

UNIFIED DESCRIPTION OF A CLASS OF MODELS FOR BOSE–EINSTEIN CORRELATION IN MULTIPLE PARTICLE PRODUCTION PROCESSES*

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Numerous models have been proposed to describe the Bose–Einstein correlations in multiple particle production process. In the present paper we describe a generalization, which includes many previous models as special cases and, therefore, can be useful for work of comparison. We apply the powerful methods of eigenfunction expansions and generating functionals, which often make the calculations much shorter than in the original papers.

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1. Introduction

Bose–Einstein correlations in multiple particle production processes at high energy are now much discussed. The reviews [1–3] contain hundreds of references and many more can be found in the papers quoted there. There is a variety of models often based on very different physical pictures of the production process. In the present paper we stress that many of these models differ only in their choice of a single particle density matrix, further called input single particle density matrix. Once this choice has been made, the calculations become model independent. Following the approach from [4, 5] we describe a general model with a free function, which contains all these models as special cases. Then we show that the further calculations can be

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greatly simplified, if one uses eigenfunction expansions for the input single particle density matrices and suitable generating functionals to summarize the information about the distributions, which can be compared with experiment. This paper is essentially a description from the point of view presented in [4, 5] of a class of models including all the multiparticle symmetrization effects.

2. Models with factorization

Many models of Bose–Einstein correlations in multiple particle production processes can be reduced to the following four steps, which we list here postponing their discussion to the following sections.

1. Guess an input single particle density matrix $\rho_1^{(0)}(\vec{p}, \vec{p}')$.
2. Construct the n -particle density matrix for distinguishable particles

$$\rho_n^0(\vec{p}_1, \dots, \vec{p}_n; \vec{p}'_1, \dots, \vec{p}'_n) = \prod_{k=1}^n \rho_1^{(0)}(\vec{p}_k, \vec{p}'_k). \quad (1)$$

3. Symmetrize this density matrix in order to obtain a density matrix for indistinguishable particles. The diagonal elements, which are enough to calculate all the momentum distributions, are

$$\rho_n(\vec{p}_1, \dots, \vec{p}_n) = \frac{1}{n!} \sum_{\sigma, \tau} \rho_n^{(0)}(\vec{p}_{\sigma 1}, \dots, \vec{p}_{\sigma n}; \vec{p}_{\tau 1}, \dots, \vec{p}_{\tau n}), \quad (2)$$

where each of the summations over σ and τ extends over all the $n!$ permutations of the set of indices $1, \dots, n$. The normalization is such that calculating the trace one integrates over the momentum space without introducing the factor $\frac{1}{n!}$.

4. Build the diagonal elements of the overall density matrix according to the formula

$$\rho = \sum_{n=1}^{\infty} p^{(0)}(n) \rho_n. \quad (3)$$

The coefficients are usually chosen Poissonian with some average ν :

$$p^{(0)}(n) = \frac{\nu^n}{n!} e^{-\nu}. \quad (4)$$

Further, models built according to this recipe will be called models with factorization. Let us now discuss the four steps.

3. Step 1 — choice of the input single particle density matrix

The superscript of the function $\rho_1^{(0)}$ indicates that this is not the observed one-particle density matrix, but an input function necessary for the construction of the true density matrix. It may be interpreted as the single particle density matrix for the unphysical case, when the particles are produced independently and there is no Bose–Einstein symmetrization. The different models differ in the inspirations used to guess this function. The shortest way is, of course, to guess directly the input density matrix $\rho_1^{(0)}$, however, a longer way making use of a model may be easier. Let us quote some examples.

One can guess a source function $S(X, K)$ and calculate the density matrix $\rho_1^{(0)}$ from the formula

$$\rho_1^{(0)}(\vec{p}, \vec{p}') = \int d^4 X e^{-iqX} S(X, K), \quad (5)$$

where

$$K = \frac{1}{2}(p + p'), \quad (6)$$

$$q = p - p' \quad (7)$$

are four-vectors with the four-vectors p, p' being on shell particle momenta. The advantage of this approach is that the source function reflects the space-time and momentum distribution of the sources of particles. Thus, there are intuitions what it should look like [1, 6, 7].

Another approach is to assume that the particles originate from a large number N of independent, incoherent sources. In order to produce a reasonable number of particles, each of the sources must be weak. A formula for the “total source” [8, 9] with a good high N limit is

$$J(\vec{p}, N, \zeta, \phi) = \frac{1}{\sqrt{N}} \sum_{k=1}^N e^{i\phi_k} e^{ipx_k} j_0(\Lambda_k p), \quad (8)$$

where $\phi = \{\phi_1, \dots, \phi_N\}$ is a set of random phases, $\zeta = \{x_1, \vec{v}_1, \dots, x_N, \vec{v}_N\}$ is a set of parameters characterizing the N sources and Λ_k are Lorentz transformations with velocities \vec{v}_k , acting on the momentum fourvector p . The “total source” is not a source in the sense of quantum field theory. It is rather a kind of single-particle wave function with the condition that the components labeled by different indices k are not allowed to interfere. This is implemented by the random phases ϕ_k . An object of this kind can be replaced by a single particle density matrix

$$\rho_1'(\vec{p}, \vec{p}', \zeta) = \frac{1}{N} \sum_{k=1}^N e^{ix_k(p-p')} j_0(\Lambda_k p) j_0^*(\Lambda_k p'). \quad (9)$$

There are two ways of proceeding further. One can assume that the density matrix ρ_1^0 is an average of this density matrix over the sets $\zeta_k = \{x_k, \vec{v}_k\}$ and obtain [9]

$$\rho_1^{(0)}(\vec{p}, \vec{p}') = \int d\zeta_k \rho(\zeta_k) e^{ix_k(p-p')} j_0(A_k p) j_0^*(A_k p'). \tag{10}$$

In the approach of Ref. [9] there are additional factors dependent on the number of sources N , which for N large drop out from the final result. We absorb the N -dependent factor into the density $\rho(\zeta_k)$ so that it does not appear explicitly in the formulae. In this approach one has to guess the spectrum of a single source at rest $j_0(p)$ and the distribution of sources $\rho(\zeta_k)$. The resulting single particle input density matrix does not depend on the unmeasurable parameters any more.

Alternatively one can postpone the averaging over the unmeasurable parameters as done in Ref. [8] (further quoted GKW) and choose the pure state input single particle density matrix

$$\rho_1^{(0)}(\vec{p}, \vec{p}') = \frac{1}{\bar{n}(N, \zeta, \phi)} J(\vec{p}, N, \zeta, \phi) J^*(\vec{p}', N, \zeta, \phi), \tag{11}$$

where

$$\bar{n}(N, \zeta, \phi) = \int d^3k |J(\vec{k}; N, \zeta, \phi)|^2 \tag{12}$$

is a normalizing constant. It is natural to assume in such models that also the input multiplicity distribution $p^{(0)}(n)$ depends on the unmeasurable parameters. Models of this kind do not belong to the class of models discussed in the present paper, unless simplifying assumptions are made. We will discuss only the simplest case, when the unmeasurable parameters are fixed or absent, and consequently no averaging is necessary. This model (*cf.* [4]) will be called pure state model, because for it the input single particle density matrix, which can be written in the form

$$\rho_1^{(0)}(\vec{p}, \vec{p}') = \psi(\vec{p}) \psi^*(\vec{p}'), \tag{13}$$

corresponds to a pure state. There are two reasons to consider this grossly oversimplified model: it is one of the very few models, where multiparticle effects can be included analytically and it is a good starting point for the discussion of the much more important GKW model.

Still another strategy is to guess a set of single particle wave packets $|\alpha\rangle$ and the distribution of such packets $\rho(\alpha)$ [10,11]. Then

$$\rho_1^{(0)}(\vec{p}, \vec{p}') = \int d\alpha \langle \vec{p} | \alpha \rangle \rho(\alpha) \langle \alpha | \vec{p}' \rangle. \tag{14}$$

Let us note for further reference that if the states $|\alpha\rangle$ form an orthonormal set, they are the eigenvectors corresponding to the eigenvalues $\rho(\alpha)$ of the matrix $\rho_1^{(0)}(\vec{p}, \vec{p}')$. Then, usually, the integration gets replaced by a summation.

Whatever the starting point, very often one finally obtains a Gaussian (cf. e.g. [6, 9, 10])

$$\rho_1^{(0)}(\vec{p}, \vec{p}') = \prod_{i=x,y,z} \frac{1}{\sqrt{2\pi\Delta_i^2}} \exp\left[-\frac{K_i^2}{2\Delta_i^2} - \frac{R_i^2 q_i^2}{2}\right]. \quad (15)$$

In order not to contradict the Heisenberg uncertainty principle, one must have

$$\Delta_i R_i \geq \frac{1}{2}. \quad (16)$$

In the wave packet approach the parameters R_i and Δ_i are expressed in terms of other parameters in such a way that this condition is automatically fulfilled. In other approaches one must impose it as a constraint.

One could include in the exponent a term $is\vec{q} \cdot \vec{K}$ [12], where s is a real constant and the factor i is required by the hermiticity of the density matrix. Since, however, $\vec{K} \cdot \vec{q} = (p_1^2 - p_2^2)/2$, this addition does not affect (in the momentum representation) the density matrix elements we are interested in [9]. It does affect the deduced size of the interaction region, but this problem is not discussed in the present paper.

Let us make some remarks about normalization. It is often convenient to use the normalization

$$\int d^3 p \rho_1^{(0)}(\vec{p}, \vec{p}) = 1. \quad (17)$$

In other cases an invariant normalization may be preferable

$$\int \frac{d^3 p}{E_p} \rho_{\text{inv}1}^{(0)}(\vec{p}, \vec{p}) = 1, \quad (18)$$

where $E_p = \sqrt{m^2 + \vec{p}^2}$. The relation between the two density matrices is

$$\rho_{\text{inv}1}^{(0)}(\vec{p}, \vec{p}') = \sqrt{E_p E_{p'}} \rho_1^{(0)}(\vec{p}, \vec{p}') \quad (19)$$

and can be used at any time to go from one normalization to the other. For definiteness we will use the matrix $\rho_1^{(0)}$ normalized according to (17).

The calculation of multiparticle effects can be greatly simplified, when for the input density matrix $\rho_1^{(0)}$ its eigenfunctions $\psi_n(\vec{p})$ and its eigenvalues λ_n are known. These are defined by the equation

$$\int d^3 p' \rho_1^{(0)}(\vec{p}, \vec{p}') \psi_n(\vec{p}') = \lambda_n \psi_n(\vec{p}), \quad (20)$$

The eigenfunctions and the eigenvalues are known both for the pure state model (13) and for the Gaussian model (15). For the pure state model obviously

$$\psi_0(\vec{p}) = \psi(\vec{p}) \quad \text{and} \quad \lambda_n = \delta_{n,0}. \tag{21}$$

with the remaining eigenfunctions constrained only by the condition that they are orthogonal to ψ_0 and to each other. For the Gaussian model in one dimension [4, 13]

$$\psi_n(p) = \sqrt{\frac{\alpha}{\sqrt{\pi}2^n n!}} \exp\left[-\frac{\alpha^2 p^2}{2}\right] H_n(\alpha p), \tag{22}$$

$$\lambda_n = (1 - z)z^n, \tag{23}$$

where $n = 0, 1, \dots$,

$$\alpha = \sqrt{\frac{R}{\Delta}}; \quad z = \frac{2R\Delta - 1}{2R\Delta + 1}. \tag{24}$$

In the three-dimensional case the index n becomes the set $\{n_x, n_y, n_z\}$. The eigenfunctions and the eigenvalues are

$$\psi_n(\vec{p}) = \psi_{n_x}(p_x)\psi_{n_y}(p_y)\psi_{n_z}(p_z); \quad \lambda_n = \lambda_{n_x}\lambda_{n_y}\lambda_{n_z}. \tag{25}$$

In terms of its eigenfunctions and eigenvalues the input density matrix is

$$\rho_1^{(0)}(\vec{p}, \vec{p}') = \sum_n \lambda_n \psi_n(\vec{p})\psi_n^*(\vec{p}'). \tag{26}$$

The normalization condition implies that

$$\sum_n \lambda_n = 1. \tag{27}$$

4. Step 2 — independent particles stage

The density matrix $\rho_n^{(0)}$ is also an auxiliary construct and does not correspond to an existing physical system. It describes a system of n independent, distinguishable particles. The particles are independent in the sense that the average of the product of any two single particle operators acting on different particles is equal to the corresponding product of averages

$$\langle \hat{O}_1(\vec{p}_1)\hat{O}_2(\vec{p}_2) \rangle_n = \langle \hat{O}_1(\vec{p}_1) \rangle_n \langle \hat{O}_2(\vec{p}_2) \rangle_n, \tag{28}$$

where the averages are defined by the standard formula

$$\langle \hat{O} \rangle_n = \text{Tr} \left[\hat{\rho}_n^{(0)} \hat{O} \right]. \tag{29}$$

One could ask what is the relation between the physical assumption that pions are emitted independently and the factorization of the weight function in the definition of the density matrix. *E.g.* in the wave packet picture does

$$\rho_n(p_1, \dots, p_n) = \sum_{\alpha_1, \dots, \alpha_n} |\alpha_1, \dots, \alpha_n\rangle \bar{\rho}_n(\alpha_1, \dots, \alpha_n) \langle \alpha_1, \dots, \alpha_n|, \quad (30)$$

$$\bar{\rho}_n(\alpha_1, \dots, \alpha_n) = \prod_{k=1}^n \bar{\rho}_1(\alpha_k), \quad (31)$$

imply that the emission is independent? As easily checked from definition (28) the answer is affirmative, if the wave functions $\langle \vec{p}_1, \dots, \vec{p}_n | \alpha_1, \dots, \alpha_n \rangle$ are products of single particle wave functions $\langle \vec{p}_k | \alpha_k \rangle$. This is the case in step two of the present model. If the n -particle wave function corresponds to correlated particles, the factorizability of the weight function $\bar{\rho}_n$ does not help. This is the case after the wave functions are symmetrized, since then the Bose–Einstein correlations appear. It is a matter of taste, whether these correlations are ascribed to the symmetrization of the states $|\alpha_1, \dots, \alpha_n\rangle$, which are part of the density operator, or to the symmetrization to the external states $|\vec{p}_1, \dots, \vec{p}_n\rangle$. The first choice was made *e.g.* by Zimányi and Csörgö [10] and the second *e.g.* in the classical paper of the Goldhabers, Lee and Pais [14]. This is not a physical distinction, however, because the scalar product defining the n particle wave function does not depend on whether the first factor, the second factor, or both factors got symmetrized.

Thus, the model is an independent emission model [4, 5, 9] in a well-defined though somewhat formal sense. When $\rho_1^{(0)}$ depends on unobservable parameters, independence holds for fixed values of these parameters and would be destroyed, if one averaged over them.

One could desire for distinguishable particles, in analogy to step 4, a density matrix, which involves all the multiplicities

$$\rho^{(0)} = \sum_{n=0}^{\infty} \frac{\nu^n}{n!} e^{-\nu} \rho_n^{(0)}. \quad (32)$$

It is easy to check that also this matrix corresponds to independent particle production. A different choice of the probabilities $p^{(0)}(n)$ would correspond to correlated production even at the stage when the particles are considered distinguishable [9]. Of course, introducing correlations by modifying the input multiplicity distribution only, without modifying the independent production assumption for each given multiplicity, is not the most general way of introducing correlations.

5. Step 3 – exclusive momentum distributions

The diagonal elements of the symmetrized n -particle density matrix obtained in step 3 yield the momentum distribution for n identical particles, when no more particles of this kind have been produced. There are no constraints on the production of particles of other kinds. Thus, strictly speaking, this is a semiinclusive distribution. Since, however, in this paper particles of other kinds are ignored and could just as well be assumed to be absent, we have called this distribution exclusive. It is in general not normalized, even when the single particle density matrix $\rho_1^{(0)}$ is normalized. We will use the notation [4]: $W_0 = 1$ and for $n > 0$

$$W_n = \text{Tr } \rho_n = \frac{1}{n!} \sum_{\sigma, \tau} \int \prod_{k=1}^n \rho_1^{(0)}(\vec{p}_{\sigma 1k}, \vec{p}_{\tau k}) d^3 p_k. \tag{33}$$

Using the matrix $\rho_{\text{inv}1}^{(0)}$ instead of $\rho_1^{(0)}$ one can make this formula invariant. The same remark applies to our further formulae and we will not repeat it. For further use it is convenient to define the generating functional [4, 5]: $W_0[u] = 1$ and for $n > 0$

$$W_n[u] = \int \rho_n(p_1, \dots, p_n) \prod_{k=1}^n u(\vec{p}_k) d^3 p_k. \tag{34}$$

For $u(\vec{p}) \equiv 1$ one recovers the parameters W_n : $W_n \equiv W_n[1]$.

Since each permutation can be decomposed into cycles, the parameters W_n for $n > 0$ can be expressed in terms of the simpler parameters C_k defined for $k > 0$ by

$$C_k = \text{Tr } [\rho_1^{(0)}]^k = \int \rho_1^{(0)}(\vec{p}_1, \vec{p}_2) \rho_1^{(0)}(\vec{p}_2, \vec{p}_3) \dots \rho_1^{(0)}(\vec{p}_k, \vec{p}_1) \prod_{j=1}^n d^3 p_j. \tag{35}$$

If the input matrix $\rho_1^{(0)}$ is normalized by (17), $C_1 = 1$. The corresponding functional is

$$C_k[u] = \int \rho_1^{(0)}(\vec{p}_1, \vec{p}_2) \rho_1^{(0)}(\vec{p}_2, \vec{p}_3) \dots \rho_1^{(0)}(\vec{p}_k, \vec{p}_1) \prod_{j=1}^n u(\vec{p}_j) d^3 p_j \tag{36}$$

with $C_k[1] = C_k$.

The functional $W_n[u]$ for $n > 0$ can be expressed in terms of the functionals $C_k[u]$ according to the formula

$$W_n[u] = n! \sum_{n_1, \dots, n_n} \prod_{k=1}^n \frac{(C_k[u]/k)^{n_k}}{n_k!}. \tag{37}$$

The sum is over all the sets of nonnegative integers $\{n_1, \dots, n_n\}$ satisfying the relation $\sum_{k=1}^n kn_k = n$, or equivalently over all the decompositions of the set of permutations of n objects into cycles, so that there are n_k cycles of length k . Note that $\text{Tr} \rho_n^{(0)} = C_1^n$ is equal to the term $n_k = n\delta_{k,1}$ of this sum. Thus, all the further terms in $W_n[u]$ can be interpreted as corrections due to symmetrization.

At this point one can write down the normalized n -particle exclusive momentum distribution

$$P(\vec{p}_1, \dots, \vec{p}_n) = \frac{1}{W_n} \rho_n(\vec{p}_1, \dots, \vec{p}_n). \quad (38)$$

The distribution is here normalized to unity, but a change of this convention would be trivial.

Using representation (26) for the input single particle density matrix, one can write the parameters C_k in the form

$$C_k = \sum_n \lambda_n^k. \quad (39)$$

For the pure state model (13)

$$C_k[u] = \left(\int d^3p |\psi(\vec{p})|^2 u(\vec{p}) \right)^k; \quad k = 1, 2, \dots, \quad (40)$$

which implies for each k : $C_k = 1$ and $W_k[u] = k! C_k[u]$.

For the Gaussian model in three dimensions, after summing three geometrical progressions,

$$C_k = \prod_{i=x,y,z}^3 \frac{(1 - z_i)^k}{1 - z_i^k}. \quad (41)$$

Two limits are here of interest [9,10]. When the phase space per particle is minimal, for $i = x, y, z$: $2\Delta_i R_i \rightarrow 1$ and $z_i \rightarrow 0$. Consequently, $C_k \rightarrow 1$ for all $k > 0$. The state becomes pure as it should, when due to Einstein's condensation practically all the particles are in the single particle state corresponding to the eigenvalues λ_{0i} . When the phase space is large, for each i : $\Delta_i R_i \rightarrow \infty$ and $z_i \rightarrow 1$. Consequently, $C_k \rightarrow \delta_{k,1}$. In this limit multiparticle effects become negligible.

Let us note that in order to calculate momentum distributions one uses only the diagonal elements of the density matrix. Therefore, for this calculation any density matrix can be replaced by a diagonal matrix with the same diagonal elements. For instance, the density operator proposed in Refs. [8,9]:

$$\hat{\rho} = e^{-\bar{n}} \exp \left[i \int dp J(\vec{p}) a_{\vec{p}}^\dagger \right] |0\rangle \langle 0| \exp \left[-i \int dp J^*(\vec{p}) a_{\vec{p}} \right], \quad (42)$$

where $a_{\vec{p}}, a_{\vec{p}}^\dagger$ are annihilation and creation operators for particles with momentum \vec{p} , yields a density matrix nondiagonal in n . All that matters for the calculation of momentum distributions, however, are the diagonal elements and in our notation one finds for each subspace of n -particle states

$$p^{(0)}(n) \langle \vec{p}_1, \dots, \vec{p}_n | \hat{\rho}_n | \vec{p}_1, \dots, \vec{p}_n \rangle = \frac{e^{-\bar{n}}}{n!} \prod_{k=1}^n |J(\vec{p}_k)|^2. \tag{43}$$

Thus choosing $p^{(0)}(n) = \exp(-\bar{n})\bar{n}^n/(n!)^2$ one can replace the density operator (42) by the much simpler density operator of the pure state case. For given multiplicity the probability distribution for the particle momenta is a product of single particle momentum distributions. Therefore, particle momenta are uncorrelated. In particular, no Bose–Einstein correlations are seen in the distributions of relative momenta. This corresponds to the coherent case of the GKW model [8]. If, on the other hand, the unobservable parameters have not been averaged out, the (unnormalized) exclusive momentum distribution is

$$\langle p^{(0)}(n) \rho_n(\vec{p}_1, \dots, \vec{p}_n) \rangle = \left\langle e^{-\bar{n}} \prod_{k=1}^n |J(\vec{p}_k)|^2 \right\rangle, \tag{44}$$

where $\langle \dots \rangle$ denotes averaging over the unobservable parameters. This distribution is not a product of single particle distributions any more and consequently correlations among the particles are present. It corresponds to the chaotic case of the GKW model.

6. Step 4 — multiplicity distribution

Let us note first that, since in general $W_n = \text{Tr } \rho_n \neq 1$, the coefficients $p^{(0)}(n)$ are not the probabilities for producing n particles. These (unnormalized) probabilities are

$$p(n) = p_n^{(0)} W_n. \tag{45}$$

Substituting the Poisson formula for $p^{(0)}(n)$ and formula (37) with $u \equiv 1$ for $\text{Tr } \rho_n$ one finds

$$\sum_n p(n) = e^{-\nu} \exp \left[\sum_{k=1}^{\infty} \frac{C_k}{k} \nu^k \right]. \tag{46}$$

For non-Poissonian input multiplicity distributions the corresponding summation usually must be done numerically [9]. In the following we limit our discussion to the Poissonian case. The probability distribution $p(n)$ can be

normalized, if and only if the series in the exponent in (46) converges. This is the case if

$$\nu \lim_{k \rightarrow \infty} \sqrt[k]{C_k} = \nu \lambda_0 < 1, \quad (47)$$

where λ_0 is the largest eigenvalue of the input single particle density matrix. The series diverges when $\nu > \lambda_0^{-1}$. This happens for the pure state model when

$$\nu > 1, \quad (48)$$

and for the Gaussian model, when

$$\nu > \nu_0 = \prod_{i=x,y,z} \left(\frac{1}{2} + \Delta_i R_i \right). \quad (49)$$

In the terminology of thermodynamics $\nu = \nu_0$ corresponds to a singularity of the (grand)partition function, *i.e.* (usually, see below) to a phase transition. We show further that this is Einstein's condensation. Note that we have given here a complete derivation of the formula for ν_0 in the Gaussian case. Without using the representation of the input density matrix in terms of its eigenvalues, this calculation would have taken many pages ([10] and references given there). As Zimányi and Csörgö put it [10] "these solutions are not easily obtained" — and they worked for simplicity with the spherically symmetric case, where Δ_i and R_i do not depend on i . For $\nu < \nu_0$ the normalized probability distribution for finding n particles is

$$P(n) = \frac{\nu^n}{n!} W_n \exp \left[- \sum_{k=1}^{\infty} \frac{C_k}{k} \nu^k \right]. \quad (50)$$

When ν tends to ν_0 from below, the sum in the exponent of the normalizing factor tends to infinity and the probability of finding $n < n_0$ particles for any finite n_0 tends to zero. This effect disappears, when the eigenvalue λ_0 and the corresponding eigenstate ψ_0 are removed from the density matrix (26). Thus, the surplus of particles condenses in the state ψ_0 — Einstein's condensation occurs. Since in the pure state model all the particles are in one pure state anyway, there is no Einstein condensation in this model for $\nu \rightarrow 1$.

For comparison with experiment one needs the multiplicity distribution, its moments, cumulants *etc.*, as well as inclusive single particle, two particle *etc.* distributions. All this information is conveniently summarized in the

generating functional¹

$$\Phi[u] = \sum_{n=0}^{\infty} p^{(0)}(n)W_n[u] = \exp \left[\sum_{k=1}^{\infty} \frac{C_k[u]}{k} \nu^k - \nu \right]. \tag{51}$$

This formula differs from the corresponding formula in Ref. [5] by a momentum independent factor, which does not affect the logarithmic derivatives of interest. For the pure state model

$$\Phi[u] = \frac{e^{-\nu}}{1 - \nu \int d^3p |\psi(\vec{p})|^2 u(\vec{p})}. \tag{52}$$

In order to calculate the moments and/or cumulants of the multiplicity distribution, one chooses the function u constant. Let us denote this constant by z . Then

$$W_n[z] = z^n W_n, \tag{53}$$

$$C_k[z] = z^k C_k. \tag{54}$$

Let us calculate for example

$$\langle n \rangle = \left(\frac{\partial \log \Phi[z]}{\partial z} \right)_{z=1} = \sum_{k=1}^{\infty} \nu^k C_k, \tag{55}$$

$$\langle n^2 \rangle - \langle n \rangle^2 = \left(\frac{\partial^2 \log \Phi[z]}{\partial z^2} \right)_{z=1} + \langle n \rangle = \sum_{k=1}^{\infty} \nu^k k C_k = \nu \frac{d}{d\nu} \langle n \rangle. \tag{56}$$

These formulae can be simplified by using the eigenvalue expansion. For instance, assuming that the geometrical series converge, one finds

$$\langle n \rangle = \sum_{n=0}^{\infty} \frac{\nu \lambda_n}{1 - \nu \lambda_n}. \tag{57}$$

For given $\rho_1^{(0)}$, *i.e.* for given eigenvalues λ_k , the parameter ν should be chosen so as to reproduce the observed value of $\langle n \rangle$. With increasing $\langle n \rangle$ the parameter ν increases. For $\langle n \rangle \rightarrow \infty$, $\nu \lambda_0 \rightarrow 1$. Each term in the sum (57), except for the first one, tends to a finite limit. Let us assume that the sum of these limits is non zero and finite. Then, with $\langle n \rangle$ increasing to large values, only the first term keeps growing significantly. The k -th term in the sum gives the average population of the k -th eigenstate of the density

¹ Many of the steps below become obvious, when one translates the argument into the language of thermodynamics [15].

operator. Thus for $\langle n \rangle$ large and growing, almost all the additional particles due to the increase of ν land in one state ($|\psi_0\rangle$) — Einstein's condensation occurs. The formula for the dispersion of the multiplicity distribution can be rewritten as

$$\langle n^2 \rangle - \langle n \rangle^2 = \sum_{n=0}^{\infty} \frac{\nu \lambda_n}{(1 - \nu \lambda_n)^2}. \quad (58)$$

In the pure state model, where each of the coefficients C_k equals one, the series for $\langle n \rangle$ converges only if $\nu < 1$ and then

$$\langle n \rangle = \frac{\nu}{1 - \nu} \quad \text{for} \quad \nu < 1. \quad (59)$$

For this model symmetrization from step 3 reduces to a multiplication of the density matrix by $n!$. The unnormalized probability for producing n particles

$$p(n) = e^{-\nu} \nu^n, \quad (60)$$

which explains the singularity in $\langle n \rangle$ for $\nu \rightarrow 1$.

As seen from these examples, when the eigenvalues λ_n are known, the numerical evaluation of the moments of the particle multiplicity distribution is easy.

7. Step 4 — inclusive distribution of momenta

When calculating the inclusive distribution of momenta it is convenient to use the functions

$$L(\vec{p}, \vec{p}') = \sum_{k=1}^{\infty} \nu^k \left[\rho_1^{(0)} \right]^k (\vec{p}, \vec{p}'), \quad (61)$$

where

$$\left[\rho_1^{(0)} \right]^k (\vec{p}, \vec{p}') = \int d^3 p_1 \dots d^3 p_n \rho_1^{(0)}(\vec{p}, \vec{p}_1) \rho_1^{(0)}(\vec{p}_1, \vec{p}_2) \dots \rho_1^{(0)}(\vec{p}_k, \vec{p}'), \quad (62)$$

or equivalently

$$\left[\rho_1^{(0)} \right]^k (\vec{p}, \vec{p}') = \sum_n \psi_n(\vec{p}) \lambda_n^k \psi_n^*(\vec{p}'). \quad (63)$$

Using the latter notation

$$L(\vec{p}, \vec{p}') = \sum_n \psi_n(\vec{p}) \psi_n^*(\vec{p}') \frac{\nu \lambda_n}{1 - \nu \lambda_n}. \quad (64)$$

As seen from this formula

$$L^*(\vec{p}, \vec{p}') = L(\vec{p}', \vec{p}). \quad (65)$$

Actually, $L(\vec{p}, \vec{p}')$ is the symmetrized single particle inclusive density matrix (see below).

The various inclusive distributions of momenta can be evaluated as functional derivatives of the generating functional $\Phi[u]$ at $u = 1$. Thus, the single particle distribution and the two particle correlation function are

$$P_1(\vec{p}) = \left(\frac{\delta}{\delta u(\vec{p})} \ln \Phi[u] \right)_{u=1} = L(\vec{p}, \vec{p}), \quad (66)$$

$$\begin{aligned} P_2(\vec{p}_1, \vec{p}_2) - P(\vec{p}_1)P(\vec{p}_2) &= \left(\frac{\delta^2}{\delta u(\vec{p}_1)\delta u(\vec{p}_2)} \ln \Phi[u] \right)_{u=1} \\ &= |L(\vec{p}_1, \vec{p}_2)|^2. \end{aligned} \quad (67)$$

For the pure state model $L(\vec{p}, \vec{p}') = \rho_1^{(0)}(\vec{p}, \vec{p}')$ and there are no symmetrization effects in the inclusive distribution of momenta. For the Gaussian model, using the formulae from Section 3, one can obtain the function $L(\vec{p}, \vec{p}')$ as a power series in ν . The coefficients of this series are explicitly known Gaussians.

8. Conclusions

Let us summarize our conclusions and add a few comments.

- Many models used for the description of Bose–Einstein correlations in high energy multiple-particle-production processes belong to the class of factorizable models described in the present paper. As example of a model which does not, let us quote the string model developed by Andersson and collaborators [16–18]. This model is based on a very different picture of the particle production process, though it has been suggested [19] that numerically its predictions might be very close to the predictions of a suitably chosen factorizable model. In practice the models are often supplemented with corrections for resonance production, final state strong interactions, Coulomb interactions *etc.* We have not discussed these problems here.
- The input consists of an input multiplicity distribution (usually Poissonian) and an input single particle density matrix in the momentum representation. There is a great variety of physical pictures used to suggest these inputs, but there is no consensus on which is the best.

- Once the input has been chosen, there are standard procedure to derive general formulae for the inclusive and exclusive distributions of momenta, multiplicity distributions and various correlation functions and coefficients. In practice, however, it is usually difficult to use these formulae beyond the few particle distributions uncorrected for multiple particle effects.
- The multiparticle effects can be more easily included, when the eigenfunctions and eigenvalues of the input single particle density matrix are known. Examples are the two cases, where exact solutions are known, *i.e.* the Gaussian model and the pure state model. In general, Einstein's condensation occurs, when the model is not the pure state model and the parameter ν of the input Poissonian multiplicity distribution is equal to the inverse of the largest eigenvalue of the input single particle density matrix.

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