CLASSICAL RADIATION OF A FINITE NUMBER OF PHOTONS

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Dedicated to Stefan Pokorski on his 60th birthday

Under certain conditions the number of photons radiated classically by a charged particle following a prescribed trajectory can be finite. An interesting formula for this number is presented and discussed.

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1. Finite number of photons and their formula

In the classical theory of radiation by a charged particle one calculates the energy radiated into the electromagnetic field. Indeed from the purely classical point of view the energy is practically the only quantity there *is* to calculate. However, in certain problems [2] one may come upon the idea of finding the *number* of photons radiated by a charged particle following a given trajectory. Although the photon is a quantum concept and so the question of finding the number might be thought to involve quantum mechanics, it does so only in the most minimal way. Given the energy radiated into a given mode of the field, it is only necessary to use Planck's relation and divide by the frequency ω to find the number n; thus the problem remains an essentially classical one.

It might be objected that n can be infinite while the energy is finite, as in the well-known "infrared catastrophe". True, but as we shall see, there is an interesting class of cases where this is *not* the case. In particular when the trajectory of a charged particle begins and ends with the same vector velocity $\boldsymbol{v}$, n is generally finite. This is because the "infrared catastrophe" results from the difference in the long flight paths for the in-and-outgoing particles, and when they are the same, the "catastrophe" is averted.

On the other hand, bremsstrahlung calculations often have an "ultraviolet" or high frequency divergence, due to the sudden appearance or deflection of a charge [4]. Evidently one cannot emit an infinite number of *finite* energy photons, so this divergence must be an artifact of the calculation. This difficulty can have either a quantum or classical resolution. In the quantum solution, as in the Feynman graph technique, one takes into account the energy-momentum conservation usually neglected at the classical level, and a suppression results for high energy photons. However, and more of interest to us here, there can also be a classical resolution: the ultraviolet divergence results from abrupt changes in the trajectory, and if the velocity changes or accelerations are sufficiently smooth, there is no divergence.

Therefore, we have the interesting situation that for a smooth trajectory, beginning and ending with the same velocities, the number of photons n radiated according to a classical calculation should be finite. But n is a dimensionless, Lorentz-invariant quantity. Thus there ought to be some simple formula for it, a relation mapping the path of a particle in space-time to the real number, n.

This relation is

$$n = \frac{\alpha}{\pi} \int \int dx_\mu \, \frac{1}{S_{i\varepsilon}^2} \, dx'_\mu \, . \tag{1}$$

In this formula x and x' are four-dimensional coordinates, referring to points on the space-time path of the charged particle, so that dx_μ is a 4-vectorial element of the path (Fig. 1). One may introduce the proper time or invariant path length τ and the four-velocity $u_\mu = dx_\mu/d\tau$ to also write the expression as

$$n = \frac{\alpha}{\pi} \int \int \frac{dx_\mu}{d\tau} \frac{1}{S_{i\varepsilon}^2} \frac{dx_\mu}{d\tau'} \; d\tau d\tau' = \frac{\alpha}{\pi} \int \int u_\mu(\tau) \frac{1}{S_{i\varepsilon}^2} \; u_\mu(\tau') \; d\tau d\tau' \, . \tag{2}$$

$S_{i\varepsilon}$ is the four-distance between the points x, x' in the following way

$$S_{i\varepsilon}^2 = (t - t' + i\varepsilon)^2 - (\boldsymbol{x} - \boldsymbol{x}')^2 \, . \tag{3}$$

As we shall explain shortly the $i\varepsilon$ is necessary to make sense of and to properly define Eq. (1).

It is interesting that the Sommerfeld fine structure constant $\alpha = e^2/\hbar c \approx 1/137$, involving Planck's quantum constant, appears. It arises, however, not as a coupling constant, but rather from the conversion of the classical energy to quanta, where we finally must divide by $\hbar\omega$. The expression is a non-perturbative. If the particle is multiply charged with Q electron charges, the formula should be multiplied by Q^2. Observe that by interchanging t

and t' in the integrations one can reverse the sign of $i\varepsilon$ to show that n is real. The expression is not an integer since it represents the average or expectation value of the number of particles radiated.

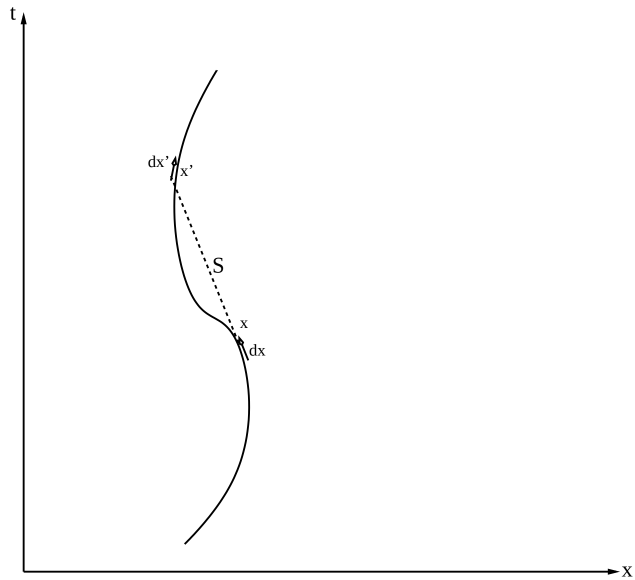

Fig. 1. Path in space-time.

2. Derivation

We can derive Eq. (1) from the standard treatments of classical radiation theory [4, 5] where one calculates $\boldsymbol{j}(\boldsymbol{k})$, the Fourier components of the three-vector current, and then finds the energy radiation rate $\sim |\boldsymbol{j}|^2 \sin^2\theta$, where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{j}$. Dividing by ω (units $\hbar = 1$) gives n.

To obtain the total number, we sum over all modes $\boldsymbol{k}$ of the radiation field

$$n \sim \int \sin^2\theta\,|\boldsymbol{j}(\boldsymbol{k})|^2\,\frac{1}{\omega}\,d^3k\,. \tag{4}$$

One could try to perform the integrations for every particular trajectory, but we would like to find a general formula in terms of the path itself. Thus we attempt to perform the d^3k integration first, before the Fourier transform. This gives at first a not well-defined expression, which leads to the need to introduce the $i\varepsilon$.

To see how this comes about, we observe that for smooth paths $\boldsymbol{x}(t)$ the Fourier transform $\boldsymbol{j}(\boldsymbol{k})$ is strongly convergent at large ω. When we introduce the current density for our classical point particle as

$$\boldsymbol{j}(t,\boldsymbol{x}) = e\,\boldsymbol{v}(t)\delta^3(\boldsymbol{x}-\boldsymbol{x}(t)) \tag{5}$$

we have $\boldsymbol{j}(\boldsymbol{k}) = e \int dt\, \boldsymbol{v}\, \mathrm{e}^{i(\boldsymbol{kx}(t)-\omega t)}$ for the Fourier transform. Since the path of the particle is time–like the exponent can never become zero, and if $\boldsymbol{x}$ or $\boldsymbol{v} = d\boldsymbol{x}/dt$ have no jumps or kinks the integral vanishes rapidly for large ω. Eq. (4) can be rewritten in a more covariant way using $\sin^2\theta = 1 - \cos^2\theta$ and noting the current conservation relation $\omega j_0 = \omega \cos\theta |\boldsymbol{j}|$, so that Eq. (4) is essentially the integral of the four-vector squared $\boldsymbol{j}^2 - j_0^2 = -j_\mu^2$. Using $dt\ \boldsymbol{v} = d\boldsymbol{x}$ one has a compact line integral expression for j_μ, namely

$$j_\mu(\boldsymbol{k}) = e \int dx_\mu \mathrm{e}^{ikx}\,, \tag{6}$$

where kx is the four dimensional scalar product $k_\nu x_\nu = k_0 t - \boldsymbol{kx}(t)$, with $k_0 = \omega = |\boldsymbol{k}|$.

Squaring Eq. (6) and writing in the constants [5] we get for the integral of $-j_\mu^2(k)$ over all modes of the radiation field:

$$n = -\frac{\alpha}{4\pi^2} \int d^3k \frac{1}{\omega} \int\int dx_\mu dx'_\mu \mathrm{e}^{ik(x-x')}\,. \tag{7}$$

Firm in the belief that we actually have a well defined, convergent expression, we allow ourselves to do the d^3k integral first and to use some $i\varepsilon$ manipulations in dealing with seemingly ill-defined expressions.

After a few steps we get to the expression $\int_0^\infty \omega d\omega \mathrm{e}^{i\omega((t-t')-R\cos\theta)}$, where $R = |\boldsymbol{x} - \boldsymbol{x'}|$ and where here θ is the angle between $\boldsymbol{k}$ and the vector $(\boldsymbol{x} - \boldsymbol{x'})$. Now, since we believe that our expressions are rapidly vanishing at large ω and ω is always positive, it should not hurt to add an $i\varepsilon$ in the exponent [6], to make the replacement $(t - t') \to (t - t' + i\varepsilon)$. The integral is then convergent, and integrating over $d\cos\theta$ and keeping track of the $i\varepsilon$, this leads to $\int d^3k(1/\omega)\mathrm{e}^{ik(x-x')} = -4\pi(1/S_{i\varepsilon}^2)$ and so Eq. (1).

Although these manipulations are similar to those with the invariant D functions in *quantum* field theory, and our function $1/S^2$ resembles a D function, our assumption of smooth paths has lead us to an "$i\varepsilon$ prescription" in position rather than in momentum space.

3. Elimination of ε

The difficult job now becomes the evaluation of the $i\varepsilon$ in Eq. (1). We look for guidance to the theory of generalized functions [6] where one has the relation

$$\int\limits_{-\infty}^{+\infty} \frac{f(t)}{(t+i\varepsilon)^2} dt = \int\limits_{-\infty}^{+\infty} \frac{f(t) - f(0)}{t^2} dt\,, \tag{8}$$

where we have specialized to the case of an even function $f(t) = f(-t)$.

One way of understanding this relation is to note that, for ε non-zero,

$$\int\limits_{-\infty}^{+\infty} \frac{1}{(t+i\varepsilon)^2} dt = 0 \,, \tag{9}$$

which just follows from explicit integration. Thus we have simply subtracted zero in Eq. (8), and have chosen the coefficient of this zero term in such a way as to cancel the singularity of the integrand. With f even, $f(t) - f(0) \sim t^2$, and the expression is indeed finite at the singularity.

Can we apply the same idea here? That is, by examining the neighborhood of the singularity for small but non-zero ε we see that our expression is finite and ε independent. We would thus like to subtract something which is zero for finite ε and which will regulate the singularity at $t = t'$ in Eq. (1), enabling us to finally dispense with ε altogether. The problem, however, would seem to be much more difficult than in Eq. (8). There we simply had to adjust one constant, namely $f(0)$. Here we need a function $f(t,t')$, such that when it is integrated over $1/S^2(t,t')$ gives zero and then can be adjusted to cancel the numerator $dx_\mu dx'_\mu$ when we make the replacement $dx_\mu dx'_\mu \to dx_\mu dx'_\mu(1-f)$. Furthermore it should do this for *any* path $\boldsymbol{x}(t)$ we care to put in the formula.

This last requirement rules out, for example, the at first likely-looking candidate replacement $dx_\mu dx'_\mu = u_\mu u'_\mu d\tau d\tau' \to (u_\mu u'_\mu - 1) d\tau d\tau'$. Since $u_\mu u'_\mu \to 1$ for $\tau \to \tau'$ this would regulate the singularity and looks promising. But it doesn't work, $\int\int d\tau d\tau'\ 1/S^2(\tau,\tau')$ is obviously path dependent and cannot be zero for all paths.

This, however, suggests a solution. If we can find something for f which is a total derivative and so can be integrated, it would depend only on the end points of the path and not on the path itself. To this end, note the following useful relation. Let G be some function of $S^2(\tau,\tau') = \Delta_\mu^2$ where $\Delta_\mu = x_\mu(\tau) - x_\mu(\tau')$, then

$$\partial_\tau \partial_{\tau'} G(S^2) = -G'' 4 \left(\Delta_\mu \frac{dx_\mu}{d\tau}\right)\left(\Delta_\nu \frac{dx_\nu}{d\tau'}\right) - 2G' \frac{dx_\mu}{d\tau}\frac{dx_\mu}{d\tau'} \,, \tag{10}$$

where G' and G'' refer to first and second derivatives with respect to S^2.

Since the expression is a total derivative and its integral is independent of the path and is to be interpreted as zero, we may look for some choice for G that leads to $\sim 1/S^2$ for $\tau \to \tau'$ and so might cancel the singularity. This occurs, in fact, if we consider $G = \ln S^2$. The above equation becomes

$$\partial_\tau \partial_{\tau'} \ln(S^2) = +\frac{4}{S^4}\left(\Delta_\mu \frac{dx_\mu}{d\tau}\right)\left(\Delta_\nu \frac{dx_\nu}{d\tau'}\right) - \frac{2}{S^2}\frac{dx_\mu}{d\tau}\frac{dx_\mu}{d\tau'} \,. \tag{11}$$

Since $\Delta_\mu/S \to dx_\mu/d\tau$ for $\tau \to \tau'$ the expression $\sim 1/S^2$ at the singularity. Now we want to add this to Eq. (2), that is to $\int dx_\mu/d\tau\, dx_\mu/d\tau'(1/S^2)\, d\tau d\tau'$, with some weight such that the singularity at $\tau = \tau'$ is canceled. To determine this weight note that since the four velocity satisfies $(dx_\mu/d\tau)^2 = 1$ the rhs of Eq. (11) goes to $+2/S^2$ for $\tau \to \tau'$. Hence we must subtract 1/2 of Eq. (11) from Eq. (2) to remove the singularity. Doing this we end up with the following nice relation

$$\int\int \frac{dx_\mu}{d\tau}\frac{1}{S^2_{i\varepsilon}}\frac{dx_\mu}{d\tau'}\; d\tau d\tau' = 2\int\int \frac{dx_\mu}{d\tau}\frac{\delta_{\mu\nu} - \left(\frac{\Delta_\mu \Delta_\nu}{S^2}\right)}{S^2}\;\frac{dx_\nu}{d\tau'}\; d\tau d\tau' \tag{12}$$

or alternatively

$$\int\int dx_\mu \frac{1}{S^2_{i\varepsilon}} dx'_\mu = 2\int\int dx_\mu \frac{\delta_{\mu\nu} - \left(\frac{\Delta_\mu \Delta_\nu}{S^2}\right)}{S^2}\; dx'_\nu\,. \tag{13}$$

The notation is meant to indicate that while the $i\varepsilon$ is in S^2 on the left, it is not needed on the right.

We can rewrite these equations in an interesting way by noting that Δ_μ/S is like an average 4-velocity connecting the points τ, τ', which we might call U_μ in analogy to the instantaneous 4-velocity $u_\mu = dx_\mu/d\tau$

$$U_\mu(\tau,\tau') = \frac{\Delta_\mu}{S} = \frac{x_\mu(\tau) - x_\mu(\tau')}{S}\,. \tag{14}$$

Then we can write

$$\begin{aligned} n \;&=\; \frac{\alpha}{\pi}2\int\int u_\mu \frac{\delta_{\mu\nu} - \left(\frac{\Delta_\mu \Delta_\nu}{S^2}\right)}{S^2}\; u_\nu\; d\tau d\tau' \\ &=\; \frac{\alpha}{\pi}2\int\int \frac{u(\tau)\; u(\tau') - (U\; u(\tau))(U\; u(\tau'))}{S^2}\; d\tau d\tau'\,. \end{aligned} \tag{15}$$

We can now introduce the notion of a "transverse vector" u^{T} associated with the points x, x' at τ, τ'. This is the vector u with the "longitudinal" part, that is the component along U, removed

$$\begin{aligned} u^{\mathrm{T}}_\mu(\tau,\tau') \;&=\; u_\mu(\tau) - U_\mu(U\; u(\tau))\,, \\ u^{\mathrm{T}}_\mu(\tau',\tau) \;&=\; u_\mu(\tau') - U_\mu(U\; u(\tau'))\,. \end{aligned} \tag{16}$$

With these definitions, $u^{\mathrm{T}}_\mu(x,x')\,u^{\mathrm{T}}_\mu(x',x) = u_\mu(x)\,(\delta_{\mu\nu} - U_\mu U_\nu)\,u_\nu(x')$, which is the numerator in Eq. (15) and we can write the above equations in terms of products of "transverse vectors"

$$n = \frac{\alpha}{\pi}\, 2 \int\!\!\int u^{\mathrm{T}}_\mu(\tau,\tau')\,\frac{1}{S^2}\,u^{\mathrm{T}}_\mu(\tau',\tau)\; d\tau d\tau' \,. \tag{17}$$

Hence we can say that our integral for n represents the "interaction" of pairs of transverse vectors along the path of the charge. The possible singularity at $S = 0$ is absent because $u^{\mathrm{T}}(\tau,\tau')$ vanishes for two points very close together.

4. Properties of the quantities

We summarize some properties of these quantities: U and u are unit vectors, $u^2 = U^2 = 1$. From their definitions U and u^{T} are orthogonal $u^{\mathrm{T}}_\mu(\tau,\tau')U_\mu = u^{\mathrm{T}}_\mu(\tau',\tau)U_\mu = 0$. Note for straight line motion or more generally for any smooth path as $\tau \to \tau'$

$$U_\mu \to u_\mu \,. \tag{18}$$

Unlike u and U, u^{T} is a space-like (or zero) vector. This follows from the fact that there is a frame where the time component of u^{T} vanishes, namely the rest-frame of the time-like U.

We note a point concerning the definition of U in Eq. (14). This point is irrelevant in all those expressions where S or U appears quadratically, but we mention it for consistency. Namely, we would like U to resemble the velocity. Therefore, it must be understood that $S(\tau,\tau') = \sqrt{S^2}$ is "directed"; an odd function, positive for $\tau > \tau'$ and negative for $\tau < \tau'$, like $(\tau - \tau')$. It implies a definition of U such that $U(\tau,\tau') = U(\tau',\tau)$ and that U is "forward pointing", *i.e.* U_0 is always positive.

An interesting expression for u^{T} follows from the definition of U in Eq. (14), reflecting the fact that the derivative of a unit vector is transverse to itself

$$\begin{aligned} \frac{1}{S}\; u^{\mathrm{T}}_\mu(\tau,\tau') &= \partial_\tau U_\mu(\tau,\tau') \,, \\ -\frac{1}{S}\; u^{\mathrm{T}}_\mu(\tau',\tau) &= \partial_{\tau'} U_\mu(\tau,\tau') \,. \end{aligned} \tag{19}$$

With this, we can also write Eq. (17) in some different looking ways

$$n = -\frac{\alpha}{\pi}\; 2 \int\!\!\int \partial_\tau U_\mu(\tau,\tau')\partial_{\tau'} U_\mu(\tau,\tau')\; d\tau d\tau' \,. \tag{20}$$

Or, since $U^2 = 1$ and so $\partial_\tau \partial_{\tau'} (U_\mu U_\mu) = 0$ we can also write

$$n = \frac{\alpha}{\pi}\, 2 \int\!\!\int U_\mu(\tau,\tau')\ \partial_\tau\ \partial_{\tau'} U_\mu(\tau,\tau')\ d\tau d\tau'\,. \tag{21}$$

Another variant results if we note that Eq. (20) looks like the cross terms of a quadratic expression: $2\partial_\tau U \partial_{\tau'} U = \frac{1}{2}[(\partial_\tau U + \partial_{\tau'} U)^2 - (\partial_\tau U - \partial_{\tau'} U)^2]$. Introducing the sum and difference variables $\tau_+ = \frac{1}{2}(\tau + \tau')$ and $\tau_- = \frac{1}{2}(\tau' - \tau)$

$$n = \frac{\alpha}{\pi} \int\!\!\int\ [(\partial_{\tau_-} U)^2 -\ (\partial_{\tau_+} U)^2]\ d\tau_+ d\tau_-\,, \tag{22}$$

where because of the rotation of the coordinates in the τ, τ' plane, the integration boundaries, in the case if finite limits, would now have the form of a diamond instead of a square.

5. Infrared behavior

The $i\varepsilon$ takes care of the high frequency or ultraviolet behavior, but there might be questions concerning the infrared region. We see this very simply if we consider the stationary particle with $\boldsymbol{v} = 0$, for which n of course should be zero. Eq. (1) leads to

$$n = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dt dt'\, \frac{1}{(t - t' + i\varepsilon)^2}\,. \tag{23}$$

There are two kinds of limits implied here $\varepsilon \to 0$, and some upper/lower limit of the integration $L \to \infty$, and the integral can depend on how the limits are taken. If we simply apply Eq. (9) we get of course zero, as desired. On the other hand if we interpret the limits at $\pm\infty$ by, say, integrating over some test function which is constant up to some large number L and then drops off, we can get an answer involving εL, which depends on the order in which we take $\varepsilon \to 0$, $L \to \infty$. The problem originates in the fact that we are turning off the charge at some L, violating charge conservation and in the process producing some photons. We are thus to handle this limit by remembering that ε is to be kept finite to the very end, and so we first mean $\varepsilon L \to \infty$. Alternatively one can subtract the integral for the stationary particle, as in Eq. (23) to regularize the infrared behavior.

Naturally if we use the form without the $i\varepsilon$, say from Eq. (15) we immediately get zero for the stationary particle. More generally, we can investigate the contribution of the final and initial long straight paths for a moving particle in Eq. (15). The dangerous regions for the infrared behavior are large

τ, τ' as the particle comes from or goes to infinity. For both τ's very large or both very small, we have in view of Eq. (18) and $u^2 = 1$ that the numerator becomes $u^2 - u^2u^2 = 0$. Similarly if one τ is at very early times and the other a very late times and the velocities at these times are the same, there is the analogy to Eq. (18)

$$U_\mu(\tau, \tau') = u_\mu + \mathcal{O}\left(\frac{T}{\tau - \tau'}\right) , \tag{24}$$

where T is the finite time period where the charge was not in uniform motion. For large $(\tau - \tau')$, $U \to u$ and the numerator will again tend to zero, as long as the initial and final velocities are the same. It is interesting that the elimination of both the long and short distance singularities can in a sense be attributed to the same relation, Eq. (18).

6. Non-relativistic limit

Consider that class of paths whose tangents are roughly parallel to some straight line on Fig. (1), meaning that the ordinary velocity $\boldsymbol{v}$ is always close to some typical or average velocity. By making a Lorentz transformation (under which n is invariant) so that this typical velocity is zero, we have as a first approximation in the ordinary velocity the non-relativistic limit, leading to expressions quadratic in velocities.

We could make a stab at the non-relativistic limit by taking our basic expression Eq. (1) and just naively expanding it for small velocities. We introduce $\boldsymbol{V}$, the average velocity vector connecting two points on the curve

$$\boldsymbol{V} = \frac{\boldsymbol{x}(t) - \boldsymbol{x}(t')}{t - t'} , \tag{25}$$

$\boldsymbol{V}$ is a symmetric quantity $\boldsymbol{V}(t, t') = \boldsymbol{V}(t', t)$, in parallel with our earlier definition of U and becomes equal to $\boldsymbol{v}$ when $t \to t'$. $\boldsymbol{V}$ plays a role analogous to U except that it does not have a fixed length and so V^2 has non-zero time derivatives. Expanding in Eq. (1) we can try to write

$$\frac{1}{S_{i\varepsilon}^2} \approx \frac{1}{(t - t' + i\varepsilon)^2} \left(1 + V_{i\varepsilon}^2(t, t') + V_{i\varepsilon}^4(t, t') + ...\right) , \tag{26}$$

where $\boldsymbol{V}_{i\varepsilon} = \frac{(\boldsymbol{x} - \boldsymbol{x}')}{(t - t' + i\varepsilon)}$ so that

$$\begin{aligned} n &= \frac{\alpha}{\pi} \iint dx_\mu \frac{1}{S_{i\varepsilon}^2} dx'_\mu \\ &\approx \frac{\alpha}{\pi} \iint dt dt' (1 - \boldsymbol{v}(t)\boldsymbol{v}(t')) \frac{1}{(t - t' + i\varepsilon)^2} \left(1 + V_{i\varepsilon}^2(t, t') + V_{i\varepsilon}^4(t, t') + ...\right) . \end{aligned} \tag{27}$$

Now, using Eq. (9) the "1" term vanishes and we have

$$n = \frac{\alpha}{\pi}\int\int dt dt' \frac{1}{(t-t')^2}[V^2 - \boldsymbol{v}(t)\boldsymbol{v}(t')] \qquad \text{(non − rel. limit)}\,. \quad (28)$$

We have dropped the $i\varepsilon$ since the expression is non-singular with $\boldsymbol{V} \to \boldsymbol{v}$ as $t \to t'$.

This is a not unreasonable-looking expression. Indeed, if we expand it for $t \approx t'$ we get

$$\left[V^2 - \boldsymbol{v}(t)\boldsymbol{v}(t')\right] \approx \frac{1}{4}a^2(t-t')^2\dots\,, \quad (29)$$

showing that the singularity is canceled and that the leading terms of the expression are positive and exhibit the familiar connection between acceleration squared and radiation. The $(\dots)$ includes terms involving time derivatives of the acceleration a ($\boldsymbol{a} = d\boldsymbol{v}/dt$) as well as higher order terms in $(t-t')^2$.

One might have some qualms about the carefree $i\varepsilon$ manipulations, and we indicate how to arrive at Eq. (28) by straightforward application of our more conventional formulas.

First note that U can be written in terms of V in the usual way relating a three-velocity and a four-velocity, $U_0 = 1/\sqrt{1-V^2}$, $\boldsymbol{U} = \boldsymbol{V}/\sqrt{1-V^2}$ with the non-relativistic limits $U_0 \approx 1 - 1/2V^2$, $\boldsymbol{U} \approx \boldsymbol{V}$. We find from either Eq. (20) or Eq. (21) to leading order in V

$$n = \frac{\alpha}{\pi}2\int\int dt dt'\ \partial_t \boldsymbol{V}(t,t')\ \partial_{t'}\boldsymbol{V}(t,t') \qquad \text{(non − rel. limit)}\,. \quad (30)$$

Now in analogy to Eq. (19) we have from the definition of $\boldsymbol{V}$

$$\begin{aligned} \partial_t \boldsymbol{V}(t,t') &= \frac{\boldsymbol{v}(t) - \boldsymbol{V}}{t-t'}\,, \\ \partial_{t'} \boldsymbol{V}(t,t') &= -\frac{\boldsymbol{v}(t') - \boldsymbol{V}}{t-t'} \end{aligned} \quad (31)$$

and Eq. (30) becomes

$$n = \frac{\alpha}{\pi}2\int\int dt dt'\left[-\boldsymbol{v}(t)\boldsymbol{v}(t') + (\boldsymbol{v}(t) + \boldsymbol{v}(t'))\boldsymbol{V} - V^2\right]\frac{1}{(t-t')^2}\,.$$

$$\text{(non − rel. limit)}\,. \quad (32)$$

One could be satisfied with this formula as it is, but to bring it into the perhaps simpler form Eq. (28), note the following identity: $\frac{1}{2}\partial_t\partial_{t'}V^2 = [-\boldsymbol{v}(t)\boldsymbol{v}(t')+2(\boldsymbol{v}(t)+\boldsymbol{v}(t'))\boldsymbol{V}-3V^2](t-t')^{-2}$, which follows from differentiating $V^2 = (\boldsymbol{x}-\boldsymbol{x}')^2(t-t')^{-2}$ twice. We split this into two parts: $\frac{1}{2}\partial_t\partial_{t'}V^2 = [-\boldsymbol{v}(t)\boldsymbol{v}(t')+(\boldsymbol{v}(t)+\boldsymbol{v}(t'))\boldsymbol{V}-V^2](t-t')^{-2}+[(\boldsymbol{v}(t)+\boldsymbol{v}(t'))\boldsymbol{V})-2V^2](t-t')^{-2}$, where we make the split so that the first part corresponds to Eq. (32). Now $\partial_t\partial_{t'}V^2$ is a total derivative whose integral may be set to zero. Therefore, the integral of the first and second parts represents the same quantity with opposite signs and we can write

$$\begin{aligned} n &= \frac{\alpha}{\pi}2\int\int dtdt'[-\boldsymbol{v}(t)\boldsymbol{v}(t')+(\boldsymbol{v}(t)+\boldsymbol{v}(t'))\boldsymbol{V}-V^2]\frac{1}{(t-t')^2} \\ &= -\frac{\alpha}{\pi}2\int\int dtdt'[(\boldsymbol{v}(t)+\boldsymbol{v}(t'))\boldsymbol{V})-2V^2]\frac{1}{(t-t')^2} \\ &\qquad\qquad\qquad\qquad (\mathrm{non-rel.\ limit})\,. \qquad (33)\end{aligned}$$

Taking the one-half the sum of the two forms we finally obtain, after much labor, Eq. (28). The cavalier $i\varepsilon$ manipulations were certainly a lot quicker!

Despite the familiar acceleration squared in Eq. (29), we shouldn't expect that n can be represented simply by an integral of some local quantity along the path. There is the $(\ldots)$, and all our expressions are bi-local in the time. The photon is a non-local concept and a certain space-time interval is necessary to define it. This corresponds to the distinctive property of relativistic local field theory that while one has local constructions for quantities like charge density or energy density, there is in fact no local quantity for particle number or photon density. Indeed, the need to have some space-time interval to define a particle, leads to the concept of the "formation zone" [3], which can be used to understand certain phenomena like the absence of "cascading" for particle production on nuclear targets.

7. Simple cases

With the simple Eq. (28) in hand we can proceed to calculate a couple of concrete examples.

Dipole radiation: We first take the classic problem of dipole radiation. Let a charged particle be oscillating in one dimension according to $x = x_0 \sin \Omega t$, so $v = x_0\Omega\cos\Omega t$. Changing variables $\Omega t \to t$ and similarly for t', Eq. (28) becomes

$$n = \frac{\alpha}{\pi}(x_0\Omega)^2\int\int dtdt'\left[\frac{(\sin\,t-\sin\,t')^2}{(t-t'+i\varepsilon)^4}-\frac{\cos\,t\,\ \cos\,t'}{(t-t'+i\varepsilon)^2}\right]. \qquad (34)$$

Although the $i\varepsilon$ is not necessary since the combination of the two terms gives something non-singular, it is convenient to keep it since it allows us to handle each term separately. Carrying out, say, the t integral first and using relations of the type $\int dt \cos t/(t-t'+i\varepsilon)^2 = 2\pi \mathrm{e}^{-it'}$ or $\int dt \sin t/(t-t'+i\varepsilon)^4 = \pi(i/6)\mathrm{e}^{-it'}$ and $\int dt \cos t'/(t-t'+i\varepsilon)^2 = 0$, we arrive at

$$\begin{aligned} n &= \frac{\alpha}{\pi}(x_0\Omega)^2 \int dt' \left[\frac{-i\pi}{3}\sin t' - (-\pi)\cos t'\right] \mathrm{e}^{-it'} \\ &= \alpha(x_0\Omega)^2 \frac{T\Omega}{3}\,, \end{aligned} \tag{35}$$

where T is the length of time the particle is in motion. The number of photons generated increases linearly with time, as was to be expected. We have neglected a contribution, not proportional to T, connected with turning the motion on and off. If we now multiply by Ω to find the energy and divide by T to get the power, we obtain $Power = \alpha\frac{1}{3}(x_0\Omega)^2\Omega^2$, which is the classical formula for dipole radiation averaged over a cycle [4, 5]. Not surprisingly we recover the classical result, as was our starting point.

Note, however, that in general the energy radiated and the number of photons do not stand in direct relation since the oscillating charge produces higher harmonics in addition to the fundamental at frequency Ω [4]. However, in the nonrelativistic limit $v/c \to 0$ these higher harmonics become negligible (since they are a retardation effect), and we expect the energy radiation to be simply proportional to n.

This example shows that despite our requirement that the initial and final velocities be equal, the method need not be of purely academic interest for practical calculations. If the effects connected with turning the motion on and off are negligible compared to some main effect, we can always return the particle to, say, zero velocity, while retaining the main effect.

Smooth deflection: Instead of an oscillator which is on for a long time we can consider a charge undergoing a smooth deflection, for example $x = x_0/(1+(t/t_0)^2)$ so that $v = -2(x_0/t_0)\,(t/t_0)\,1/(1+(t/t_0)^2)$. This leads to

$$\begin{aligned} n &= \frac{\alpha}{\pi}\int\int dt\,dt'\,\frac{1}{(t-t'+i\varepsilon)^2}\left[V^2 - \boldsymbol{v}(t)\boldsymbol{v}(t')\right] \\ &= \frac{\alpha}{\pi}\left(\frac{x_0}{t_0}\right)^2\left(\frac{-1}{8}\pi^2 - \frac{-3}{8}\pi^2\right) = \frac{\alpha\pi}{4}\left(\frac{x_0}{t_0}\right)^2 . \end{aligned} \tag{36}$$

8. Further questions

Eq. (1) should have some general symmetry properties with respect to changing the path. There are the evident invariances under Lorentz transformations, translation, 3D rotation, reflection, and time-reversal. Since there are no dimensional quantities except the path itself involved, there is also an invariance under rescaling of all 4-coordinates simultaneously, $x_\mu \to \lambda x_\mu$. That is to say, if the path is expanded in space and time proportionally so that the velocities remain unchanged, n is unchanged, as we see in the examples. It would be interesting to know if there are further invariances and what the full invariance group is.

Also, we might consider the problem for gravitons instead of photons. Presumably one will find, in analogy to Eq. (2), and in view of the tensorial character of the source of gravitons

$$n \sim Gm^2 \int\int u_\mu(\tau)u_\nu(\tau)\frac{1}{S^2_{i\varepsilon}}u_\mu(\tau')\mu_\nu(\tau')\ d\tau d\tau'\ ,$$

where G is the gravitational constant and m the mass of the radiating particle (recall $\hbar, c = 1$, so $Gm^2 = (m/M_{\rm pl})^2$), but it would be interesting to investigate this more closely.

Finally, a number of interesting mathematical problems suggest themselves. Our n gives an invariant characterization of the "wiggly-ness" of a curve. There appears to be no reason why it should not be also used in Euclidean space where $S^2(\tau,\tau') = \Sigma(x_i(\tau) - x_i(\tau'))^2$. The main difference would seem to be that the curve can now go "backwards", opening the possibility of closed curves. As well, there is the possibility of new singularities when two parts of the curve, with remote values of τ, come close together.

For a closed plane curve it is plausible that the minimum value of n obtains for the circle. Then defining $n = -2\int\int \frac{1}{S^2}\boldsymbol{u}^{\rm T}\boldsymbol{u}'^{\rm T} d\tau d\tau'$ (we use Eq. (17), leave away α/π, and the natural sign is now minus), we thus conjecture that the minimum value of n for any plane curve is $2\pi^2$, which is what we obtain for the circle.

Also, there may be a "topological" aspect to n, connected with knots. For an open path the minimum value of n, namely zero, is reached for a straight line. If the path has a knot, however, it cannot be continuously deformed to a straight line, and must go "backwards" somewhere, suggesting that the minimum value of n continuously attainable is related to the presence of knots.

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REFERENCES

[1] We use a notation where in the 4-vector product such as $dx_\mu dx'_\mu$ the time component is positive: $dx_\mu dx'_\mu = dtdt' - \boldsymbol{dx dx'}$, where boldface stands for 3-vectors.

[2] L. Stodolsky, *Decoherence in the Radiation Field*, in preparation.

[3] L. Stodolsky, *Formation Zone Description in Multiproduction*, Proc. 7th Int. Colloquium on Multiparticle Reactions, Oxford, 1974. The concept was first introduced by L. Landau, I. Pomeranchuk, *Doklady Akademi Nauk SSR* **92**, 535 (1953); *Doklady Akademi Nauk SSR* **92**, 735 (1953).

[4] J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, 1965.

[5] See L.D. Landau, E M. Lifshitz, *The Classical Theory of Fields*, Pergamon Press, 1962. For the manipulations leading to Eq. (7) see the problem at the end of Section 66.

[6] The introduction of the $i\varepsilon$ amounts to the use of the ω_+ function of the theory of generalized functions. Indeed it would be the *same* relation were we to make the non-relativistic approximation $S^2 \approx (t-t')^2$. See *Verallgemeinerte Funktionen* vol. I, by I.M. Gelfand and G.E. Schilow, VEB Deutscher Verlag der Wissenschaften, Berlin 1967. See Section 4, and in particular the formula $\int_0^\infty d\omega\omega e^{i\sigma\omega} = -\frac{2}{(\sigma+i\varepsilon)^2}$, (appendix, table of Fourier transforms, no. 21).