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CLIFFORD ALGEBRA IMPLYING THREE FERMION GENERATIONS REVISITED*

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Dedicated to Stefan Pokorski on his 60th birthday

The author's idea of *algebraic compositeness* of fundamental particles, allowing to understand the existence in Nature of three fermion generations, is revisited. It is based on two postulates. *Primo*, for all fundamental particles of matter the Dirac square-root procedure $\sqrt{p^2} \rightarrow \Gamma^{(N)} \cdot p$ works, leading to a sequence $N = 1, 2, 3, \ldots$ of Dirac-type equations, where four Dirac-type matrices $\Gamma^{(N)}_\mu$ are embedded into a Clifford algebra *via* a Jacobi definition introducing four "centre-of-mass" and $(N-1)\times$four "relative" Dirac-type matrices. These define one "centre-of-mass" and $N-1$ "relative" Dirac bispinor indices. *Secundo*, the "centre-of-mass" Dirac bispinor index is coupled to the Standard Model gauge fields, while $N-1$ "relative" Dirac bispinor indices are all free indistinguishable physical objects obeying Fermi statistics along with the Pauli principle which requires the full antisymmetry with respect to "relative" Dirac indices. This allows only for *three* Dirac-type equations with $N = 1, 3, 5$ in the case of N odd, and *two* with $N = 2, 4$ in the case of N even. The first of these results implies unavoidably the existence of *three and only three* generations of fundamental fermions, namely leptons and quarks, as labelled by the Standard Model signature. At the end, a comment is added on the possible shape of Dirac 3×3 mass matrices for four sorts of spin-1/2 fundamental fermions appearing in three generations. For charged leptons a prediction is $m_\tau = 1776.80$ MeV, when the input of experimental m_e and m_μ is used.

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One of the most important theoretical achievements in the history of physics was Dirac's algebraic discovery of the particle's spin 1/2, inherently connected with the linearisation of relativistic wave equation through his famous square-root procedure $\sqrt{p^2} \to \gamma \cdot p$ [1]. As a result, some physical particles, called later on spin-1/2 fermions, got — in addition to their spatial coordinates $\vec{r}$ — new algebraic degrees of freedom, described with the use of Dirac bispinor index $\alpha = 1, 2, 3, 4$. This was acted on by the 4×4 Dirac matrices, in particular, by the spin-1/2 matrix $\frac{1}{2}\vec{\sigma}$ which supplemented the orbital angular momentum operator $\vec{r} \times \vec{p}$ to the operator of particle's total angular momentum. Thus, through an act of abstraction from the particle's spatial properties, its spin 1/2 was recognised as an algebraic analogue of its spatial, orbital angular momentum, satisfying the same rotation-group commutation relations.

The starting point of the author's idea of *algebraic compositeness* [2] was a proposal of a new act of abstraction from the particle's spatial properties, where a wave function $\psi^{(N)}_{\alpha_1 \alpha_2 \ldots \alpha_N}(\vec{r})$ of several Dirac bispinor indices $\alpha_1, \alpha_2, \ldots, \alpha_N$ (as representing N physical objects correlated with one material point of spatial coordinates $\vec{r}$) was introduced in an analogy with the wave function $\psi^{(N)}(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ of several spatial coordinates $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ (as representing N physical objects being, this time, material points).

In order to construct these several Dirac bispinor indices the observation was made [2] that, generically, the Dirac algebra

$$\left\{ \Gamma^{(N)}_\mu \, , \, \Gamma^{(N)}_\nu \right\} = 2 g_{\mu\nu} \tag{1}$$

can be realized through Dirac-type matrices of the form

$$\Gamma^{(N)}_\mu \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma^{(N)}_{i\mu} \tag{2}$$

built up linearly from N elements of the Clifford algebra

$$\left\{ \gamma^{(N)}_{i\mu} \, , \, \gamma^{(N)}_{j\nu} \right\} = 2 \delta_{ij} g_{\mu\nu} \, , \tag{3}$$

where $N = 1, 2, 3, \ldots$, $i, j = 1, 2, \ldots, N$ and $\mu, \nu = 0, 1, 2, 3$. Then, the Dirac square-root procedure $\sqrt{p^2} \to \Gamma^{(N)} \cdot p$ leads in the interaction-free case to the sequence $N = 1, 2, 3, \ldots$ of Dirac-type equations

$$\left(\Gamma^{(N)} \cdot p - M^{(N)} \right) \psi^{(N)}(x) = 0 \tag{4}$$

with $\psi^{(N)}(x) \equiv \left(\psi^{(N)}_{\alpha_1 \alpha_2 \ldots \alpha_N}(x) \right)$, where the meaning of Dirac bispinor indices is explained as in Eq. (12) later on. Here, each $\alpha_i = 1, 2, 3, 4$ $(i =$

$1,2,\ldots,N$). The mass $M^{(N)}$ is independent of $\Gamma^{(N)}_\mu$. In general, the mass $M^{(N)}$ should be replaced by a mass matrix of elements $M^{(N,N')}$ which would couple $\psi^{(N)}(x)$ with all appropriate $\psi^{(N')}(x)$, and it might be natural to assume for $N \neq N'$ that $\gamma^{(N)}_{i\mu}$ and $\gamma^{(N')}_{j\nu}$ commute, and so do $\Gamma^{(N)}_\mu$ and $\Gamma^{(N')}_\nu$.

For $N=1$, Eq. (4) is evidently the usual Dirac equation and for $N=2$ it is known as the Dirac form [3] of Kähler equation [4], while for $N \geq 3$ Eqs. (4) give us *new* Dirac-type equations [2]. They describe some spin-halfinteger or spin-integer particles for N odd or N even, respectively.

The Dirac-type matrices $\Gamma^{(N)}_\mu$ for any N can be embedded into the new Clifford algebra

$$\left\{\Gamma^{(N)}_{i\mu}\,,\,\Gamma^{(N)}_{j\nu}\right\} = 2\delta_{ij}g_{\mu\nu}\,, \tag{5}$$

isomorphic with the Clifford algebra (3) of $\gamma^{(N)}_{i\mu}$, if $\Gamma^{(N)}_{i\mu}$ are defined by the properly normalised Jacobi linear combinations of $\gamma^{(N)}_{i\mu}$:

$$\begin{aligned}\Gamma^{(N)}_{1\mu} &\equiv \Gamma^{(N)}_\mu \equiv \frac{1}{\sqrt{N}}\left(\gamma^{(N)}_{1\mu}+\ldots+\gamma^{(N)}_{N\mu}\right)\,,\\ \Gamma^{(N)}_{i\mu} &\equiv \frac{1}{\sqrt{i(i-1)}}\left[\gamma^{(N)}_{1\mu}+\ldots+\gamma^{(N)}_{i-1\,\mu}-(i-1)\gamma^{(N)}_{i\mu}\right]\end{aligned} \tag{6}$$

for $i=1$ and $i=2,\ldots,N$, respectively. So, $\Gamma^{(N)}_{1\mu}$ and $\Gamma^{(N)}_{2\mu},\ldots,\Gamma^{(N)}_{N\mu}$, respectively, present the "centre-of-mass" and "relative" Dirac-type matrices. Note that the Dirac-type equation (4) for any N does not involve the "relative" Dirac-type matrices $\Gamma^{(N)}_{2\mu},\ldots,\Gamma^{(N)}_{N\mu}$, including solely the "centre-of-mass" Dirac-type matrix $\Gamma^{(N)}_{1\mu} \equiv \Gamma^{(N)}_\mu$. Since $\Gamma^{(N)}_{i\mu} = \sum_{j=1}^N O_{ij}\gamma^{(N)}_{j\mu}$, where $O=(O_{ij})$ is an orthogonal $N\times N$ matrix ($O^T = O^{-1}$), we obtain for the total spin tensor the equality

$$\sum_{i=1}^N \sigma^{(N)}_{i\mu\nu} = \sum_{i=1}^N \Sigma^{(N)}_{i\mu\nu}\,, \tag{7}$$

where

$$\sigma^{(N)}_{j\mu\nu} \equiv \frac{i}{2}\left[\gamma^{(N)}_{j\mu}\,,\,\gamma^{(N)}_{j\nu}\right]\,,\;\; \Sigma^{(N)}_{j\mu\nu} \equiv \frac{i}{2}\left[\Gamma^{(N)}_{j\mu}\,,\,\Gamma^{(N)}_{j\nu}\right]\,. \tag{8}$$

The total spin tensor (7) is the generator of Lorentz transformations for $\psi^{(N)}(x)$.

In place of the chiral representations for individual $\gamma^{(N)}_j = \left(\gamma^{(N)}_{j\mu}\right)$, where

$$\gamma^{(N)}_{j5} \equiv i\gamma^{(N)}_{j0}\gamma^{(N)}_{j1}\gamma^{(N)}_{j2}\gamma^{(N)}_{j3}\,,\;\; \sigma^{(N)}_{j3} \equiv \sigma^{(N)}_{j12} \tag{9}$$

are diagonal, it is convenient to use for any N the chiral representations of Jacobi $\Gamma_j^{(N)} = \left(\Gamma_{j\mu}^{(N)}\right)$, where now

$$\Gamma_{j5}^{(N)} \equiv i\Gamma_{j0}^{(N)}\Gamma_{j1}^{(N)}\Gamma_{j2}^{(N)}\Gamma_{j3}^{(N)}\,, \quad \Sigma_{j3}^{(N)} \equiv \Sigma_{j12}^{(N)} \tag{10}$$

are diagonal (all matrices (9) and similarly (10) commute simultaneously, both with equal and different j).

When using the Jacobi chiral representations, the "centre-of-mass" Dirac-type matrices $\Gamma_{1\mu}^{(N)} \equiv \Gamma_\mu^{(N)}$ and $\Gamma_{15}^{(N)} \equiv \Gamma_5^{(N)} \equiv i\Gamma_0^{(N)}\Gamma_1^{(N)}\Gamma_2^{(N)}\Gamma_3^{(N)}$ can be taken in the reduced forms

$$\Gamma_\mu^{(N)} = \gamma_\mu \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{N-1 \text{ times}}\,, \quad \Gamma_5^{(N)} = \gamma_5 \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{N-1 \text{ times}}\,, \tag{11}$$

where γ_μ, $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3$ and $\mathbf{1}$ are the usual 4×4 Dirac matrices.

Then, the Dirac-type equation (4) for any N can be rewritten in the reduced form

$$\left(\gamma \cdot p - M^{(N)}\right)_{\alpha_1\beta_1} \psi^{(N)}_{\beta_1\alpha_2\ldots\alpha_N}(x) = 0\,, \tag{12}$$

where α_1 and $\alpha_2\,, \ldots\,, \alpha_N$ are the "centre-of-mass" and "relative" Dirac bispinor indices, respectively ($\alpha_i = 1,2,3,4$ for any $i = 1,2,\ldots,N$). Note that in the Dirac-type equation (12) for any $N > 1$ there appear the "relative" Dirac indices $\alpha_2\,, \ldots\,, \alpha_N$ which are free from any coupling, but still are subjects of Lorentz transformations.

The Standard Model gauge interactions can be introduced to the Dirac-type equations (12) by means of the minimal substitution $p \to p - gA(x)$, where p plays the role of the "centre-of-mass" four-momentum, and so, x — the "centre-of-mass" four-position. Then,

$$\left\{\gamma \cdot [p - gA(x)] - M^{(N)}\right\}_{\alpha_1\beta_1} \psi^{(N)}_{\beta_1\alpha_2\ldots\alpha_N}(x) = 0\,, \tag{13}$$

where $g\gamma \cdot A(x)$ symbolises the Standard Model gauge coupling that involves within $A(x)$ the familiar weak-isospin and colour matrices, the weak-hypercharge dependence as well as the usual Dirac chiral matrix γ_5. The last arises from the "centre-of-mass" Dirac-type chiral matrix $\Gamma_5^{(N)}$, when a generic $g\Gamma^{(N)} \cdot A(x)$ is reduced to $g\gamma \cdot A(x)$ in Eqs. (13) [see Eq. (11)]. Note that then $A_\mu(x) \equiv A_\mu(x,\gamma_5) \equiv A_\mu(x,0) + A'_\mu(x,0)\gamma_5$ depends linearly on γ_5.

In Eqs. (13) the Standard Model gauge fields interact only with the "centre-of-mass" index α_1 that, therefore, is distinguished from the physically unobserved "relative" indices $\alpha_2, \ldots, \alpha_N$. This was the reason, why some time ago we conjectured that the "relative" Dirac bispinor indices

$\alpha_2, \ldots, \alpha_N$ are all indistinguishable physical objects obeying Fermi statistics along with the Pauli principle requiring the full antisymmetry of wave function $\psi^{(N)}_{\alpha_1 \alpha_2 \ldots \alpha_N}(x)$ with respect to $\alpha_2, \ldots, \alpha_N$ [2]. Hence, due to this "intrinsic Pauli principle", only five values of N satisfying the condition $N - 1 \leq 4$ are allowed, namely $N = 1, 3, 5$ for N odd and $N = 2, 4$ for N even. Then, from the postulate of relativity and the probabilistic interpretation of $\psi^{(N)}(x) \equiv \left(\psi^{(N)}_{\alpha_1 \alpha_2 \ldots \alpha_N}(x)\right)$ we were able to infer that these N odd and N even correspond to states with total spin 1/2 and total spin 0, respectively [2].

Thus, the Dirac-type equation (13), jointly with the "intrinsic Pauli principle", if considered on a fundamental level, justifies the existence in Nature of *three and only three* generations of spin-1/2 fundamental fermions coupled to the Standard Model gauge bosons (they are identified with leptons and quarks). In addition, there should exist *two and only two* generations of spin-0 fundamental bosons also coupled to the Standard Model gauge bosons (they are not identified yet). Note that one cannot hope here for a construction of the full supersymmetry. At most, there might appear a partial supersymmetry: two to two, broken by the absence of one boson generation (the question is of which).

The wave functions or fields of spin-1/2 fundamental fermions (leptons and quarks) of three generations $N = 1, 3, 5$ can be presented in terms of $\psi^{(N)}_{\alpha_1 \alpha_2 \ldots \alpha_N}(x)$ as follows:

$$\begin{aligned}
\psi^{(f_1)}_{\alpha_1}(x) &= \psi^{(1)}_{\alpha_1}(x)\,, \\
\psi^{(f_3)}_{\alpha_1}(x) &= \tfrac{1}{4}\left(C^{-1}\gamma_5\right)_{\alpha_2 \alpha_3} \psi^{(3)}_{\alpha_1 \alpha_2 \alpha_3}(x) = \psi^{(3)}_{\alpha_1 12}(x) = \psi^{(3)}_{\alpha_1 34}(x)\,, \\
\psi^{(f_5)}_{\alpha_1}(x) &= \tfrac{1}{24}\varepsilon_{\alpha_2 \alpha_3 \alpha_4 \alpha_5} \psi^{(5)}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5}(x) = \psi^{(5)}_{\alpha_1 1234}(x)\,,
\end{aligned} \tag{14}$$

where $\psi^{(N)}_{\alpha_1 \alpha_2 \ldots \alpha_N}(x)$ carries also the Standard Model (composite) label, suppressed in our notation, and C denotes the usual 4×4 charge-conjugation matrix. Here, writing explicitly, $f_1 = \nu_e\,, e^-\,, u\,, d\,,$ $f_3 = \nu_\mu\,, \mu^-\,, c\,, s$ and $f_5 = \nu_\tau\,, \tau^-\,, t\,, b$, thus each f_N corresponds to the same suppressed Standard Model (composite) label. We can see that, due to the full antisymmetry in α_i indices for $i \geq 2$, the wave functions or fields $N = 1, 3$ and 5 appear (up to the sign) with the multiplicities 1, 4 and 24, respectively. Thus, for them, there is defined the weighting matrix

$$\rho^{1/2} = \frac{1}{\sqrt{29}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{24} \end{pmatrix}, \tag{15}$$

where $\mathrm{Tr}\,\rho = 1$.

For each bispinor wave function or field $\psi^{(f_N)}_{\alpha_1}(x)$ $(N=1,3,5)$ defined in Eqs. (14), the Dirac-type equation (13) can be reduced to the usual Dirac equation

$$\left\{\gamma\cdot[p-gA(x)]-M^{(N)}\right\}_{\alpha_1\beta_1}\psi^{(f_N)}_{\beta_1}(x)=0\,. \tag{16}$$

This gives in turn the relativistic covariant conserved current of the usual Dirac form

$$j^{(f_N)}_{\mu\mathrm{D}}(x)\equiv\psi^{(f_N)*}_{\alpha_1}(x)\,(\gamma_0\gamma_\mu)_{\alpha_1\beta_1}\,\psi^{(f_N)}_{\beta_1}(x)\,. \tag{17}$$

In fact, $\partial^\mu j^{(f_N)}_{\mu\mathrm{D}}(x)=0$ since $A^\dagger_\mu(x)=A_\mu(x)$ where $\gamma^\dagger_5=\gamma_5$.

Concluding the first part of this note, we would like to point out that our algebraic construction of *three and only three* generations of leptons and quarks may be interpreted *either* as ingenuously algebraic (much like the famous Dirac's algebraic discovery of spin 1/2), *or* as a summit of an iceberg of really composite states of N spatial partons with spin 1/2 whose Dirac bispinor indices manifest themselves as our Dirac bispinor indices $\alpha_1,\alpha_2,\ldots,\alpha_N$ $(N=1,3,5)$ which thus may be called "algebraic partons", as being algebraic building blocks for leptons and quarks. Among all N "algebraic partons" in any generation N of leptons and quarks, there are one "centre-of-mass algebraic parton" (α_1) and $N-1$ "relative algebraic partons" $(\alpha_2,\ldots,\alpha_N)$, the latter undistinguishable from each other and so, obeying our "intrinsic Pauli principle".

Now, we pass to some more formal discussion. It is not difficult to see that both for N odd and N even the Dirac-type equation (13) implies the local conservation of the following relativistic covariant structure:

$$j^{(N)}_{\mu\alpha_2\ldots\alpha_N,\beta_2\ldots\beta_N}(x)\equiv\psi^{(N)*}_{\alpha_1\alpha_2\ldots\alpha_N}(x)\,(\gamma_0\gamma_\mu)_{\alpha_1\beta_1}\,\psi^{(N)}_{\beta_1\beta_2\ldots\beta_N}(x)\,. \tag{18}$$

In fact, $\partial^\mu j_{\mu\alpha_2\ldots\alpha_N,\beta_2\ldots\beta_N}(x)=0$ because $A^\dagger_\mu(x)=A_\mu(x)$. The local conservation of the currents (17) for $N=1,3,5$ follows immediately from Eq. (18), since

$$\begin{aligned}
j^{(f_1)}_{\mu\mathrm{D}}(x) &= j^{(1)}_\mu(x)\,,\\
j^{(f_3)}_{\mu\mathrm{D}}(x) &= \tfrac{1}{4}\left(C^{-1}\gamma_5\right)^*_{\alpha_2\alpha_3} j^{(3)}_{\mu\alpha_2\alpha_3,\beta_2\beta_3}(x)\tfrac{1}{4}\left(C^{-1}\gamma_5\right)_{\beta_2\beta_3}\,,\\
j^{(f_5)}_{\mu\mathrm{D}}(x) &= \tfrac{1}{24}\varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5} j^{(5)}_{\mu\alpha_2\alpha_3\alpha_4\alpha_5,\beta_2\beta_3\beta_4\beta_5}(x)\tfrac{1}{24}\varepsilon_{\beta_2\beta_3\beta_4\beta_5}\,.
\end{aligned} \tag{19}$$

In general, the relativistic covariant Dirac-type currents both for N odd and N even must have the form

$$\begin{aligned}
j^{(N)}_{\mu\mathrm{D}}(x) &\equiv \psi^{(N)*}_{\alpha_1\alpha_2\ldots\alpha_N}(x)\xi^{(N)}\\
&\quad\times\left(\Gamma^{(N)}_{10}\Gamma^{(N)}_{20}\ldots\Gamma^{(N)}_{N0}\Gamma^{(N)}_{1\mu}\right)_{\alpha_1\alpha_2\ldots\alpha_N,\beta_1\beta_2\ldots\beta_N}\psi^{(N)}_{\beta_1\beta_2\ldots\beta_N}(x)\,,
\end{aligned} \tag{20}$$

where $\Gamma^{(N)}_{i\mu}$ are the Dirac-type matrices in their Jacobi version, introduced in Eqs. (6), while $\xi^{(N)}$ is a phase factor making Hermitian the $N \times N$ bispinor matrix appearing in this current. For N even this definition is trivial, as then the Dirac-type current vanishes. In the case of N odd, we are going to show that

$$j^{(N)}_{\mu \mathrm{D}}(x) = j^{(N)}_{\mu\alpha_2\ldots\alpha_N,\beta_2\ldots\beta_N}(x)(\gamma_0)_{\alpha_2\beta_2}\ldots(\gamma_0)_{\alpha_N\beta_N}\,. \tag{21}$$

Thus, $\partial^\mu j^{(N)}_{\mu \mathrm{D}}(x) = 0$ for N odd.

To prove Eq. (21), we observe that the Dirac-type matrices $\Gamma^{(N)}_{i\mu}$ $(i = 1, 2, \ldots, N)$, satisfying the anticommutation relations of Clifford algebra (5), can be represented in terms of the usual 4×4 Dirac matrices as follows:

$$\begin{aligned}
\Gamma^{(N)}_{1\mu} &= \gamma_\mu \otimes \underbrace{\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \mathbf{1}}_{N-1 \text{ times}}\,, \\
\Gamma^{(N)}_{2\mu} &= \gamma_5 \otimes i\gamma_\mu\gamma_5 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \mathbf{1}\,, \\
\Gamma^{(N)}_{3\mu} &= \gamma_5 \otimes \gamma_5 \otimes \gamma_\mu \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \mathbf{1}\,, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\Gamma^{(N)}_{N\mu} &= \underbrace{\gamma_5 \otimes \gamma_5 \otimes \gamma_5 \otimes \gamma_5 \otimes \ldots \otimes \gamma_5}_{N-1 \text{ times}} \otimes \begin{cases} \gamma_\mu & \text{for } N \text{ odd} \\ i\gamma_\mu\gamma_5 & \text{for } N \text{ even}\,, \end{cases}
\end{aligned} \tag{22}$$

what is an extension of the representation (11) for $\Gamma^{(N)}_{1\mu} \equiv \Gamma^{(N)}_{\mu}$ leading to the form (13) of Dirac-type equation for any N. Forming their product for $\mu = 0$,

$$\Gamma^{(N)}_{10}\Gamma^{(N)}_{20}\ldots\Gamma^{(N)}_{N0} = \begin{cases} i^{\frac{N-1}{2}}\;\gamma_0 \otimes \gamma_0 \otimes \ldots \otimes \gamma_0 & \text{for } N \text{ odd} \\ (-i)^{\frac{N}{2}} i\gamma_0\gamma_5 \otimes i\gamma_0\gamma_5 \otimes \ldots \otimes i\gamma_0\gamma_5 & \text{for } N \text{ even}\,, \end{cases} \tag{23}$$

and multiplying from the right by $\Gamma^{(N)}_{1\mu}$, we obtain

$$\Gamma^{(N)}_{10}\Gamma^{(N)}_{20}\ldots\Gamma^{(N)}_{N0}\Gamma^{(N)}_{1\mu} = \begin{cases} i^{\frac{N-1}{2}}\;\gamma_0\,\gamma_\mu \otimes \gamma_0 \otimes \ldots \otimes \gamma_0 & \text{for } N \text{ odd} \\ (-i)^{\frac{N-2}{2}} \gamma_0\gamma_5\gamma_\mu \otimes i\gamma_0\gamma_5 \otimes \ldots \otimes i\gamma_0\gamma_5 & \text{for } N \text{ even.} \end{cases} \tag{24}$$

Hence, we can define the phase factors in Eq. (20) as follows:

$$\xi^{(N)} = \begin{cases} (-i)^{\frac{N-1}{2}} & \text{for } N \text{ odd} \\ (-i)^{\frac{N-2}{2}} & \text{for } N \text{ even.} \end{cases} \tag{25}$$

Thus, in the case of N odd we can represent the Dirac-type current (20) in the form

$$j^{(N)}_{\mu \mathrm{D}}(x) = \psi^{(N)*}_{\alpha_1 \alpha_2 \ldots \alpha_N}(x) (\gamma_0 \gamma_\mu)_{\alpha_1 \beta_1} (\gamma_0)_{\alpha_2 \beta_2} \ldots (\gamma_0)_{\alpha_N \beta_N} \psi^{(N)}_{\beta_1 \beta_2 \ldots \beta_N}(x) \,. \tag{26}$$

From Eqs. (18) and (26) the relationship (21) follows.

For N odd, the Dirac-type current (20) or (26) is locally conserved, but such is also the relativistic noncovariant structure $j^{(N)}_{\mu \alpha_2 \ldots \alpha_N, \alpha_2 \ldots \alpha_N}(x)$ calculated from the definition (18) by summing over $\alpha_2 = \beta_2, \ldots, \alpha_N = \beta_N$. It follows that the Hermitian, not explicitly covariant $N \times N$ bispinor matrix

$$\xi^{(N)} \Gamma^{(N)}_{20} \ldots \Gamma^{(N)}_{N0} = \mathbf{1} \otimes \underbrace{\gamma_0 \otimes \gamma_0 \otimes \ldots \otimes \gamma_0}_{N-1 \text{ times}} = \left(\delta_{\alpha_1 \beta_1} (\gamma_0)_{\alpha_2 \beta_2} \ldots (\gamma_0)_{\alpha_N \beta_N}\right) \tag{27}$$

is a constant of motion. This matrix may be called the total “relative” internal parity. Imposing on $\psi^{(N)}(x) \equiv \left(\psi^{(N)}_{\alpha_1 \alpha_2 \ldots \alpha_N}(x)\right)$ the stationary constraint in the form of the eigenvalue equation

$$\xi^{(N)} \Gamma^{(N)}_{20} \ldots \Gamma^{(N)}_{N0} \psi^{(N)}(x) = \psi^{(N)}(x) \,, \tag{28}$$

requiring that the eigenvalue of total “relative” internal parity must be always equal to $+1$, we simplify the relativistic covariant Dirac-type current (20) or (26) to the form

$$\begin{aligned} j^{(N)}_{\mu \mathrm{D}} &= \psi^{(N)*}_{\alpha_1 \alpha_2 \ldots \alpha_N}(x) \left(\Gamma^{(N)}_{10} \Gamma^{(N)}_{1\mu}\right)_{\alpha_1 \alpha_2 \ldots \alpha_N, \beta_1 \beta_2 \ldots \beta_N} \psi^{(N)}_{\beta_1 \beta_2 \ldots \beta_N}(x) \\ &= \psi^{(N)*}_{\alpha_1 \alpha_2 \ldots \alpha_N}(x) \left(\gamma_0 \gamma_\mu\right)_{\alpha_1 \beta_1} \psi^{(N)}_{\beta_1 \alpha_2 \ldots \alpha_N}(x) \end{aligned} \tag{29}$$

that is not explicitly covariant in the world of “relative” Dirac degrees of freedom. The form (29) leads to the positive-definiteness of $\psi^{(N)}(x)$,

$$j^{(N)}_{0 \mathrm{D}}(x) = \psi^{(N)*}_{\alpha_1 \alpha_2 \ldots \alpha_N}(x) \psi^{(N)}_{\alpha_1 \alpha_2 \ldots \alpha_N}(x) > 0 \,, \tag{30}$$

which is a natural requirement for $\psi^{(N)}(x)$.

It is not difficult to demonstrate that wave functions or fields of spin-1/2 fundamental fermions, $\psi^{(f_N)}_{\alpha_1}(x)$ $(N = 1, 3, 5)$ defined in Eqs. (14), satisfy the constraint (28). In fact, writing

$$\psi^{(3)}_{\alpha_1\alpha_2\alpha_3}(x) = (\gamma_5 C)_{\alpha_3\alpha_2}\psi^{(f_3)}_{\alpha_1}(x)\,, \quad \psi^{(5)}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}(x) = \varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5}\psi^{(f_5)}_{\alpha_1}(x)\,, \tag{31}$$

we check that

$$\delta_{\alpha_1\beta_1}(\gamma_0)_{\alpha_2\beta_2}(\gamma_0)_{\alpha_3\beta_3}\psi^{(3)}_{\beta_1\beta_2\beta_3}(x) = \psi^{(3)}_{\alpha_1\alpha_2\alpha_3}(x) \tag{32}$$

and

$$\delta_{\alpha_1\beta_1}(\gamma_0)_{\alpha_2\beta_2}(\gamma_0)_{\alpha_3\beta_3}(\gamma_0)_{\alpha_4\beta_4}(\gamma_0)_{\alpha_5\beta_5}\psi^{(5)}_{\beta_1\beta_2\beta_3\beta_4\beta_5}(x) = \psi^{(5)}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}(x)\,, \tag{33}$$

where in the chiral representation

$$\gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \; C = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$
$$\gamma_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{34}$$

Here, $(\gamma_5 C)^T = -\gamma_5 C$.

In conclusion, we would like to emphasise that the phenomenon of existence in Nature of three generations of fundamental fermions (leptons and quarks) can be understood in a satisfactory way on the base of two postulates:

(*i*) For all fundamental particles of matter the Dirac square-root procedure $\sqrt{p^2} \to \Gamma^{(N)} \cdot p$ works, leading in the interaction-free case to the sequence $N = 1, 2, 3, \ldots$ of Dirac-type equations (4), satisfied by the sequence $N = 1, 2, 3, \ldots$ of wave functions or fields $\psi^{(N)}(x) \equiv \left(\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N}(x)\right)$, where α_1 is a "centre-of-mass" Dirac bispinor index and $\alpha_2, \ldots, \alpha_N$ are "relative" Dirac bispinor indices.

(*ii*) The "centre-of-mass" Dirac bispinor index α_1 is coupled to the Standard Model gauge fields through the term $g\gamma_{\alpha_1\beta_1} \cdot A(x)$ in the Dirac-type equation (13), while the "relative" Dirac bispinor indices $\alpha_2, \ldots,$

α_N are all free indistinguishable physical objects obeying Fermi statistics along with the Pauli principle (called then "intrinsic Pauli principle") which requires the full antisymmetry of $\psi^{(N)}_{\alpha_1\alpha_2\ldots\alpha_N}(x)$ with respect to $\alpha_2,\ldots,\alpha_N$.

This antisymmetry allows only for *three* $N = 1,3,5$ in the case of N odd, and for *two* $N = 2,4$ in the case of N even. Hence, unavoidably, there follow *three and only three* generations of fundamental fermions (leptons and quarks) and *two and only two* generations of some fundamental bosons (not recognised yet). The former carry spin $1/2$ and the latter spin 0 (as was argued in Ref. [2]). All possess a conventional Standard Model signature, suppressed in our notation.

The Dirac-type matrices $\Gamma^{(N)}_\mu$, appearing in the square-root procedure $\sqrt{p^2} \to \Gamma^{(N)}\cdot p$, are here constructed by means of the Clifford algebra (3) of matrices $\gamma^{(N)}_{i\mu}$, and then embedded into of the Clifford algebra (5) of matrices $\Gamma^{(N)}_{i\mu}$ *via* a Jacobi definition (6), where $\Gamma^{(N)}_{1\mu} \equiv \Gamma^{(N)}_\mu$ and $\Gamma^{(N)}_{2\mu},\ldots,\Gamma^{(N)}_{N\mu}$ play the role of "centre-of-mass" and "relative" Dirac-type matrices, respectively, defining one "centre-of-mass" and $N-1$ "relative" Dirac bispinor indices (α_1 and $\alpha_2,\ldots,\alpha_N$).

Finally, a few words about a possible shape of the Dirac 3×3 mass matrices $M^{(f)} = \left(M^{(f)}_{N\,N'}\right)$ $(N,N' = 1,3,5)$ for four sorts of spin-$1/2$ fundamental fermions $f = \nu\,,\, e\,,\, u\,,\, d$ (leptons and quarks). Their three generations $f_N = \nu_N\,,\, e_N\,,\, u_N\,,\, d_N$ $(N = 1,3,5)$ are described in our formalism by the wave functions or fields (14) appearing with the weighting matrix (15). Some time ago, we introduced a simple specific ansatz

$$M^{(f)} = \rho^{1/2} h^{(f)} \rho^{1/2}\,, \tag{35}$$

where $\rho^{1/2}$ is the weighting matrix given in Eq. (15) and

$$h^{(f)} = \mu^{(f)}\left[N^2 - (1-\varepsilon^{(f)})N^{-2}\right] + \alpha^{(f)}(a + a^\dagger) \tag{36}$$

with $\mu^{(f)} > 0$ and $\varepsilon^{(f)} > 0$ being parameters. Here, the matrix

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} = 1 + 2n \tag{37}$$

describes the number of all α_i indices with $i = 1,2,\ldots,N$ (all "algebraic partons") present in any of three fermion generations $N = 1,3,5$, while

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}\,,\; a^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{38}$$

play the role of "truncated" annihilation and creation matrices for pairs of "relative" indices $\alpha_i\alpha_j$ with $(i,j) = (2,3),\ldots,(N-1,N)$ (pairs of "relative algebraic partons"):

$$[a\,,\,n] = a\,,\; [a^\dagger\,,\,n] = -a^\dagger\,,\; n = a^\dagger a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\,, \tag{39}$$

where the "truncation" condition $a^3 = 0 = a^{\dagger\,3}$ is satisfied (implying that $N = 1,3,5$ or $n = 0,1,2$ only). The formulae (35) and (36) lead to the specific form of Dirac mass matrices:

$$M^{(f)} = \frac{1}{29}\begin{pmatrix} \mu^{(f)}\varepsilon^{(f)} & 2\alpha^{(f)} & 0 \\ 2\alpha^{(f)} & 4\mu^{(f)}(80+\varepsilon^{(f)})/9 & 8\sqrt{3}\,\alpha^{(f)} \\ 0 & 8\sqrt{3}\,\alpha^{(f)} & 24\mu^{(f)}(624+\varepsilon^{(f)})/25 \end{pmatrix}. \tag{40}$$

In particular for charged leptons ($f = e$), if in this case $\alpha^{(e)}$ can be neglected, the mass matrix (40) gives the following masses of electron, muon and tauon (corresponding to $N = 1,3$ and 5, respectively):

$$m_e = \frac{\mu^{(e)}}{29}\varepsilon^{(e)}\,,\; m_\mu = \frac{\mu^{(e)}}{29}\frac{4}{9}\left(80+\varepsilon^{(e)}\right)\,,\; m_\tau = \frac{\mu^{(e)}}{29}\frac{24}{25}\left(624+\varepsilon^{(e)}\right)\,. \tag{41}$$

Taking as an input the experimental values of m_e and m_μ, one determines from Eqs. (41) that [2]

$$\mu^{(e)} = 85.9924 \text{ MeV}\,,\; \varepsilon^{(e)} = 0.172329\,,\; m_\tau = 1776.80 \text{ MeV}\,, \tag{42}$$

the last prediction being really close to the experimental value $m_\tau^{\rm exp} = 1777.03^{+0.30}_{-0.26}$ MeV [5]. Correcting Eqs. (41) by means of the lowest perturbation with respect to $\alpha^{(e)}$, one gets from the mass matrix $M^{(e)}$ as given in Eq. (40) the experimental value $m_\tau^{\rm exp}$ by putting $\left(\alpha^{(e)}/\mu^{(e)}\right)^2 = 0.023^{+0.029}_{-0.025}$ (not inconsistently with zero).

The above impressive prediction for m_τ from the experimental m_e and m_μ seems to justify some speculations about the physical origin of the ansatz (36). In the kernel (36) of the Dirac mass matrix (35), the first term $\mu^{(f)}N^2$ may be intuitively interpreted as coming from an interaction of all N "algebraic partons" treated on equal footing, while the second term $-\mu^{(f)}(1-\varepsilon^{(f)})N^{-2}$ may be considered as being a correcting term caused by the fact that there is one "centre-of-mass algebraic parton" distinguished (due to its external coupling to the Standard Model gauge fields) among all N "algebraic partons" of which $N-1$ are "relative algebraic partons", indistinguishable from each other. This distinguished "algebraic parton" appears, therefore, with the probability $[N!/(N-1)!]^{-1} = N^{-1}$ that, when squared, leads to the additional term $\mu^{(f)}(1-\varepsilon^{(f)})N^{-2}$ [with an, in principle, independent coefficient $\mu^{(f)}(1-\varepsilon^{(f)})$] which should be subtracted in the kernel (36) from the former term in order to obtain a small mass matrix element $M_{11}^{(f)} = \mu^{(f)}\varepsilon^{(f)}/29$, tending to the zeroth lowest mass m_{f_1} if $\alpha^{(f)} \to 0$ and $\varepsilon^{(f)} \to 0$. Eventually, the third term $\alpha^{(f)}(a+a^\dagger)$ in the kernel (36) annihilates and creates pairs of "relative algebraic partons" and so, is responsible in a natural way for mixing of three fermion generations in the Dirac mass matrix $M^{(f)}$.

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