) W_\mu^\pm(y), &(22)\\ W_5^\pm(-y) &= -\exp\left(\pm i\frac{y}{R}\right) W_5^\pm(y), &(23)\\ A_\mu^3(-y) &= A_\mu^3(y), &(24)\\ A_5^3(-y) &= -A_5^3(y) + \frac{1}{R}. &(25)\end{aligned}$$

To summarize, using a more compact notation, we have the following constraints on the fields (valid for all modes, except the constant piece of A_5^3). Under the generalized parity transformation the fields ϕ_P (with $P = \pm 1$) satisfy:

$$\phi_P(-y) = P\exp\left(i\frac{y}{R}I_3\right)\phi_P(y) \tag{26}$$

with isospin eigenvalue $I_3 = \pm 1, 0$. The periodicity condition is given by:

$$\phi_P(y + 2\pi R) = \phi_P(y). \tag{27}$$

We then obtain the following decomposition into KK modes:

$$\begin{aligned} \phi_+(x_\mu, y) &= \sum_{n=0}^{\infty}\phi_+^{(n)}(x_\mu)\,\exp\left(-i\frac{y}{2R}I_3\right)\,\cos n\frac{y}{R} && \text{for even } I_3, &(28)\\ \phi_+(x_\mu, y) &= \sum_{n=0}^{\infty}\phi_+^{(n)}(x_\mu)\,\exp\left(-i\frac{y}{2R}I_3\right)\,\cos\left(n+\tfrac{1}{2}\right)\frac{y}{R} && \text{for odd } I_3, &(29)\\ \phi_-(x_\mu, y) &= \sum_{n=0}^{\infty}\phi_+^{(n)}(x_\mu)\,\exp\left(-i\frac{y}{2R}I_3\right)\,\sin(n+1)\frac{y}{R} && \text{for even } I_3, &(30)\\ \phi_-(x_\mu, y) &= \sum_{n=0}^{\infty}\phi_+^{(n)}(x_\mu)\,\exp\left(-i\frac{y}{2R}I_3\right)\,\sin\left(n+\tfrac{1}{2}\right)\frac{y}{R} && \text{for odd } I_3. &(31)\end{aligned}$$

From transformations (22)–(25) we see that the KK mode expansion of A_μ^3 [(+) field with $I_3 = 0$] is given in Eq. (28) with corresponding masses n/R. This is the only field which has a zero mode. It corresponds to the gauge field of the unbroken U(1). The expansion of $W_\mu^\pm$ [(+) field with $I_3 = \pm 1$] is given in Eq. (29) with corresponding masses $(n+1/2)/R$. Similarly, the expansion of $W_5^{\pm}$ [$(-)$ field with $I_3 = \pm 1$] is given in Eq. (31) with corresponding masses $(n+1/2)/R$. And finally, the expansion of A_5^3 [$(-)$ field with $I_3 = 0$] is given by Eq. (30) up to the value of the constant background:

$$A_5^3(x_\mu, y) = \frac{1}{2R} \; + \; \sum_{n=0}^{\infty} A_5^{3(n)}(x_\mu) \; \sin(n+1)\frac{y}{R}. \tag{32}$$

The holonomy T in this case is given by

$$T = \exp(i \oint A_5^3 T^3) = \exp(i\pi T^3) = \mathrm{diag}\,(-1,-1,1). \tag{33}$$

Hence $T^2 = \mathbb{1}$ or the set of possible holonomies $\{\mathbb{1}, T\}$ maps the gauge group into the discrete group $\mathbb{Z}_2$. Unlike the case of Wilson loops on S^1 discussed in section 2, the background gauge field and consequently the holonomy on S^1/Z_2 can only take discrete values.

Now let us consider the gauge picture with vanishing background gauge field. As in the case of compactification on a circle, we can gauge away the constant background by the non-single valued gauge transformation given in Eq. (12). The transformations under the generalized parity are now those of Eqs. (15) and (16). In addition the non-single valued gauge transformation changes the periodicity condition as in Eq. (13) with $\alpha = \pi$.

To obtain the spectrum of KK modes of a field ϕ we consider both the transformation under parity and the effect of a non-trivial holonomy. Under parity,

$$\mathcal{P}: \quad \phi_{PT}(y) \;\to\; \phi_{PT}(-y) = P\phi_{PT}(y), \tag{34}$$

with $P^2 = 1$ or $P = \pm 1$. When going around the circle, the fields transform in the following way:

$$\mathcal{T}: \quad \phi_{PT}(y) \;\to\; \phi_{PT}(y + 2\pi R) = T\phi_{PT}(y) \tag{35}$$

with $T^2 = \mathbb{1}$ or $T = \pm 1$. Therefore there are four different kinds of fields $\phi_{\pm\pm}$ corresponding to the four different combinations of (P, T). It is easy to see that a field with given (P, T) can be expanded into the following modes:

$$\begin{aligned}
\xi_n(+,+) &= \cos n\frac{y}{R}\,, \\
\xi_n(+,-) &= \cos\left(n + \tfrac{1}{2}\right)\frac{y}{R}\,, \\
\xi_n(-,+) &= \sin(n+1)\frac{y}{R}\,, \\
\xi_n(-,-) &= \sin\left(n + \tfrac{1}{2}\right)\frac{y}{R}\,.
\end{aligned} \tag{36}$$

Only the $(+,+)$ fields have massless zero modes. Of all the gauge fields only A^3_μ is a $(+,+)$ field with a zero mode. $W^\pm_\mu$, A^3_5 and $W^\pm_5$ are $(+,-)$, $(-,+)$ and $(-,-)$ fields , respectively. Clearly the mode expansion and the corresponding KK masses are the same as in the previous picture. Note, our gauge transformation parameters (Eq. (4)) are constrained to satisfy $\theta^3(x_\mu,y) = \theta^3_n(x_\mu)\xi_n(+,+)$ and $\theta^{1,2}(x_\mu,y) = \theta^{1,2}_n(x_\mu)\xi_n(+,-)$. Hence, SO(3) is the symmetry everywhere in the five dimensions, EXCEPT on the boundary at $y = \pi R$.

3.2. *Correspondence to $S^1/(Z_2 \times Z_2')$ orbifold*

The S^1/Z_2 orbifold with holonomy T in the gauge picture without a constant background gauge field is directly related to the $S^1/(Z_2 \times Z_2')$ orbifold used recently in the literature [1–5]. This correspondence is also evident in the work of Ref. [14,15]. We just need to identify the $S^1/(Z_2 \times Z_2')$ orbifold with S^1, a circle of circumference $4\pi R$, divided by the Z_2 transformation $y \to -y$ and Z_2' transformation $y' \to -y'$, where $y' \equiv y - \pi R$. The physical space is again the line segment $y \in [0, \pi R]$ with orbifold fixed points at $y = 0$ and πR. It is easy to see that $\mathcal{P}' \in Z_2'$ in this picture corresponds to the combined translation and parity transformation in the previous picture, namely $\mathcal{P}' = \mathcal{T}\mathcal{P}$. Note, a point at $y = y_0$ which corresponds to $y' = y_0 - \pi R$ is transformed by Z_2' into the point $y' = -(y_0 - \pi R)$ corresponding to $y = -y_0 + 2\pi R$; this is equivalent to the action of $\mathcal{T}Z_2$ on the point at $y = y_0$, see Fig. 1.

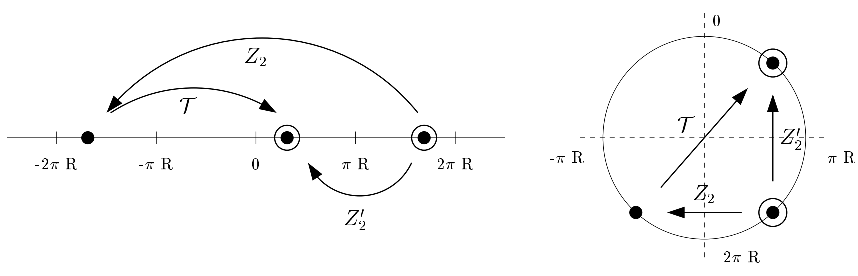

Fig. 1. The Z_2' parity transformation is equivalent to the combined Z_2 parity transformation and translation $\mathcal{T}$.

The action of Z_2 on the fields is given by

$$\mathcal{P}: \quad \phi_{PP'}(y) \to \phi_{PP'}(-y) = P\phi_{PP'}(y), \tag{37}$$

with $P^2 = 1$ or $P = \pm 1$. Similarly, under Z_2' we have

$$\mathcal{P}': \quad \phi_{PP'}(\pi R + y') \to \phi_{PP'}(\pi R - y') = P'\phi_{PP'}(\pi R + y') \tag{38}$$

with $P' = TP$ and $(P')^2 = \mathbb{1}$ or $P' = \pm 1$.

It is easy to see what the holonomy means in this picture. Since points y_0 and $y_0+2\pi R$ are identified, the closed loop corresponds to going around half of the circle (the circumference of the circle in this picture is $4\pi R$). Going around the whole circle (from y_0 to $y_0+2\pi R$ and then from $y_0+2\pi R$ to $y_0+4\pi R$) clearly corresponds to T^2. From Eq. (16) we see that going from $y_0+2\pi R$ to $y_0+4\pi R$ is equivalent to going backwards from $y_0+2\pi R$ to y_0. Therefore $T^2=\mathbb{1}$ and there are only two possibilities for holonomy, $T=+1$ and $T=-1$, the same as in the S^1/Z_2 picture. Hence we have $T\in\mathbb{Z}_2$. Note, in the above we have assumed that P and T can be simultaneously diagonalized. In general however P and T do not commute. In this case we would have $P\ T\ P=T^{-1}$.

3.3. Monopole string on S^1/Z_2

We saw that the gauge theory in 5-dimensions becomes a "gauge-Higgs" theory after the 5th dimension is compactified. The Higgs potential which breaks the SO(3) gauge symmetry to U(1) is absent, however its effect can be replaced by the Wilson loop along the compactified dimension. It was shown by 't Hooft and Polyakov [8] that the Georgi–Glashow model has a magnetic monopole solution to the equations of motion. It is natural to ask whether magnetic monopoles are present in the compactified 5-dimensional gauge theory and what is the correspondence with the usual 't Hooft–Polyakov solution.

The equations of motion corresponding to the Lagrangian (5) are:

$$D_\mu D^\mu \widetilde{\Phi}=0\ \ ,\quad D_\nu F^{\mu\nu}=ie^2[\widetilde{\Phi},D^\mu\widetilde{\Phi}]. \tag{39}$$

They correspond to the equations of motion of the Georgi–Glashow model in the absence of the Higgs potential.

Consider the ansatz (for $0<y<\pi R$):

$$\frac{A_5}{e}\equiv\widetilde{\Phi}=\frac{1}{2Re}(\hat{\vec{r}}\cdot\vec{T})\ F(r)\ , \tag{40}$$

$$A_i=\frac{1}{r}(\vec{T}\times\hat{\vec{r}})_i\ G(r)\ ,\quad A_0=0\ , \tag{41}$$

where $r=\sqrt{x_i^2}$, $\hat{r}_i=x_i/r$ and $F(r)$ and $G(r)$ are dimensionless functions. Asymptotically, for $r\to\infty$ we have $G(r)\to 1$. Note, the constant $\frac{1}{2R}$ in the normalization of A_5 has been fixed by the vacuum boundary conditions with the choice $F(r)\to 1$ as $r\to\infty$ (see discussion below). This is exactly the 't Hooft–Polyakov ansatz, and therefore it is a solution to the equations of motion, Eq. (39) with

$$V \equiv \lim_{r\to\infty} \sqrt{\frac{\mathrm{Tr}\,(\widetilde{\Phi}^2)}{k}} = \frac{1}{2Re}\,. \tag{42}$$

In order to complete the solution we need to extend the above solution to negative y (*i.e.* $-\pi R < y < 0$). As in the case with a constant background field A_5 we use the generalized parity operation, Eqs. (17) and (18), now with

$$U(y) = \exp\left(-i\frac{y}{R}\hat{\vec{r}}\cdot\vec{T}\right), \tag{43}$$

we obtain

$$\frac{A_5(-y)}{e} \equiv \widetilde{\Phi}(-y) = \frac{-F(r)+2}{2Re}\,(\hat{\vec{r}}\cdot\vec{T})\,, \tag{44}$$

$$\begin{aligned} A_i(-y) &= \frac{G(r)-1}{r}(\vec{T}\times\hat{\vec{r}})_i\cos\frac{y}{R} \\ &\quad + \frac{G(r)-1}{r}(T_i - \hat{r}_i(\hat{\vec{r}}\cdot\vec{T}))\sin\frac{y}{R} + \frac{1}{r}(\vec{T}\times\hat{\vec{r}})_i\,. \end{aligned} \tag{45}$$

Note, that the asymptotic values of A_i and A_5, normalized as in Eq. (40), for $r\to\infty$ satisfy $A_i(-y) = A_i(y)$ and $A_5(-y) = A_5(y)$ [2]. Hence we obtain the asymptotic holonomy

$$\lim_{r\to\infty} T(r) = \exp(i\pi\hat{\vec{r}}\cdot\vec{T}) \tag{46}$$

satisfying the condition $T^2 = \mathbb{1}$, *i.e.* $T\in\mathbb{Z}_2$. Moreover in any given spatial direction $\hat{\vec{r}}$, the asymptotic holonomy is gauge equivalent to the vacuum value, Eq. (33). It is this physical requirement, that asymptotically far away from the monopole we recover the vacuum holonomy, which fixes the asymptotic magnitude of A_5, Eq. (40). Note, in the case of a simple circle, discussed in Section 2.1, $T\in\mathbb{Z}$ and the magnitude of A_5 is arbitrary. In this case, the monopole mass can be taken continuously to zero. Hence monopoles on S^1 are unstable.

Although the form of the gauge fields for $-\pi R < y < 0$, defined by the generalized parity transformation of the 't Hooft ansatz for $0 < y < \pi R$ is quite complicated, it is easy to see that they are a solution to the field

[2] For any finite r, the field $A_i(y)$ is discontinuous in y at the point $y = \pi R$. This results in some singular behavior for F_{i5} at $y = \pi R$. This may be a serious problem for the construction of monopoles on orbifolds or there may be a simple way of removing the singularity. At any rate, this problem is presently under investigation.

equations. This is because the action is both parity and gauge invariant. In fact the action

$$S \equiv \int d^4x \int\limits_{-\pi R}^{+\pi R} dy \mathcal{L} = 2 \int d^4x \int\limits_{0}^{+\pi R} dy \mathcal{L} \tag{47}$$

is completely defined in terms of the fields in the fundamental domain $0 \leq y \leq \pi R$.

The asymptotic ($r \to \infty$) gauge field strength is given by

$$F_{ij} = \frac{\varepsilon_{ijk} \; \hat{r}_k \; (\hat{\vec{r}} \cdot \vec{T})}{r^2} \,. \tag{48}$$

The asymptotic U(1) Abelian magnetic field is then given by

$$B_i \equiv -\frac{1}{2ek} \; \varepsilon_{ijk} \; \mathrm{Tr} \left((\hat{\widetilde{\Phi}}) \; F_{jk} \right) = -\frac{\hat{r}_i}{e \, r^2} \,, \tag{49}$$

where $\hat{\widetilde{\Phi}} \equiv \widetilde{\Phi}/V$. Therefore the solution is a magnetic monopole string with total magnetic charge $g = -1/e$ or equivalently a magnetic charge per unit length in the 5th direction given by $g/\pi R$.

The monopole string energy density is given by

$$\mathcal{H} = \frac{1}{2\pi R} \; \left[\frac{1}{4e^2 k} \mathrm{Tr} \, (F_{ij} F^{ij}) + \frac{1}{2k} \mathrm{Tr} \, (D_i \widetilde{\Phi} D_i \widetilde{\Phi}) \right] . \tag{50}$$

It is a constant function of y and thus we should really talk about a monopole string stretched in the 5th direction from $y = 0$ to $y = \pi R$. The energy density, Eq. (50), is the usual four dimensional energy density divided by the length of the fifth dimension and the energy per unit length of the monopole string is obtained by integrating $\mathcal{H}$ over the three flat spatial dimensions. Note, the integrated energy density from Eq. (50) can be written as

$$H = \int d^3x \frac{1}{k} \mathrm{Tr} \; \left[\frac{1}{4} \left(\frac{1}{e} F_{ij} \mp \varepsilon_{ijk} D_k \widetilde{\Phi} \right)^2 \pm \frac{1}{2e} \; \varepsilon_{ijk} \; F_{ij} \; D_k \widetilde{\Phi} \right] , \tag{51}$$

where the integration over y has been performed. The second term can be rewritten using Bianchi identity as $\frac{1}{2ek} \varepsilon_{ijk} \partial_k \mathrm{Tr} \, (F_{ij} \widetilde{\Phi})$ and its contribution to the energy of the monopole is

$$\pm \frac{1}{2ek} \; \varepsilon_{ijk} \int d^3x \, \partial_k \mathrm{Tr} \, (F_{ij} \widetilde{\Phi}) = \pm V \int \vec{B} \cdot d\vec{S} = \pm 4\pi V g. \tag{52}$$

When the first term in (51) vanishes the monopole solution is said to satisfy the Bogomol'nyi bound and such monopoles are called BPS monopoles. In fact, the general 't Hooft–Polyakov monopole solution reduces to a BPS monopole in the limit that the Higgs potential for the adjoint scalar vanishes. Hence our monopole strings are in fact BPS monopole strings and their mass is given by

$$M_m = \frac{4\pi V}{e} = \frac{M_W}{\alpha} = \frac{1}{2\alpha R}\,, \tag{53}$$

where $\alpha = e^2/4\pi$ is the dimensionless fine structure constant at the scale $1/R$, and R is the orbifold radius.

It is also important to express the equations for the BPS condition and the monopole energy density in an explicitly gauge invariant and 5D covariant form. The BPS condition is

$$F_{ij} = \pm\varepsilon_{ijk} F_{k5} \tag{54}$$

and the energy density is given by

$$\begin{aligned}\mathcal{H} = \pm\frac{1}{2e_5 k}\,\varepsilon_{ijk}\,\mathrm{Tr}\,(F_{ij}\;D_k\Phi) \;&=\; \pm\frac{1}{2e_5^2 k}\varepsilon_{ijk}\,\mathrm{Tr}\,(F_{ij}\;F_{k5}) \\ &=\; \pm\frac{1}{8e_5^2 k}\varepsilon_{0NPQR}\,\mathrm{Tr}\,(F^{NP}\;F^{QR})\,.\end{aligned} \tag{55}$$

Note it is then clear that the five dimensional Hamiltonian density is the time component of a five vector given by

$$\mathcal{P}^M = \pm\frac{1}{8e_5^2 k}\,\varepsilon^{MNPQR}\,\mathrm{Tr}\,(F_{NP}\;F_{QR}) \equiv \partial_N K^{MN} \tag{56}$$

with

$$K^{MN} = \pm\frac{1}{4e_5^2 k}\,\varepsilon^{MNPQR}\,\mathrm{Tr}\,(A_P\,F_{QR} - i\frac{2}{3}A_P\,A_Q\,A_R)\,. \tag{57}$$

Hence P^M satisfies the topological conservation law $\partial_M P^M \equiv 0$.

As a final note we can also consider the monopole solution in the gauge with vanishing background gauge field, *i.e.* $\langle A_5\rangle \equiv 0$. We find (for $0 < y < \pi R$)

$$\frac{A_5}{e} \;\equiv\; \widetilde{\Phi} = \frac{F(r)-1}{2Re}\,(\hat{\vec{r}}\cdot\vec{T})\,, \tag{58}$$

$$\begin{aligned}A_i \;=\; &\frac{G(r)-1}{r}\,(\vec{T}\times\hat{\vec{r}})_i\,\cos\frac{y}{2R} \\ &+\frac{G(r)-1}{r}\,(T_i - \hat{r}_i(\hat{\vec{r}}\cdot\vec{T}))\,\sin\frac{y}{2R} + \frac{1}{r}\,(\vec{T}\times\hat{\vec{r}})_i\,.\end{aligned} \tag{59}$$

Then for $-\pi R < y < 0$ we obtain, by explicitly gauge transforming the fields in Eqs. (44) and (45), $A_5(-y) = -A_5(y)$ and $A_i(-y) = A_i(y)$ as expected from "naive" parity, Eqs. (15) and (16).

4. Conclusions

In this letter we discussed Wilson loop symmetry breaking on orbifolds in five dimensions. We have also cleared up the mathematical correspondence between two apparently distinct orbifolds considered in the literature, namely S^1/Z_2 with a background gauge field and $S^1/(Z_2 \times Z_2')$. In fact they are identical upon rescaling the radius by a factor of 2. Although our analysis has been in non-supersymmetric gauge theories, it should be easy to extend to the case of orbifold symmetry breaking in supersymmetric gauge theories.

We have constructed monopole string solutions for an SO(3) gauge group; valid when SO(3) is broken to U(1). Our construction can be extended to any SU(N) gauge group on an $M \times S^1/Z_2$ orbifold with background gauge field. Such monopole strings may have interesting phenomenological consequences for grand unified scenarios with large extra dimensions [1]. They would be expected to have mass of order $1/2\alpha R$, with a compactification scale $1/R$ as small as a few TeV. Note that a GUT monopole string can lead to catalysis of baryon number violating processes [16].

Another interesting example would be in the case of the SU(3) electroweak unification model recently discussed in the literature [5]. It is easy to show that this model also contains monopole strings when the symmetry is broken to either $\mathrm{SU}(2) \times \mathrm{U}(1)_Y$ or directly to $\mathrm{U}(1)_{\mathrm{EM}}$ with the addition of a Higgs multiplet in the triplet representation. Such a monopole string will have mass of order $60/R$.

Clearly in light of the results presented here, it will be interesting to study monopole string production at high energy accelerators and at finite temperatures in the early universe.

I would like to take this opportunity to congratulate S. Pokorski on the occasion of his 60$^{\mathrm{th}}$ birthday. I would like to thank him for his valued friendship and also for the many fruitful discussions we have had over the years. Partial support for this work came from DOE contract DOE/ER/01545.

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[17] Consider a background field gauge with $A_M = B_M + a_M$ where B_M is the background value of the gauge field and a_M are the small fluctuations. The background covariant derivative is given by $D_M \equiv \partial_M + i[B_M, \;]$. If we use the covariant gauge fixing condition $D^M a_M \equiv 0$, then the gauge field equations of motion are given by $D^M D_M a_N + 3i[B_{NM}, a^M] = 0$. Note, for a constant background gauge field $B_{NM} \equiv 0$.

[18] For any finite r, the field $A_i(y)$ is discontinuous in y at the point $y = \pi R$. This results in some singular behavior for F_{i5} at $y = \pi R$. This may be a serious problem for the construction of monopoles on orbifolds or there may be a simple way of removing the singularity. At any rate, this problem is presently under investigation.