ON THE HIERARCHY OF NEUTRINO MASSES*

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(Received July 1, 2002)

Dedicated to Stefan Pokorski on his 60th birthday

We present a model of neutrino masses combining the seesaw mechanism and strong Dirac mass hierarchy and at the same time exhibiting a significantly reduced hierarchy at the level of active neutrino masses. The heavy Majorana masses are assumed to be degenerate. The suppression of the hierarchy is due to a symmetric and unitary operator R whose role is discussed. The model gives realistic mixing and mass spectrum. The mixing of atmospheric neutrinos is attributed to the charged lepton sector whereas the mixing of solar neutrinos is due to the neutrino sector. Small U_{e3} is a consequence of the model. The masses of the active neutrinos are given by $\mu_3 \approx \sqrt{\Delta m^2_{@}}$ and $\mu_1/\mu_2 \approx \tan^2\theta_{\odot}$.

PACS numbers: 14.60.Pq, 12.15.Ff, 14.60.St

1. Introduction

In this talk the results on neutrino masses and mixings are presented which were obtained in our recent publications [1, 2].

In view of the recent results from SNO [3] and SuperKamiokande [4] and owing to developments in theory [5], see also [6, 7] and references therein, there has emerged a unique solution to the problem of neutrino oscillations. The only allowed solution is now LMA MSW, all others being excluded at

* Presented at *Planck 2002*, the Fifth European Meeting, *From the Planck Scale to the Electroweak Scale* "Supersymmetry and Brane Worlds", Kazimierz, Poland, May 25–29, 2002. Special session dedicated to S. Pokorski on the occasion of his 60-th birthday.

the 3σ level [8]. Thus we know the pattern of the oscillations of the active neutrinos, that is those observed in experiment. Simultaneously it becomes more and more clear that the oscillations of atmospheric neutrinos are due to $\nu_\mu \rightleftarrows \nu_\tau$ transitions [9, 10]. The third important piece of information is the CHOOZ limit [11] indicating that the element U_{e3} of the Maki–Nakagawa–Sakata (MNS) lepton mixing matrix [12] is small.

The experimental data mentioned above can be described by a model based on the seesaw mechanism [13] and a large hierarchy of the Dirac masses of neutrinos. Such a model was considered in [1,2]. We present it here including some technical details related to the derivation of the formulae.

The main idea of the present model is that a large hierarchy of the Dirac masses of neutrinos is possible even though the hierarchy of masses indicated by experimental data is far less pronounced than that expected from comparison with the quark or charged lepton mass spectra. This nontrivial fact is interesting since it hints at a possible similarity between the observed hierarchy of quark and charged lepton masses and that of neutrino Dirac masses. The disappearance of this hierarchy at the level of the observable masses of the active neutrinos is caused by the seesaw mechanism as well as by the algebraic structure of the low energy effective mass operator $\mathcal{N}$ describing the masses of the active neutrinos. The latter is due to a symmetric and unitary operator R acting in the flavor space and related to the unitary transformations of the right-handed neutrinos. This operator has for the first time been considered in [1]. It has been pointed out in [1, 2] that R plays a crucial role in the low energy physics of neutrinos. In fact, it affects the form of the $U_{\rm MNS}$ mixing matrix. In the model we consider, we find a form of the operator R leading to a reduction of the underlying Dirac mass hierarchy and thus producing realistic mass spectra. The resulting mixing matrix is naturally exhibiting a small value of the U_{e3} element. It also follows from our model that the mass ratio of the two lighter neutrinos is given by $\tan^2\theta_\odot$, $\theta_\odot$ denoting the solar mixing angle.

2. Mass hierarchy

Our aim is to explain the observed mass spectrum of neutrinos starting from a hierarchy of Dirac masses comparable with that of the corresponding up quark masses. Let us begin with a look at data and at the expectations from the simplest version of the seesaw mechanism. What is known are the squared mass differences affecting the oscillation pattern of neutrinos. Denote the masses of the active neutrinos by μ_1, μ_2 and μ_3. Then we can define the ratio

$$\rho = \frac{\Delta m^2_\odot}{\Delta m^2_@} = \frac{\mu_2^2 - \mu_1^2}{\mu_3^2 - \mu_2^2} \tag{1}$$

of the solar to atmospheric mass splitting. The experimental value of the parameter ρ is

$$\rho_{\rm exp} \approx \frac{5\times 10^{-5}{\rm eV}^2}{2.5\times 10^{-3}{\rm eV}^2} = 2\times 10^{-2}\,. \tag{2}$$

Although this might well be called a hierarchy, we must compare it to the predictions offered by the seesaw mechanism. Choose a reference frame where the heavy Majorana mass matrix $M_{\rm R}$ is diagonal and assume furthermore that it is proportional to the unit matrix. Of course, the hierarchy of the active neutrino masses can be reduced by assuming a hierarchy in $M_{\rm R}$, partly compensating for the hierarchy originating from the Dirac masses, see *e.g.* [14, 15]. However, we will not have to abandon the simple assumption of degenerate right-handed Majorana masses to destroy the hierarchy. So, we do not consider the more general case, although it is not hard to do so. Thus,

$$M_{\rm R} = M\cdot \mathbf{1}\,. \tag{3}$$

At the same time, the Dirac mass matrix for neutrinos is written as

$$N = U_{\rm R} m^{(\nu)} U_{\rm L} \tag{4}$$

with

$$m^{(\nu)} = {\rm diag}(m_1, m_2, m_3)\,. \tag{5}$$

Then the mass spectrum of the active neutrinos is given by the effective operator $\mathcal{N}$ of dimension five

$$\mathcal{N} = N^{\rm T} M_{\rm R}^{-1} N = U_{\rm L}^{\rm T} m^{(\nu){\rm T}} U_{\rm R}^{\rm T} M_{\rm R}^{-1} U_{\rm R} m^{(\nu)} U_{\rm L}\,\,. \tag{6}$$

With the simplifying assumption (3), the mass spectrum obtained from the matrix $\mathcal{N}$ in Eq. (6) is seen to depend crucially on the following matrix R,

$$R = U_{\rm R}^{\rm T} U_{\rm R}\,, \tag{7}$$

which is symmetric and unitary. The matrix $U_{\rm R}$ satisfying the equation above for our final choice of R, see Eq. (10), can be found as described in Appendix A. The predictions of the simplest seesaw model correspond to assuming that $R = 1$. Then the resulting spectrum of the active neutrino masses is

$$\mu_1 = \frac{m_1^2}{M} \ll \mu_2 = \frac{m_2^2}{M} \ll \mu_3 = \frac{m_3^2}{M}\,. \tag{8}$$

Since we also require a hierarchy for the Dirac masses

$$m_1 \ll m_2 \ll m_3 \tag{9}$$

it becomes evident that the active neutrino hierarchy is even stronger. The ratio ρ can now be estimated by letting the mass ratio m_3/m_2 be of the order of the corresponding mass ratios for other fundamental fermions, *i.e.* $m_b/m_s \sim 30$, $m_\tau/m_\mu \approx 17$ or $m_t/m_c \sim 100$. We would obtain a quantity of the order of 10^{-8}–10^{-4}, which is much less than the observed hierarchy. So, if we are to succeed in describing reality with a seesaw model, we must find some way of hiding this huge hierarchy.

Now we see that the operator R cannot be a unit matrix. In fact, one can easily convince oneself that its element $(R)_{33}$ must vanish in order to prevent the Dirac mass hierarchy from showing up in the observable mass ratio.

Now consider what happens if the element $(R)_{23} = (R)_{32}$ is non-vanishing. It turns out that the resulting mass spectrum for the active neutrinos is acceptable from the phenomenological point of view if $(R)_{23} = \mathcal{O}(1)$ is assumed. This spectrum corresponds to the case of the so-called inverted hierarchy. However, the resulting structure of the lepton mixing matrix does not resemble the experimentally observed one [2].

The only remaining case is $R_{33} = R_{23} = 0$ which implies

$$R = \begin{pmatrix} 0 & 0 & \exp i\phi_1 \\ 0 & \exp i\phi_2 & 0 \\ \exp i\phi_1 & 0 & 0 \end{pmatrix} . \tag{10}$$

The complex phase factors in Eq. (10) can be of crucial importance for lepton number violating processes like neutrino-less double beta decays. However, these phase factors do not affect our discussion which concentrates on neutrino oscillations. So, for the sake of simplicity, in the following considerations we take the same form of R as in [1]:

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \equiv P_{13} . \tag{11}$$

It turns out that for strongly hierarchical Dirac masses Eq. (11) is a necessary condition for a realistic mixing and mass spectrum. Therefore we assume that some symmetry underlying flavor dynamics forces U_{R} to fulfill Eq. (11). The matrix R can drastically reduce the hierarchy of the mass spectrum for the active neutrinos. So, R is observable, in principle at least, if a large hierarchy of the Dirac masses is a common feature of all quarks and leptons. In this sense R is a physical object which is imprinted in low energy physical quantities, namely the masses of the active neutrinos. Unlike the quark sector with its Cabibbo–Kobayashi–Maskawa mixing matrix [16] the lepton sector has therefore two important matrices in the flavor space. One is the lepton mixing matrix U_{MNS} [12] which affects the form of the weak charged current. Another is the matrix R defined in Eq. (7). R affects the form of U_{MNS}. Moreover, it is also reflected in the low energy neutrino mass spectrum. In our phenomenological approach we use the experimental input to fix the form of R. One may hope that this is a first step towards an underlying theory of flavor.

3. Lepton mixing matrix

In the previous Section we have arrived at a way of resolving the problem of strong hierarchy of active neutrino masses. However, we must show that the model describes correctly the mixing pattern. We now study the mixing matrix U_{MNS}. In our model, the mixing is due to both charged leptons and neutrinos. The mass matrix for the charged leptons can be written as

$$L = V_{\mathrm{R}}\, \mathrm{diag}(m_e, m_\mu, m_\tau)\, V_{\mathrm{L}} \equiv V_{\mathrm{R}}\, m^{(l)}\, V_{\mathrm{L}}\,. \tag{12}$$

The matrix V_{R} multiplying $m^{(l)}$ from the left side can be made equal to one by an appropriate redefinition of the right-handed charged leptons. This has no observable consequences because at our low energies only left-handed weak charged currents can be studied. The corresponding Dirac mass matrix for the neutrinos is given in Eq. (4). Let $\mathcal{O}$ be a unitary matrix such that

$$\mathcal{O}^{\mathrm{T}} \mathcal{N} \mathcal{O} = \mathrm{diag}(\mu_1, \mu_2, \mu_3)\,. \tag{13}$$

Note that in Eq. (13) the diagonalization is done by multiplying by the transposed matrix $\mathcal{O}^{\mathrm{T}}$, rather than the Hermitian conjugate $\mathcal{O}^\dagger$, from the left. We explain in Appendix B how to perform such a diagonalization. Eq. (12) implies that $M_{\mathrm{L}}^2 = L^\dagger L$ is diagonalized by V_{L}, *i.e.*

$$V_{\mathrm{L}} M_{\mathrm{L}}^2 V_{\mathrm{L}}^\dagger = \mathrm{diag}(m_e^2, m_\mu^2, m_\tau^2)\,. \tag{14}$$

Then from Eqs. (6), (7) one derives

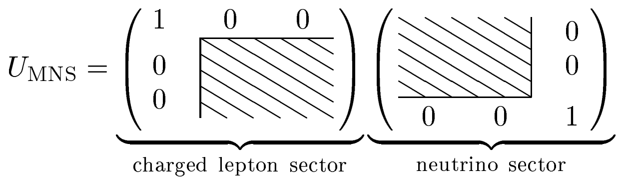
$$U_{\mathrm{MNS}} = V_{\mathrm{L}} \mathcal{O} = V_{\mathrm{L}} U_{\mathrm{L}}^{-1} \mathcal{O}'\,, \tag{15}$$

where the unitary matrix $\mathcal{O}'$ is such that

$$\frac{1}{M} \mathcal{O}'^{\mathrm{T}} m^{(\nu)\mathrm{T}}\, R\, m^{(\nu)} \mathcal{O}' = \mathrm{diag}(\mu_1, \mu_2, \mu_3) \tag{16}$$

with $R = P_{13}$, *cf.* Eq. (11).

We narrow our search for a realistic neutrino mass matrix by concentrating on the following structure of the MNS matrix, which is known to successfully describe data,



(17)

Since we choose a simple form of $U_{\mathrm{L}} = 1$, Eq. (15) implies that the second matrix on the right hand side of Eq. (17) is equal to the matrix $\mathcal{O}'$ diagonalizing the light neutrino Majorana mass matrix. Such a form of this matrix can be obtained even if the mass matrix $\mathcal{N}$ is not of the block diagonal form suggested by Eq. (17). In fact, assuming a diagonal Dirac mass matrix as in Eq. (5), one obtains the light Majorana mass matrix

$$\mathcal{N} = \mu \begin{pmatrix} 0 & 0 & r \\ 0 & 1 & 0 \\ r & 0 & 0 \end{pmatrix}, \tag{18}$$

where

$$r = \frac{m_1 m_3}{m_2^2}, \qquad \mu = \frac{m_2^2}{M}. \tag{19}$$

The way to sidestep this difficulty is to exchange the eigenvectors of the mass matrix with a permutation P_{23},

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{20}$$

see [1, 2] for details. Then $\mathcal{N}$ in Eq. (18) is diagonalized by the matrix

$$\mathcal{O}' = P_{23} U_{12}\left(\frac{\pi}{4}\right), \tag{21}$$

where

$$U_{12}(\alpha) = \begin{pmatrix} i\cos\alpha & \sin\alpha & 0 \\ -i\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{22}$$

This way, the matrix P_{23} will appear sandwiched between the two matrices in Eq. (17). In principle, such an insertion could destroy the structure of the mixing matrix. However, due to the particular form of the charged lepton matrix, nothing wrong happens since the effect of P_{23} acting to the left is that of exchanging the second and third column. But this may be seen to correspond to an innocuous relabeling of flavors, made irrelevant especially for the model of the charged lepton matrix we are using,

$$V_{\rm L} = O_{23}\left(\pm\frac{\pi}{4}\right) , \qquad (23)$$

the right hand side meaning the rotation about the first axis by the angle of $\pm\pi/4$ [2]. This form of $V_{\rm L}$ has been considered in many published models [17], in particular in the models based on the so-called lopsided form of the charged lepton mass matrix [18].

4. A realistic model

The construction shown above lets us get rid of the factor of m_3^2/M in the active neutrino spectrum, *cf.* Eq. (8), but the mass splitting ratio is now zero due to the twofold degeneracy of the lighter states, so the hierarchy problem appears to have actually been aggravated. On the other hand, the mixing pattern corresponds to the so-called bimaximal mixing [19] which is not acceptable for the solar neutrinos [8]. Both problems can be cured by an appropriate perturbation of the Dirac matrix, whose form was found in [2]. The resulting low energy neutrino mass matrix is

$$\mathcal{M} = P_{23} U_{\rm L}^{*} \mathcal{N} U_{\rm L}^{-1} P_{23} = \mu \begin{pmatrix} 0 & r & 0 \\ r & 2ar & 0 \\ 0 & 0 & 1 \end{pmatrix} , \qquad (24)$$

where $a^{-1} = \tan 2\theta_{\odot}$. For $a > 0$ the matrix $\mathcal{M}$ in (24) is diagonalized by

$$U_{12}^{\rm T}(\theta_{\odot}) \mathcal{M} U_{12}(\theta_{\odot}) = {\rm diag}\left(\mu_1, \mu_2, \mu_3\right) . \qquad (25)$$

Realistic spectra and mixing are obtained for the following range of the parameters a and r

$$0.35 \le a \le 0.75 \quad \text{and} \quad 0.05 \le r \le 0.25 \qquad (26)$$

with the best fits corresponding to a between 0.46 and 0.57 and r between 0.09 and 0.10. It is interesting that the value of $r \approx 0.08$ is obtained if the Dirac masses of neutrinos are assumed to be proportional to the corresponding masses of the charged leptons, see the footnote after Eq. (42) in [2]. The lepton mixing matrix becomes

$$U_{\mathrm{MNS}} = V_{\mathrm{L}} U_{\mathrm{L}}^{-1} P_{23} U_{12}\left(\theta_{\odot}\right), \tag{27}$$

and our model leads to the following mass spectrum, see [2]:

$$\mu_1 \approx \sqrt{\Delta m^2_{\odot}}\ \tan^2\theta_{\odot}\ /\ \sqrt{1-\tan^4\theta_{\odot}}\,, \tag{28}$$

$$\mu_2 \approx \sqrt{\Delta m^2_{\odot}}\ /\ \sqrt{1-\tan^4\theta_{\odot}}\,, \tag{29}$$

$$\mu_3 \approx \sqrt{\Delta m^2_{@}}. \tag{30}$$

From the presented model we can derive the lightest neutrino mass. One obtains about 3 meV at $\tan^2\theta_{\odot} \approx 0.4$. This mass range can be probed by the 10t version of the GENIUS project [20]. Finally, let us note that the mass scale of the Majorana masses is between 10^{10} and 10^{11} GeV if $m_2 \sim m_c$ is assumed. It has been pointed out in [21] that this is exactly the range of Majorana masses which may be important for baryogenesis; see [22] and references therein.

5. Concluding remarks

The observed neutrino mass splitting ratio exhibits little hierarchy compared to the expectations from a simple seesaw model and the assumption of a Dirac mass hierarchy of the order typical for quarks. Nevertheless, we have shown that seesaw models exist which succeed in suppressing the underlying strong Dirac mass hierarchy leaving only the weak hierarchy of effective light Majorana masses of active neutrinos. This reduction of hierarchy is caused by the symmetric unitary operator R, whose effect is to modify both the low energy mass spectrum and the lepton mixing matrix U_{MNS}. Realistic mixing pattern and masses are obtained with the form of R proposed in [1] after introducing a proper perturbation of the diagonal Dirac mass matrix [2]. The resulting model predicts a relation between the two lighter active neutrino masses, $\mu_1/\mu_2 \approx \tan^2\theta_{\odot}$, which is stable under small perturbations. Furthermore, the mass of the heaviest neutrino is related to the mass scale $\sqrt{\Delta m^2_{@}}$ governing the oscillations of atmospheric neutrinos. A small U_{e3} follows naturally from the model. The mass of the lightest neutrino is predicted to be about 3 meV and can be tested by the 10t GENIUS detector if Majorana phases are not too small and there are no strong cancellations between contributions to the mass parameter $\langle m_{\nu_e}\rangle$.

We would like to express our admiration for Stefan Pokorski's works in physics and wish him many further successes in the future. This work was done during our stay in the Institut für Theoretische Teilchenphysik, Universität Karlsruhe (TH). We would like to thank the Alexander-von-Humboldt Foundation for grants which made this possible. A warm atmosphere in TTP is gratefully acknowledged. Work supported in part by the European Community's Human Potential Programme under contract HPRN-CT-2000-00149 Physics at Colliders, and by the Polish State Committee for Scientific Research (KBN) grant no. 5P03B09320.

Appendix A

Solution for $U_{\rm R}$

The unitary matrix $U_{\rm R}$ defined in Eq. (4) is to satisfy the relation

$$U_{\rm R}^{\rm T} U_{\rm R} = \begin{pmatrix} 0 & 0 & \exp i\phi_1 \\ 0 & \exp i\phi_2 & 0 \\ \exp i\phi_1 & 0 & 0 \end{pmatrix} . \tag{A.1}$$

Note that if

$$U_{\rm R}'^{\rm T} U_{\rm R}' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{A.2}$$

and

$$U_{\rm R} = U_{\rm R}' \begin{pmatrix} \exp\frac{i\phi_1}{2} & 0 & 0 \\ 0 & \exp\frac{i\phi_2}{2} & 0 \\ 0 & 0 & \exp\frac{i\phi_1}{2} \end{pmatrix} , \tag{A.3}$$

then the condition (A.1) is satisfied. It is thus enough to find the matrix $U_{\rm R}'$ fulfilling Eq. (A.2), which can be represented as

$$U_{\rm R}'^{\rm T} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} U_{\rm R}'^{*\rm T} \quad . \tag{A.4}$$

Denoting the elements of $U_{\rm R}'^{\rm T}$ as

$$(U_{\rm R}'^{\rm T})_{ij} = a_{ij} \tag{A.5}$$

one can write equations for the a_{ij} following from Eq. (A.4)

$$a_{3i} = a_{1i}^*, \quad a_{2i} = a_{2i}^*, \quad i = 1 \ldots 3 . \tag{A.6}$$

Another set of relations follows from the unitarity of U'_{R} and can conveniently be written in terms of the real vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ defined as

$$\boldsymbol{u} \equiv (a_{21}, a_{22}, a_{23}), \quad \boldsymbol{v} + i\boldsymbol{w} \equiv (a_{11}, a_{12}, a_{13})\,. \tag{A.7}$$

Then,

$$\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{u}\cdot\boldsymbol{w} = \boldsymbol{v}\cdot\boldsymbol{w} = 0\,, \tag{A.8}$$

$$\boldsymbol{u}^2 = 1, \quad \boldsymbol{v}^2 = \boldsymbol{w}^2 = \frac{1}{2}\,. \tag{A.9}$$

Obviously, the conditions (A.8), (A.9) do not change under a rotation of the system

$$\begin{pmatrix} \boldsymbol{v}+i\boldsymbol{w} \\ \boldsymbol{u} \\ \boldsymbol{v}-i\boldsymbol{w} \end{pmatrix} \longrightarrow \begin{pmatrix} \boldsymbol{v}+i\boldsymbol{w} \\ \boldsymbol{u} \\ \boldsymbol{v}-i\boldsymbol{w} \end{pmatrix} \mathcal{O}\,, \tag{A.10}$$

where $\mathcal{O}$ is an arbitrary orthogonal matrix. We can therefore choose

$$\boldsymbol{u} = (0,1,0), \quad \boldsymbol{v} = \frac{1}{\sqrt{2}}(1,0,0)\,, \qquad \boldsymbol{w} = \frac{1}{\sqrt{2}}(0,0,1) \tag{A.11}$$

to get

$$U'_{\mathrm{R}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \end{pmatrix}\,. \tag{A.12}$$

If we take the rotation matrix

$$\mathcal{O} = \begin{pmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\,, \tag{A.13}$$

we obtain another solution to Eq. (A.2)

$$\mathcal{O}U'_{\mathrm{R}} = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega & 1 & \omega^* \\ 1 & 1 & 1 \\ \omega^* & 1 & \omega \end{pmatrix}\,, \tag{A.14}$$

where $\omega = \exp\frac{2\pi i}{3}$.

Appendix B

Diagonalization with transposed matrices

When diagonalizing the neutrino mass matrix, one must do it by multiplying a unitary matrix V to the right and its *transpose*, V^{T}, rather than the Hermitian conjugate, to the left. That is, we are faced with the problem of finding a unitary matrix V such that a given symmetrical matrix M is diagonalized

$$V^{\mathrm{T}} M V = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,. \tag{B.1}$$

Since the eigenvalues are to be interpreted as masses, we furthermore require $\lambda_i \in \boldsymbol{R}$ and $\lambda_i \geq 0$. In general, this problem is different from the usual procedure of diagonalization. In this appendix we show that the matrix V satisfying Eq. (B.1) has the form

$$V = (\boldsymbol{v}_1\ \boldsymbol{v}_2\ \ldots\ \boldsymbol{v}_n) = (\boldsymbol{u}_1\ \boldsymbol{u}_2\ \ldots\ \boldsymbol{u}_n) - i\,(\boldsymbol{w}_1\ \boldsymbol{w}_2\ \ldots\ \boldsymbol{w}_n)\ , \tag{B.2}$$

where $\boldsymbol{u}_i, \boldsymbol{v}_i, \boldsymbol{w}_i$ are column vectors, *e.g.*

$$\boldsymbol{v}_i = \begin{pmatrix} v_{i,1} \\ \vdots \\ v_{i,n} \end{pmatrix}\ , \tag{B.3}$$

and they satisfy the equation

$$\mathcal{M} \begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{w}_i \end{pmatrix} \equiv \begin{pmatrix} M_{\mathrm{R}} & M_{\mathrm{I}} \\ M_{\mathrm{I}} & -M_{\mathrm{R}} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{w}_i \end{pmatrix} = \lambda_i \begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{w}_i \end{pmatrix} \text{ for } \lambda_1, \ldots, \lambda_n \geq 0\,. \tag{B.4}$$

In the formula above, M_{R} and M_{I} are the real and imaginary parts of the matrix M, respectively:

$$M = M_{\mathrm{R}} + i M_{\mathrm{I}}\,. \tag{B.5}$$

Note that if

$$\begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{w}_i \end{pmatrix} \tag{B.6}$$

is the eigenvector with the eigenvalue λ_i then

$$\begin{pmatrix} \boldsymbol{w}_i \\ -\boldsymbol{u}_i \end{pmatrix} \tag{B.7}$$

is the eigenvector with the eigenvalue $-\lambda_i$. The unitarity of V requires the orthogonality relations for the vectors $\boldsymbol{u}_i, \boldsymbol{w}_i$:

$$\boldsymbol{v}_i^{\dagger} \boldsymbol{v}_j = \boldsymbol{u}_i^{\mathrm{T}} \boldsymbol{u}_j + \boldsymbol{w}_i^{\mathrm{T}} \boldsymbol{w}_j = (\boldsymbol{u}_i, \boldsymbol{w}_i)^{\mathrm{T}} \begin{pmatrix} \boldsymbol{u}_j \\ \boldsymbol{w}_j \end{pmatrix} = \delta_{ij}\,. \tag{B.8}$$

The conditions (B.8) are fulfilled due to the fact that the $2n \times 2n$ matrix $\mathcal{M}$ appearing in Eq. (B.4) is symmetric and real, so it has orthogonal eigenvectors, which can be normalized.

To prove Eq. (B.2), solve the equation

$$MV = V^* \Lambda, \text{ where } \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)\,. \tag{B.9}$$

Writing

$$V = V_{\mathrm{R}} - iV_{\mathrm{I}}\,, \tag{B.10}$$

where

$$V_{\mathrm{R}} = (\boldsymbol{u}_1\ \boldsymbol{u}_2\ \ldots\ \boldsymbol{u}_n)\ , V_{\mathrm{I}} = (\boldsymbol{w}_1\ \boldsymbol{w}_2\ \ldots\ \boldsymbol{w}_n)\ , \tag{B.11}$$

we obtain $(M_{\mathrm{R}} + iM_{\mathrm{I}})(V_{\mathrm{R}} - iV_{\mathrm{I}}) = (V_{\mathrm{R}} + iV_{\mathrm{I}})\Lambda$ and so

$$M_{\mathrm{R}}V_{\mathrm{R}} + M_{\mathrm{I}}V_{\mathrm{I}} \ = \ V_{\mathrm{R}}\Lambda\,, \tag{B.12}$$

$$M_{\mathrm{I}}V_{\mathrm{R}} - M_{\mathrm{R}}V_{\mathrm{I}} \ = \ V_{\mathrm{I}}\Lambda\,. \tag{B.13}$$

This system of equations can be rewritten in terms of the matrix $\mathcal{M}$

$$\mathcal{M}\begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{w}_i \end{pmatrix} = \lambda_i \begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{w}_i \end{pmatrix}, \quad i = 1 \ldots n\,. \tag{B.14}$$

Therefore, the problem is reduced to finding the eigenvalues and eigenvectors of the matrix $\mathcal{M}$. There are $2n$ of them and the eigenvalues are of the form $\pm\lambda_1, \pm\lambda_2, \ldots \pm \lambda_n$. Of those, we choose the n non-negative ones.

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