κ-DEFORMED KINEMATICS AND ADDITION LAW FOR DEFORMED VELOCITIES*

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In κ-deformed relativistic framework we consider three different definitions of κ-deformed velocities and introduce corresponding addition laws. We show that one of the velocities has classical relativistic addition law. The relation of velocity formulae with the coproduct for fourmomenta and noncommutative space-time structure is exhibited.

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1. Introduction

Recently due to increasing interest in deformed relativistic space-time framework (see *e.g.* [1–6]) it is important to understand the deformation of Einsteinian relativistic kinematics. In particular the problems occur if the classical Poincaré symmetries are replaced by quantum ones, with modification of classical Abelian addition law for the momenta. Here we shall consider as distinguished example the so–called κ-deformed quantum Poincaré symmetries (see *e.g.* [7–11]), which recently were also used as possible framework for describing the quantum gravity effects (see *e.g.* [12–16]).

The κ-deformed relativistic Hopf algebra framework in bicrossproduct basis is characterized by classical Lorentz algebra of $O(3,1)$ generators

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$M_{\mu\nu} = (M_i, N_i)$, commuting fourmomenta $P_\mu = (P_i, P_0 = E/c)$ and κ-deformed commutation relations of threemomenta P_i with boost generators N_i:

$$[N_i, P_j] = \frac{i}{2}\delta_{ij}\left[\kappa c(1 - \mathrm{e}^{(-2E/\kappa c^2)}) + \frac{1}{\kappa c}\vec{P}^{\,2}\right] - \frac{i}{\kappa c}P_i P_j \,. \tag{1}$$

All remaining Poincaré algebra relations remain classical. The mass square Casimir M^2 for κ-deformed Poincaré algebra in bicrossproduct basis looks as follows [3,8,9]

$$\cosh\frac{E}{\kappa c^2} - \frac{1}{2\kappa^2 c^2}\,\mathrm{e}^{(E/\kappa c^2)}\,\vec{p}^{\,2} = 1 + \frac{M^2}{2\kappa^2}\,. \tag{2}$$

One can consider the formula (2) as describing the deformation of classical energy-momentum dispersion relation $\omega(\vec{p}) = (\vec{p}^{\,2} + m_0^2 c^2)^{1/2}$ for free particles [3]

$$\begin{aligned}\omega(\vec{p}) \to E_\kappa(\vec{p}) &= -\kappa c^2 \ln\left[1 + \frac{M^2}{2\kappa^2} - \sqrt{\frac{M^2}{\kappa^2}\left(1 + \frac{M^2}{4\kappa^2}\right)^2 + \frac{\vec{p}^{\,2}}{\kappa^2 c^2}}\right] \\ &= -\kappa c^2 \ln\left[\cosh\left(\frac{m_0}{\kappa}\right) - \sqrt{\sinh^2\left(\frac{m_0}{\kappa}\right) + \frac{\vec{p}^{\,2}}{\kappa^2 c^2}}\right], \end{aligned} \tag{3}$$

where the value M^2 is related with the rest mass of particle as follows [15]

$$M^2 = 2\kappa^2(\cosh\frac{m_0}{\kappa} - 1)\,. \tag{4}$$

The quantum group structure of κ-deformed Poincaré algebra is provided by nonprimitive coproducts, which for threemomenta P_i and energy E take the following form

$$\begin{aligned}\Delta P_i &= P_i \otimes 1 + \mathrm{e}^{(-E/\kappa c^2)} \otimes P_i\,, \\ \Delta E &= E \otimes 1 + 1 \otimes E\,.\end{aligned} \tag{5}$$

The κ-deformed relativistic framework is described by dual pair of Hopf algebras describing κ-deformed Poincaré algebra and κ-deformed Poincaré group [17–19]. Considering noncommutative translations $\widehat{x}_\mu$ of κ-deformed Poincaré group as describing noncommutative space-time coordinates one can show that

$$[\widehat{x}_0, \widehat{x}_i] = -\frac{i\hbar}{\kappa c}\widehat{x}_i\,, \qquad [\widehat{x}_i, \widehat{x}_j] = 0\,. \tag{6}$$

The relations (6) describe κ-deformed Minkowski space [17,9,3]. Further, by considering semidirect product of κ-deformed Poincaré algebra and κ-Poincaré group (so-called Heisenberg double [18]) one obtains κ-deformed generalized phase space [18–19].

The aim of this note is to consider the possible definitions of κ-deformed velocities and study their addition law. Due to non-primitive coproduct (5) one can introduce three different velocity formulae:

(i) The one following from classical Hamilton equation

$$\dot{X}_i = V_i = \frac{\partial E_\kappa(\vec{p})}{\partial p_i} \,. \tag{7}$$

By considering κ-deformed phase space in bicrossproduct basis (see *e.g.* [18–20]) one confirms that the symplectic form defining Hamiltonian equations (7) are not deformed.

(ii) Two other types of velocities are linked with non-Abelian addition law of three-momenta. By considering $P_i \otimes 1 = p_i$, $1 \otimes P_i = \delta^{\mathrm{L}} p_i$ one can write

$$\vec{p} + \mathrm{e}^{(-E/\kappa c^2)} \delta^{\mathrm{L}} \vec{p} \ = \ \vec{p} + \delta\vec{p} \Rightarrow \delta^{\mathrm{L}}\vec{p} \ = \ \mathrm{e}^{(E/\kappa c^2)} \delta\vec{p} \,. \tag{8}$$

We define the left-covariant velocity as follows:

$$\begin{aligned} V_i^{\mathrm{L}} &= \lim_{\delta p_i \to 0} \frac{E_\kappa(\vec{p} + \delta\vec{p}) - E_\kappa(\vec{p})}{\delta^{\mathrm{L}} p_i} \\ &= \mathrm{e}^{(-E(\vec{p})/\kappa c^2)} \frac{\partial E_\kappa(\vec{p})}{\partial p_i} = \mathrm{e}^{(-E(\vec{p})/\kappa c^2)} V_i \,. \end{aligned} \tag{9}$$

Using the assignment $P_i \otimes 1 = \delta^{\mathrm{R}} p_i, 1 \otimes P_i = p_i$ one obtains

$$\begin{aligned} \delta^{\mathrm{R}}\vec{p} + \mathrm{e}^{(-\delta E/\kappa c^2)}\vec{p} &= \vec{p} + \delta\vec{p} \Longrightarrow \delta^{\mathrm{R}}\vec{p} = \delta\vec{p} + \left(1 - \mathrm{e}^{(-\delta E/\kappa c^2)}\right)\vec{p} \\ &= \left(1 + \frac{1}{\kappa c^2}\vec{p}\vec{\nabla}E\right)\delta\vec{p}\,, \end{aligned} \tag{10}$$

where assuming that momenta and velocities are parallel, *i.e.* $\vec{p} \parallel \vec{V} = \vec{\nabla}E(\vec{p})$ we used the relation

$$\vec{p}(\vec{\nabla}E \cdot \delta\vec{p}_i) \ = \ (\vec{p} \cdot \vec{\nabla}E)\delta\vec{p}_i \,. \tag{11}$$

Using (10) one can introduce right-covariant velocity

$$V_i^{\mathrm{R}} = \lim_{\delta p_i \to 0} \frac{E_\kappa(\vec{p} + \delta\vec{p}_i) - E_\kappa(\vec{p})}{\delta^{\mathrm{R}} p_i} \ = \ \left(1 + \frac{1}{\kappa c^2}\vec{p}\vec{V}\right)^{-1} \frac{\partial E_\kappa(\vec{p})}{\partial p_i} \,. \tag{12}$$

The velocities (7) have been introduced in standard basis still in 1992 by Bacry [21] (in bicrossproduct basis see [22]). The velocities $V_i^{\rm L}$ were introduced in [23,24] and both velocities (9) and (12) for massless case in [24] as left and right group velocities.

We shall show that the most interesting with its properties is the velocity $V_i^{\rm R}$ — it has classical velocity addition law, which for parallel velocities $(0,0,V_1^{\rm R})$, $(0,0,V_2^{\rm R})$ looks as follows

$$V_{12}^R = \frac{V_1^R + V_2^R}{1 + \frac{V_1^R V_2^R}{c^2}} \,. \tag{13}$$

We shall discuss below all the three velocities described by formulae (7), (9) and (12) for arbitrary value of mass parameter M (see also (4)).

2. Three velocities — general properties

Three velocities (7), (9) and (12) are related with each other by the following formulae

$$V_i^{\rm L} \;=\; {\rm e}^{(-E_\kappa(\vec{p})/\kappa c^2)} V_i \,, \tag{14a}$$

$$V_i^{\rm R} \;=\; \frac{V_i}{1 + \frac{1}{\kappa c^2}\vec{p}\vec{\rm v}} \,, \tag{14b}$$

where from (7) and (1)–(2) follows that[1]

$$\vec{V} = \frac{\vec{p}}{\frac{\kappa}{2}\left(1 - {\rm e}^{(-2E/\kappa c^2)} - \frac{\vec{p}^{\,2}}{\kappa^2 c^2}\right)} = \frac{\vec{p}\,{\rm e}^{(E/\kappa c^2)}}{\kappa\left[\cosh(\frac{m_0}{\kappa}) - {\rm e}^{(-E/\kappa c^2)}\right]} \,. \tag{15}$$

One can calculate that[1]

$$V^2 = c^2 {\rm e}^{(2E/\kappa c^2)} \left[1 - \frac{\sinh^2(\frac{m_0}{\kappa})}{\left(\cosh(\frac{m_0}{\kappa}) - {\rm e}^{(-E/\kappa c^2)}\right)^2}\right] \,. \tag{16}$$

and one obtains

$$\lim_{E\to\infty} V = \infty \,. \tag{17}$$

In the interval $M \leq E \leq \infty$ the function (16) increases monotonically.

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[1] Further we denote $V = |\vec{V}|$, $E = E_\kappa(\vec{p})$ and $p^2 \equiv \vec{p}^{\,2}$.

From the formulae (14) and (15) one gets

$$\vec{V}^{\rm L} = \frac{{\rm e}^{(-E/\kappa c^2)}\,\vec{p}}{\frac{\kappa}{2}\left(1-{\rm e}^{(-2E/\kappa c^2)}-\frac{\vec{p}^2}{\kappa^2c^2}\right)} = \frac{\vec{p}}{\kappa\left[\cosh(\frac{m_0}{\kappa})-{\rm e}^{(-E/\kappa c^2)}\right]}\,. \tag{18}$$

and after simple algebraic manipulation

$$\vec{V}^{\rm R} = \frac{\vec{p}}{\frac{\kappa}{2}\left(1-{\rm e}^{(-2E/\kappa c^2)}+\frac{\vec{p}^{\,2}}{\kappa^2c^2}\right)} = \frac{\vec{p}\,{\rm e}^{(E/\kappa c^2)}}{\kappa\left[{\rm e}^{(E/\kappa c^2)}-\cosh(\frac{m_0}{\kappa})\right]}\,. \tag{19}$$

We get

$$(V^{\rm L})^2 \;=\; c^2\left[1-\frac{\sinh^2(\frac{m_0}{\kappa})}{\left(\cosh(\frac{m_0}{\kappa})-{\rm e}^{(-E/\kappa c^2)}\right)^2}\right], \tag{20}$$

$$(V^{\rm R})^2 \;=\; c^2\left[1-\frac{\sinh^2(\frac{m_0}{\kappa})}{\left(\cosh(\frac{m_0}{\kappa})-{\rm e}^{(E/\kappa c^2)}\right)^2}\right]. \tag{21}$$

One can show:

(i) for all energies $(V^{\rm L})^2 \leq c^2$ and $(V^{\rm R})^2 \leq c^2$;

(ii) for $M < E < \infty$ we get $\frac{d(V^{\rm L})^2}{dE} > 0$ and $\frac{d(V^{\rm R})^2}{dE} > 0$, *i.e.* both functions (20) and (21) are monotonic;

(iii) if $M = 0$ (equivalent to $m_0 = 0$) both velocities $V_i^{\rm L}$ and $V_i^{\rm R}$ have classical absolute value *i.e.* $(V^{\rm L})^2 = (V^{\rm R})^2 = c^2$ [2].

3. Right group velocity — addition formula

Let us write the formula (19) as follows:

$$\vec{p} = \kappa\, B(m_0, E)\vec{V}^{\rm R}\,, \tag{22}$$

where

$$B(m_0, E) = 1 - \cosh(\frac{m_0}{\kappa})\,{\rm e}^{(-E/\kappa c^2)}\,. \tag{23}$$

[2] The observation *(iii)* has been made also in [24].

The function $B(m_0, E)$ enters into the κ-deformed Lorentz transformations determined by the boost parameter $\vec{\alpha} = \alpha\vec{n}$ ($\vec{n}^{\,2} = 1$) [25,16]

$$\begin{aligned} E(\alpha) &= E + \kappa c^2 \ln W(E, \vec{n}\vec{p}\,; \alpha)\,, && \text{(24a)} \\ \vec{p}(\alpha) &= W^{-1}(E, \vec{n}\vec{p}\,; \alpha)\,\{\vec{p} + [(\vec{n}\vec{p})(\cosh\alpha - 1) \\ &\quad -\kappa\, c B(m_0, E)\sinh\alpha]\,\vec{n}\}\;, && \text{(24b)} \end{aligned}$$

where

$$W(E, \vec{n}\vec{p}\,; \alpha) = 1 - \frac{1}{\kappa c}(\vec{n}\vec{p})\sinh\alpha + B(m_0, E)(\cosh\alpha - 1)\,. \tag{25}$$

The function (25) satisfies the relation

$$W(E, \vec{n}\vec{p}\,; \alpha) = \frac{B(m_0, E) - 1}{B(m_0, E(\alpha)) - 1}\,. \tag{26}$$

Let us define the velocity $\vec{W}^{\rm R}$ as the velocity (22) in the frame Lorentz-transformed by the boost parameter $\vec{\alpha} = \alpha\vec{n}$.

Using (22) we have the formula

$$\vec{p}(\alpha) = \kappa B(m_0, E(\alpha))\overrightarrow{W}^{\rm R}\,. \tag{27}$$

From (27) and (24b) one gets

$$\begin{aligned} \overrightarrow{W}^{\rm R} &= \frac{B(m_0, E)}{W(E, \vec{n}\vec{p}\,; \alpha) + B(m_0, E) - 1} \\ &\quad \times \left\{\vec{V}^{\rm R} + \vec{n}\left[(\vec{n}\vec{V}^{\rm R})(\cosh\alpha - 1) - c\sinh\alpha\right]\right\}\;, \end{aligned} \tag{28}$$

and further we obtain

$$\overrightarrow{W}^{\rm R} = \frac{\vec{V}^{\rm R} + \vec{n}[(\vec{n}\vec{u})(\cosh\alpha - 1) - c\sinh\alpha]}{\cosh\alpha[1 - \frac{1}{c}(\vec{n}\vec{u})\tanh\alpha]}\,. \tag{29}$$

Introducing the relative velocity $\vec{u}$ of two κ-deformed Lorentz frames, one at rest ($\alpha = 0$) and second described by boost parameter $\vec{\alpha} = \vec{n}\alpha$

$$\vec{u} = -c\vec{n}\tanh\alpha\,, \tag{30}$$

the relation (29) can be written in the form (see *e.g.* [26])

$$\begin{aligned}\overrightarrow{W}^{\rm R} &= \left(1+\frac{\vec{V}^{\rm R}\vec{u}}{c^2}\right)^{-1}\left\{\vec{V}^{\rm R}\left(1-\frac{u^2}{c^2}\right)^{1/2}\right.\\ &\qquad \left.+\vec{u}\left[1+\frac{\vec{V}^{\rm R}\vec{u}}{u^2}-\frac{\vec{V}^{\rm R}\vec{u}}{u^2}\left(1-\frac{u^2}{c^2}\right)^{1/2}\right]\right\}. \qquad (31)\end{aligned}$$

The formula (31) describes the general Einsteinian composition law of two arbitrary three-velocities $\vec{V}^{\rm R}, \vec{u}$, which for parallel velocities $\vec{V}^{\rm R}\|\vec{u}$ reduces to the formula (13).

It should be stressed that basic ingredient in the derivation of classical addition law (31) is the relation (22). If we use other two formulae (15) or (18) for deformed velocities, analogous reasoning leads to the deformation of classical addition formulae (13) and (31).

To complete the argument we shall show that the definition (30) is equivalent to the relation (22). Indeed, let us solve the relation $\vec{p}(\alpha)=0$ describing the transformation from the moving system with nonvanishing momentum $\vec{p}(\alpha=0)$ to the rest system with $\vec{p}=0$ $(\alpha\neq 0)$. From (24b) one gets

$$\vec{p}(\alpha)=0 \Longrightarrow \vec{p}+(\vec{n}\vec{p})(\cosh\alpha-1)-\kappa c\, B(m_0,E)\sinh\alpha=0\,. \qquad (32)$$

Further if $\vec{n}\|\vec{p}$ one obtains ($p\equiv|\vec{p}|$)

$$p\cosh\alpha-\kappa c\, B(m_0,E)\sinh\alpha=0\,. \qquad (33)$$

We see that inserting in (33) the velocity u from formula (30) ($u=c\tanh\alpha$) we obtain the relation (22) ($p=\kappa B(m_0,E)u$).

4. Velocity and noncommutative space-time

In order to describe velocity formula in noncommutative space-time one should use the corresponding deformed Hamiltonian formalism. For κ-deformed relativistic phase space such a framework has been proposed in [3][3]

The κ-deformed noncommutative phase space kinematics is determined by basic Poisson brackets of relativistic phase space variables $Y_A=(x_\mu,p_\mu)$ given in bicrossproduct basis by the following relations [18,19]:

$$\begin{aligned}\{p_i,x_j\} &= \delta_{ij}\,,\\ \{p_0,x_0\} &= -1\,,\\ \{p_0,x_i\} &= 0\,,\end{aligned}$$

[3] See formulae (1.5)–(1.10) and (4.2) in [3].

$$\begin{aligned}\{x_0, x_i\} &= \frac{x_i}{\kappa c}\,,\\ \{x_0, p_i\} &= -\frac{p_i}{\kappa c}\,,\\ \{p_\mu, p_\nu\} &= 0\,.\end{aligned} \tag{34}$$

Writing down (34) in compact form

$$\{Y_A, Y_B\} = \omega^{(\kappa)}_{AB}(x,p)\,.$$

we obtain the κ-deformed Hamilton equations describing evolution with respect to the parameter s

$$\frac{dY_A}{ds} = \omega^{(\kappa)}_{AB}\frac{\partial\mathcal{H}^{(\kappa)}}{\partial Y_B}\,. \tag{35}$$

where $\mathcal{H}^{(\kappa)} = \mathcal{H}^{(\kappa)}(x,p)$ determines the dynamics. Assuming translational invariance one should take one-particle Hamiltonian as $\mathcal{H}^{(\kappa)} = \mathcal{H}^{(\kappa)}(p_0,\vec{p})$ and explicitely from (35) one gets

$$\frac{dx_i}{ds} = -\frac{\partial\mathcal{H}^{(\kappa)}}{\partial p_i}\,, \tag{36a}$$

$$\frac{dx_0}{ds} = -\frac{1}{\kappa c}\,p_i\frac{\partial\mathcal{H}^{(\kappa)}}{\partial p_i} + \frac{\partial\mathcal{H}^{(\kappa)}}{\partial p_0} \tag{36b}$$

and $\frac{dp_\mu}{ds} = 0$ *i.e.* p_μ are s-independent.

The physical interpretation of the parameter s depends on the choice of the Hamiltonian $\mathcal{H}^{(\kappa)}$. We can consider the following two basic cases:

(i) The Hamiltonian H^κ describes the energy dispersion relation $E = E_\kappa(\vec{p})$ by means of the formula $\mathcal{H}^{(\kappa)} \equiv H^\kappa(p_0,\vec{p}) = cp_0 - E_\kappa(\vec{p})$. In such a case if $\kappa \to \infty$ (standard relativistic framework) *i.e.* $H^\infty(p_0,\vec{p}) = cp_0 - c(\vec{p}^{\,2} + m_0^2c^2)^{1/2}$ one can identify the parameter s with time variable ($\frac{dx_0}{ds} = 1 \longleftrightarrow x_0 = s + \text{const.}$) and one obtains the standard velocity formula

$$\frac{dx_i}{dx_0} = \frac{p_i}{(\vec{p}^{\,2} + m_0^2c^2)^{1/2}}\,. \tag{37}$$

If $\kappa \neq 0$ such identification is not possible, because from (36b) it follows that

$$\frac{dx_0}{ds} = c - \frac{1}{\kappa c}\,p_i\frac{\partial H^\kappa}{\partial p_i}\,, \tag{38}$$

and further we get

$$\frac{dx_i}{dx_0} = \frac{\frac{dx_i}{ds}}{\frac{dx_0}{ds}} = \frac{-\frac{\partial H^\kappa}{\partial p_i}}{c\left(1 - \frac{1}{\kappa c^2} p_i \frac{\partial H^\kappa}{\partial p_i}\right)}\,, \tag{39}$$

i.e. after using (7) we obtain our favored velocity formula (14b).

(ii) One can use also as the Hamiltonian $\mathcal{H}^{(\kappa)}$ the κ-invariant mass Casimir by assuming that $\mathcal{H}^{(\kappa)} \equiv M^2 c^4$. In classical relativistic case $\mathcal{H}^{(\infty)} = E^2 - c^2 \vec{p}^{\,2} = c^2 \left(p_0^2 - \vec{p}^{\,2}\right)$ the parameter s corresponds to Poincaré-invariant length parameter. One gets

$$\frac{dx_i}{ds} = 2c^2 p_i\,, \qquad \frac{dx_0}{ds} = 2c^2 p_0 \tag{40}$$

and on the mass-shell ($\mathcal{H}^{(\infty)} = m_0^2 c^4 = \text{const.}$) we obtain the formula (37). In general case ($\kappa \leq \infty$) the deformed mass shell condition (2) can be written as identity $\mathcal{H}^{(\kappa)}(E_\kappa(\vec{p}), \vec{p}) \equiv M^2 c^4$ where $E_\kappa(\vec{p})$ is given by (3). One obtains

$$\frac{\partial \mathcal{H}^{(\kappa)}}{\partial p_0} \frac{\partial E_\kappa}{\partial p_i} + \frac{\partial \mathcal{H}^{(\kappa)}}{\partial p_i} = 0 \Rightarrow \frac{\partial \mathcal{H}^{(\kappa)}}{\partial p_i} = -V_i \frac{\partial \mathcal{H}^{(\kappa)}}{\partial p_0}\,. \tag{41}$$

After inserting (41) into (36a)–(36b) one gets

$$\frac{dx_i}{ds} = V_i \frac{\partial \mathcal{H}^{(\kappa)}}{\partial p_0}\,, \frac{dx_0}{ds} = \left(1 + \frac{1}{\kappa c} p_i V_i\right) \frac{\partial \mathcal{H}^{(\kappa)}}{\partial p_0} \tag{42}$$

and we obtain again the formula (14b).

5. Final remarks

Following the philosophy advocated firstly by Majid [27] quantum-deformed space-time kinematics describes generalized symmetries and non-commutative geometries at ultra-short Planck scales. The real challenge is to find observable physical effects caused by such modification of short distance behavior of quantum phenomena. The study presented here has much more modest aim: it provides a contribution to the description of kinematics obtained in the framework of deformed quantum theories.

It should be pointed out that such a kinematical framework requires still several problems to be solved, in particular understanding the relation between κ-deformed kinematics and description of macroscopic bodies. The basic question is to understand how the κ-effects cancel if we consider very large sum of elementary objects described by deformed κ-kinematics. We would like to stress that in such a case we should obtain for macroscopic bodies the classical relativistic kinematics.

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