POISSON REDUCTION, POISSON BIALGEBRAS AND COMPLETE INTEGRABILITY

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We consider Poisson bialgebras on symplectic leaves of a Poisson manifold. New classes of completely integrable Hamiltonian systems with arbitrary many degrees of freedom are presented. Their Hamiltonians are defined as the k^{th} coproduct of arbitrary smooth functions on symplectic foliations. We also consider modifications of the Poisson bialgebras by introducing the deformed coproduct and the deformed Poisson tensor.

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1. Introduction

One of the main feature of dynamical systems is their nonintegrability. The study of completely integrable Hamiltonian systems started with the pioneering work of Liouville on finding local solutions by quadratures. Integrable systems were typically discovered by chance or through techniques specifically prepared for the particular problems. After Poincaré had recognized that integrability is an exceptional phenomenon of Hamiltonian systems and began the study of their qualitative properties, the interest in integrable Hamiltonian systems vanished. Integrable Hamiltonian systems play a fundamental role in the study and description of physical systems, due to their many interesting properties, both from the mathematical and physical points of view. Indeed, beyond the obvious interest of finding first integrals, the concept of integrability seems necessary for more thorough understanding of the nonintegrability phenomenon. To date, however, there exists no general method for determining whether or not a given system is integrable. Even in the simplest nontrivial case, *i.e.*, in the two-degree of freedom Hamiltonian system our knowledge is far from the desired goal.

In recent years there has been a renewed interest in completely integrable Hamiltonian systems, specially in conjunction with the study of quantum integrable systems and quantum groups. Integrable Hamiltonian systems have always a hidden algebraic structure that is responsible for their integrability. Therefore, the most fascinating problem in the study of dynamical systems is to give such general algebraic structure which provide a hidden treasure. In a recent paper Ballesteros and Ragnisco [1] have proposed a beautiful idea for proving the complete integrability of a large collection of Hamiltonian systems by using coproducts in Poisson Hopf algebras. This construction was put into a geometrical perspective in Refs. [2–4].

This paper presents a procedure in order to construct complete integrable Hamiltonian systems with arbitrary many degrees of freedom from a Poisson bialgebra $(\mathcal{F}(\mathcal{L}_\delta), \Lambda, \triangle)$ on symplectic leaves $\mathcal{L}_\delta$ of a Poisson manifold (R^3, Λ) with the Poisson tensor $\Lambda = (\alpha x_1 - \beta/2)\partial_2 \wedge \partial_3 + x_2 \partial_3 \wedge \partial_1 + x_3 \partial_1 \wedge \partial_2$. This paper is organized as follows. Section 2 contains the reduction of a Poisson manifold (M, Λ) by a Casimir function $\mathcal{C}$. In Section 3, the basic definitions of Poisson bialgebras are reviewed. In Section 4, starting with a Poisson manifold we construct a Poisson bialgebra. This bialgebra defines a family of completely integrable Hamiltonian systems with two degrees of freedom. In Section 5, we give an example of a Poisson structure $\Lambda = (\alpha x_1 - \beta/2)\partial_2 \wedge \partial_3 + x_2 \partial_3 \wedge \partial_1 + x_3 \partial_1 \wedge \partial_2$, and show how it can be used to construct new families of integrable Hamiltonian systems. In Section 6, we generalize the results obtained in the previous Section. In the last Section some conclusions are drawn.

2. Poisson reduction

Let M be a finite-dimensional differentiable manifold and let $\mathcal{F}(M)$ be a space of smooth functions on M. A Poisson structure on a manifold M is a skew-symmetric bilinear map defined by

$$\{\ ,\ \} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$$

such that for every $\phi, \varphi, \psi \in \mathcal{F}(M)$ we have

(i) $\{\{\phi, \varphi\}, \psi\} + \{\{\varphi, \psi\}, \phi\} + \{\{\psi, \phi\}, \varphi\} = 0,$

(ii) $\{\phi, \varphi\psi\} = \{\phi, \varphi\}\psi + \{\phi, \psi\}\varphi,$

(iii) $\{\phi, \varphi\} = -\{\varphi, \phi\}.$

The pair $(M, \{\ ,\ \})$ is called a Poisson manifold, and conditions *(i)* – *(iii)* make $(\mathcal{F}(M), \{\ ,\ \})$ into a Poisson algebra. The local expression for the Poisson bracket is

$$\{\phi, \varphi\} = \{x_a, x_b\} \frac{\partial \phi}{\partial x_a} \frac{\partial \varphi}{\partial x_b}, \tag{1}$$

where summation on repeated indices is understood. The expression for $\{\phi, \varphi\}$ defines a twice covariant skew-symmetric tensor Λ by

$$\Lambda(d\phi, d\varphi) = \{\phi, \varphi\}_\Lambda \tag{2}$$

then

$$\Lambda = \Lambda_{ab} \frac{\partial}{\partial x_a} \wedge \frac{\partial}{\partial x_b}. \tag{3}$$

The bivector Λ, called a Poisson tensor, will play the crucial role in this paper. If $\psi \in \mathcal{F}(M)$, the associated Hamiltonian vector field takes the form $\mathcal{X}_\psi = \Lambda(d\psi)$. Poisson structures for which the rank of Λ is everywhere equal to the dimension of M is called symplectic with symplectic structure ω, the inverse of the tensor Λ. For the case of a degenerate Poisson structure there will exist nonconstant Casimir functions $\mathcal{C}$, such that

$$\Lambda(d\psi, d\mathcal{C}) \equiv 0 \qquad \forall\, \psi \in \mathcal{F}(M). \tag{4}$$

By the Symplectic Stratification Theorem [5] any Poisson manifold is partitioned into symplectic leaves and, therefore, is a natural setting for the study of families of Hamiltonian systems. A Casimir is constant along each leaf, and the symplectic leaves are exactly common level manifolds of the Casimir functions.

Since Poisson structures correspond to possibly degenerate bivector fields, one might hope for a theory which also includes degenerate 2-forms. This is provided by the theory of Dirac structures [6]. These are subbundles of a direct sum $TM \oplus T^*M$ which are maximal isotropic for a natural symmetric bilinear form and which are closed under a bracket discovered by Courant [6] and which has become the prototype for an object known as a *Courant algebroid* [7]. Let (P, ω) be a symplectic manifold. A coisotropic submanifold of the phase space is called a *first-class constraint set.* A submanifold $N \subset P$ is called a *second-class constraint set* if the symplectic form ω restricted to N is nondegenerate. These terms are consistent with Dirac's terminology [8].

Assume $M = R^3$, with coordinates x_1, x_2, x_3, a Poisson structure (3) is defined by the component functions $\Lambda_{ab} = \{x_a, x_b\}$ satisfying the identity $\Lambda_{ab} = -\Lambda_{ba}$ and $\Lambda_{da}(\partial\Lambda_{cb}/\partial x_d) + \Lambda_{db}(\partial\Lambda_{ac}/\partial x_d) + \Lambda_{ba}(\partial\Lambda_{ba}/\partial x_d) = 0$. One can easily check that the Casimir $\mathcal{C}$ for the Poisson structure (3) obeys the condition

$$\Lambda_{ab} \frac{\partial \mathcal{C}}{\partial x_b} = 0\,. \tag{5}$$

Suppose $\mathcal{C} : R^3 \to R^1$ is a submersion, then $\mathcal{C}^{-1}(\delta) = \mathcal{L}_\delta,\ \delta \in R^1$, is a submanifold of R^3 with codimension one. Let $\{\mathcal{L}_\delta\}$ be a regular foliation defined by $\mathcal{C}$ and let (U, Ψ) be a distinguished chart at $x \in R^3$. Then $\Psi : U \to R^2 \times R^1 : (x_1, x_2, x_3) \mapsto (p, q, z)$, where z is constant on each leaf $\mathcal{L}_\delta$.

The submanifolds $\mathcal{L}_\delta$ corresponding to constant value of the distinguished coordinate z are easily seen to be Poisson submanifolds [9], with the natural reduced Poisson bracket with respect to remaining coordinates p, q. Since a Poisson structure is determined by its local character, we can assume that flat local coordinates p, q, z with $\mathcal{L}_\delta = \{(p, q, z)|z = 0\}$. Let $\tilde{\psi} : \mathcal{L}_\delta \to R^1$ be any smooth function. Then we can extend $\tilde{\psi}$ to a smooth function $\psi : R^3 \to R^1$, defined in U, with $\tilde{\psi} = \psi|\mathcal{L}_\delta$. In the local coordinates $\tilde{\psi} = \psi(p, q, 0)$. If $\tilde{\varphi} : \mathcal{L}_\delta \to R^1$ has similar extension ψ, then the Poisson bracket of $\tilde{\varphi}$ and $\tilde{\psi}$ is defined by restriction $\{\varphi, \psi\}$ to $\mathcal{L}_\delta$

$$\{\tilde{\varphi}, \tilde{\psi}\}_{\mathcal{F}} := \{\tilde{\varphi}, \tilde{\psi}\}_{\tilde{\Lambda}} = \{\varphi, \psi\}_{\Lambda}|\mathcal{L}_\delta, \tag{6}$$

with $\tilde{\Lambda}_{ab} = \Lambda_{ab}(p, q, 0)$. Since $\{\mathcal{C}, \psi\}_\Lambda = 0$ for each $\psi \in \mathcal{F}(R^3)$, it follows

$$\{\tilde{\mathcal{C}}, \tilde{\psi}\}_{\mathcal{F}} = \{\mathcal{C}, \psi\}_{\Lambda}|\mathcal{L}_\delta \equiv 0\,. \tag{7}$$

A submanifold $\mathcal{L}_\delta$ is defined by the mapping

$$\tilde{x}_1 = \tilde{x}_1(p, q), \qquad \tilde{x}_2 = \tilde{x}_2(p, q), \qquad \tilde{x}_3 = \tilde{x}_3(p, q)\,. \tag{8}$$

From (6) it follows that the Poisson structure on a symplectic leaf $\mathcal{L}_\delta$ is given by $\{\tilde{x}_a, \tilde{x}_b\}_{\tilde{\Lambda}}$. In local coordinates on $\mathcal{L}_\delta$ the Poisson structure is

$$\tilde{\Lambda}_{ab} = \frac{\partial \tilde{x}_a}{\partial p}\frac{\partial \tilde{x}_b}{\partial q} - \frac{\partial \tilde{x}_a}{\partial q}\frac{\partial \tilde{x}_b}{\partial p}\,. \tag{9}$$

3. Poisson bialgebras

Let us start with some algebraic preliminaries that will be also useful to establish notation. Detailed exposition of the theory can be found in Refs. [10–12].

A unital associative algebra over K is a linear space A together with two linear maps $m : A \otimes A \to A$ and $\eta : K \to A$ such that

$$m(m \otimes 1) \;=\; m(1 \otimes m)\,, \tag{10}$$

$$m(1 \otimes \eta) \;=\; m(\eta \otimes 1) = \mathrm{id}\,. \tag{11}$$

Here $A \otimes A$ is tensor product of two algebras, 1 is the unit element of A and id means the entity map. The usual notation is simply $ab := m(a \otimes b)$, and we will use such notation.

Let (A_1, m_1, η_1) and (A_2, m_2, η_2) be algebras, then the tensor product $A_1 \otimes A_2$ is naturally endowed with the structure of an algebra. The multiplication $m_{A_1 \otimes A_2}$ is defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)\,. \tag{12}$$

A coalgebra is a triple $(A, \triangle, \epsilon)$ with a linear space A over K, $\triangle : A \to A \otimes A$ a linear map called coproduct and $\epsilon : A \to K$ a linear morphism called counit with property

$$\begin{aligned} \triangle(ab) &= \triangle(a)\triangle(b) \qquad \forall\, a, b \in A\,, & (13)\\ (\triangle \otimes \mathrm{id}) \circ \triangle &= (\mathrm{id} \otimes \triangle) \circ \triangle\,, & (14)\\ (\mathrm{id} \otimes \epsilon) \circ \triangle &= (\epsilon \otimes \mathrm{id}) \circ \triangle = \mathrm{id}\,. & (15)\end{aligned}$$

We note that $A \otimes A$ is both an algebra and coalgebra [10].

A bialgebra $(A, m, \triangle, \eta, \epsilon)$ is a linear space over K with maps $m, \triangle, \eta, \epsilon$ which satisfy all the above properties.

One can define a tensor product of Poisson algebras $\mathcal{F}(\mathcal{L}_\delta) \otimes \mathcal{F}(\mathcal{L}_\delta)$. $\mathcal{F}(\mathcal{L}_\delta) \otimes \mathcal{F}(\mathcal{L}_\delta)$ is again a Poisson algebra structure on vector space $\mathcal{F}(\mathcal{L}_\delta) \otimes \mathcal{F}(\mathcal{L}_\delta)$ with the tensor product algebra structure and the tensor product coalgebra structure. We have to define a Poisson structure on $\mathcal{F}(\mathcal{L}_\delta) \otimes \mathcal{F}(\mathcal{L}_\delta)$ such that the axioms of Poisson algebra are satisfied. For our purpose the maps are defined as follows.

The multiplication $m_{\mathcal{F}(\mathcal{L}_\delta) \otimes \mathcal{F}(\mathcal{L}_\delta)}$

$$(\phi \otimes \varphi)(\chi \otimes \psi) = (\phi\chi) \otimes (\varphi\psi)\,. \tag{16}$$

The primitive coproduct on $\mathcal{F}(\mathcal{L}_\delta)$

$$\triangle(\tilde{x}_a) = \tilde{x}_a \otimes 1 + 1 \otimes \tilde{x}_a\,. \tag{17}$$

We note that as $\triangle$ is a homomorphism, we have $\triangle(\tilde{x}_a^n) = (\triangle(\tilde{x}_a))^n$.

Given the Poisson structure on $\mathcal{F}(\mathcal{L}_\delta)$

$$\{\varphi, \psi\}_{\mathcal{F}} = \{\tilde{x}_a, \tilde{x}_b\}_{\tilde{\Lambda}} \left(\frac{\partial \varphi}{\partial \tilde{x}_a} \frac{\partial \psi}{\partial \tilde{x}_b} \right) \tag{18}$$

one defines the following Poisson bracket on $\mathcal{F}(\mathcal{L}_\delta) \otimes \mathcal{F}(\mathcal{L}_\delta)$

$$\{\phi \otimes \varphi, \chi \otimes \psi\}_{\mathcal{F} \otimes \mathcal{F}} = \{\phi, \chi\}_{\mathcal{F}} \otimes \varphi\psi + \phi\chi \otimes \{\varphi, \psi\}_{\mathcal{F}}\,. \tag{19}$$

This is easily seen to give the following condition on the coproduct

$$\{\triangle(\varphi), \triangle(\psi)\}_{\mathcal{F} \otimes \mathcal{F}} = \triangle(\{\varphi, \psi\}_{\mathcal{F}})\,. \tag{20}$$

We will say that the set $(\mathcal{F}(\mathcal{L}_\delta), m, \triangle, \epsilon, \{\ ,\ \}_{\mathcal{F}})$ is a Poisson bialgebra.

4. Complete integrability

Let (P, ω) be a smooth symplectic $2n$-dimensional manifold and let $H : P \to R^1$ a smooth Hamiltonian with associated Hamiltonian vector field $\xi_H : P \to TP$. The Hamiltonian system (P, ω, H) is said to be *completely integrable* if there exist n smooth functions (called *first integrals*) $F_1 = H, \cdots, F_n$ defined on P so that:

(i) $\{F_i, F_j\} = \omega(\xi_{F_i}, \xi_{F_j}) = 0 \qquad i, j = 1, \cdots, n,$

(ii) $F_1, \cdots, F_n$ *are functionally independent almost everywhere in* P.

The integrability in twodegrees-of-freedom Hamiltonian system means that a second integral $F_2 = G$ exists, which is not equal to $H^*\psi$ for any $\psi : R^1 \to R^1$. So if $\mathcal{C}$ is a Casimir function and $\tilde{h}$ is an arbitrary smooth function on $\mathcal{L}_\delta$, then the Hamiltonian system defined by the Hamiltonian $H = \triangle(\tilde{h})$ is completely integrable.

Indeed, assume $\mathcal{C}$ is a Casimir for Λ, then for any $h \in \mathcal{F}(R^3)$

$$\{h, \mathcal{C}\}_\Lambda | \mathcal{L}_\delta = \{\tilde{h}, \tilde{\mathcal{C}}\}_\mathcal{F} \equiv 0\,. \tag{21}$$

Since the coproduct is a Poisson map, (21) gives

$$\{\triangle(\tilde{\mathcal{C}}), \triangle(\tilde{h})\}_{\mathcal{F}\otimes\mathcal{F}} = \triangle(\{\tilde{\mathcal{C}}, \tilde{h}\})_\mathcal{F} \equiv 0\,. \tag{22}$$

If we define

$$\tilde{x}_a(p,q) \otimes 1 = \tilde{x}_a(p_1, p_2) \qquad 1 \otimes \tilde{x}_a(p,q) = \tilde{x}_a(p_2, q_2)\,, \tag{23}$$

or more generally

$$f(p,q) \otimes g(p,q) = f(p_1, q_1) g(p_2, q_2), \tag{24}$$

we get

$$\{\triangle(\tilde{\mathcal{C}}), \triangle(\tilde{h})\}_{\mathcal{F}\otimes\mathcal{F}} = \{F, H\}_\omega\,, \tag{25}$$

with $F = \triangle(\tilde{\mathcal{C}}),\ H = \triangle(\tilde{h})$, and $\omega^{-1} = \{\ ,\ \}_\mathcal{F} \otimes 1 + 1 \otimes \{\ ,\ \}_\mathcal{F}$. Since

$$\{\triangle(h), \triangle(\tilde{\mathcal{C}})\}_{\mathcal{F}\otimes\mathcal{F}} = \{H, F\}_\omega \equiv 0$$

for any $h \in \mathcal{F}(\mathcal{L}_\delta)$, the Hamiltonian system

$$(R^4, \omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2, H = \triangle(\tilde{h}(\tilde{x}(p,q))))$$

is completely integrable if and only if $dF \wedge dH \neq 0$ almost everywhere in R^4. Because $h \in \mathcal{F}(\mathcal{L}_\delta)$ is an arbitrary smooth function, there is some class of functions in $\mathcal{F}(\mathcal{L}_\delta)$ whose coproduct is functionally independent almost everywhere in R^4 of $F = \triangle(\tilde{\mathcal{C}})$.

5. Applications

We specialize our previous considerations to the case of the Poisson manifold (R^3, Λ) with

$$\Lambda = (\alpha x_1 - \beta/2)\partial_2 \wedge \partial_3 + x_2 \partial_3 \wedge \partial_1 + x_3 \partial_1 \wedge \partial_2 \,, \tag{26}$$

where α, β are constants, and $\partial_a = \partial/\partial_a$.

5.1. Case: $\alpha = \beta = 0$

This case was considered in paper [14]. We start with the Lie algebra $e(2)$

$$[e_1, e_2] = e_3 , \qquad [e_2, e_3] = 0 \,, \qquad [e_3, e_1] = e_2 \,. \tag{27}$$

This solvable algebra is of the type VII_0 in Bianchi's classifications and is isomorphic to the Euclidean algebra of the plane. We consider the dual $e(2)^*$ to $e(2)$ equipped with the linear Poisson–Lie structure

$$\Lambda = x_3 \partial_1 \wedge \partial_2 + x_2 \partial_3 \wedge \partial_1 \,, \tag{28}$$

A Casimir for (28) is

$$\mathcal{C} = x_2^2 + x_3^2 \,. \tag{29}$$

Hence

$$\{\tilde{x}_1, \tilde{x}_2\}_{\mathcal{F}} = \tilde{x}_3 \qquad \{\tilde{x}_2, \tilde{x}_3\}_{\mathcal{F}} = 0 \qquad \{\tilde{x}_3, \tilde{x}_1\}_{\mathcal{F}} = \tilde{x}_2 \,. \tag{30}$$

From (30) we obtain

$$\tilde{x}_1 = p \qquad \tilde{x}_2 = \sin q \qquad \tilde{x}_3 = \cos q \,. \tag{31}$$

Finally we get

$$\triangle(\mathcal{C}) = F(p_1, p_2, q_1, q_2) = 1 + \cos(q_2 - q_1) \,. \tag{32}$$

Thus any Hamiltonian system (R^4, ω, H) on the symplectic manifold $(R^4, \omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2)$, with an arbitrary Hamiltonian $H(p_1, p_2, q_1, q_2) = \triangle(\tilde{h}(p, \sin q, \cos q))$ is completely integrable if and only if $dH \wedge dF \neq 0$.

5.2. Case: $\alpha = 0,\ \beta \neq 0$

This case was discussed in paper [3] hence simple calculations will be omitted. One can easily show that for Poisson structure

$$\Lambda = x_3 \partial_1 \wedge \partial_2 + x_2 \partial_3 \wedge \partial_1 - (\beta/2)\, \partial_2 \wedge \partial_3 \,, \tag{33}$$

the Casimir is

$$\mathcal{C} = x_2^2 + x_3^2 - \beta x_1 . \tag{34}$$

The reduced Poisson brackets are fulfilled by the following functions

$$\tilde{x}_1 = \frac{p}{\beta}\,, \quad \tilde{x}_2 = \sqrt{p}\,\sin(\beta q)\,, \quad \tilde{x}_3 = \sqrt{p}\,\cos(\beta q)\,. \tag{35}$$

Using formulas (17), (20) and (35), we obtain

$$\begin{aligned}
\{\triangle(\tilde{x}_1), \triangle(\tilde{x}_2)\}_{\mathcal{F}\otimes\mathcal{F}} &= \triangle(\{\tilde{x}_1, \tilde{x}_2\}_{\mathcal{F}}) = \triangle(\tilde{x}_3) \\
&= \sqrt{p_1}\cos(\beta q_1) + \sqrt{p_2}\cos(\beta q_2)\,, \\
\{\triangle(\tilde{x}_2), \triangle(\tilde{x}_1)\}_{\mathcal{F}\otimes\mathcal{F}} &= \triangle(\{\tilde{x}_2, \tilde{x}_1\}_{\mathcal{F}}) = \triangle(\tilde{x}_3) \\
&= \sqrt{p_1}\sin(\beta q_1) + \sqrt{p_2}\sin(\beta q_2)\,, \\
\{\triangle(\tilde{x}_2), \triangle(\tilde{x}_3)\}_{\mathcal{F}\otimes\mathcal{F}} &= \triangle(\{\tilde{x}_2, \tilde{x}_3\}_{\mathcal{F}}) = -\beta\,.
\end{aligned}$$

In this case

$$\triangle(\tilde{\mathcal{C}}) = (1-\beta)(p_1 + p_2) + 2\sqrt{p_1 p_2}\cos\beta(q_1 - q_2)\,. \tag{36}$$

5.3. Case: $\alpha \neq 0,\ \beta = 0$

Without loss of generality we can assume $\alpha = 1$, then the Poisson structure is generated by the Lie algebra $su(2)$. Therefore the Poisson–Lie structure on $su(2)^* \cong R^3$ is given by the bivector

$$\Lambda = \varepsilon^i_{sk} x_i \partial_s \wedge \partial_k\,, \tag{37}$$

with Levi–Civita tensor ε^i_{sk}. The Casimir function for the Poisson–Lie tensor (37) is

$$\mathcal{C} = x_1^2 + x_2^2 + x_3^2\,. \tag{38}$$

The functions $\tilde{x}_a$ are defined by the partial differential equations

$$\frac{\partial \tilde{x}_i}{\partial p}\frac{\partial \tilde{x}_j}{\partial q} - \frac{\partial \tilde{x}_i}{\partial q}\frac{\partial \tilde{x}_j}{\partial p} = \varepsilon^k_{ij}\tilde{x}_k\,.$$

A small calculation gives

$$\tilde{x}_1 = \sqrt{1-p^2}\cos q\,, \qquad \tilde{x}_2 = \sqrt{1-p^2}\sin q\,, \qquad \tilde{x}_3 = p\,, \tag{39}$$

and the coproduct of $\tilde{\mathcal{C}}$ reads

$$\triangle(\tilde{\mathcal{C}}) = 2 + 2p_1p_2 + 2\sqrt{(1-p_1^2)(1-p_2^2)}\cos(q_1 - q_2)\,. \tag{40}$$

5.4. Case: $\alpha, \beta \neq 0$

In this general case the Poisson tensor is

$$\Lambda = (\alpha x_1 - \beta/2)\partial_2 \wedge \partial_3 + x_2 \partial_3 \wedge \partial_1 + x_3 \partial_1 \wedge \partial_2 \,. \tag{41}$$

We note that only for $\beta = 0$ and $\alpha = \beta = 0$ the Poisson structures may be identified with Poisson–Lie structures. One can easily check that the Casimir for this Poisson structure is

$$\mathcal{C} = x_1(\alpha x_1 - \beta) + x_2^2 + x_3^2 \,. \tag{42}$$

From the relations

$$\{\tilde{x}_2, \tilde{x}_3\}_{\mathcal{F}} = \alpha \tilde{x}_1 - \frac{\beta}{2}\,, \tag{43}$$

$$\{\tilde{x}_1, \tilde{x}_2\}_{\mathcal{F}} = \tilde{x}_3\,, \tag{44}$$

$$\{\tilde{x}_3, \tilde{x}_1\}_{\mathcal{F}} = \tilde{x}_2\,, \tag{45}$$

we find

$$\tilde{x}_1 = \frac{p}{\beta}\,, \tag{46}$$

$$\begin{pmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \sqrt{1 + p - \alpha p^2 \beta^{-2}} \times \begin{pmatrix} \sin(\beta q) \\ \cos(\beta q) \end{pmatrix} . \tag{47}$$

The coproduct of $\tilde{\mathcal{C}}$ is

$$\begin{aligned} \triangle(\tilde{\mathcal{C}}) &= 2 + 2\alpha\beta^{-2} p_1 p_2 \\ &\quad + 2\sqrt{1 - p_1 + \alpha p_1^2 \beta^{-2}}\, \sqrt{1 - p_2 + \alpha p_2^2 \beta^{-2}}\, \cos\beta(q_1 - q_2)\,. \end{aligned} \tag{48}$$

The above relations define a class of integrable Hamiltonian systems with two degrees of freedom

$$\left(R^4, \omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2, \triangle(\tilde{h}), \triangle(\tilde{\mathcal{C}})\right)\,,$$

where $h \in \mathcal{F}(\mathcal{L}_\delta)$ is an arbitrary smooth function, such that $d(\triangle(\tilde{h})) \wedge d(\triangle(\tilde{\mathcal{C}})) \neq 0$.

6. Generalizations

The procedure to obtain integrable Hamiltonian systems with two degrees of freedom can be generalized to any degrees of freedom by making use of the kth coproduct. Letting $\triangle = \triangle^1$ we find [10]: $\triangle^{k+1} : \mathcal{F}(\mathcal{L}_\delta) \to (\mathcal{F}(\mathcal{L}_\delta))^{\otimes k+2}$ by extending

$$\triangle^{k+1} = (\triangle \otimes \mathrm{id}^k) \circ \triangle^k \,, \tag{49}$$

i.e. diagonalizing on the first factor after applying $\triangle^k$. Hence for arbitrary $k \geq 2$, we have

$$\triangle^{k-1}(\zeta) = \sum_{i=1}^{k} \zeta_i \,, \tag{50}$$

$$\{\triangle^{k-1}(\zeta), \triangle^{k-1}(\eta)\}_{\mathcal{F}^{\otimes k}} = \sum_{i=1}^{k} \{\zeta, \eta\}_i \,, \tag{51}$$

where ζ and η are linear coordinates on $\mathcal{L}_\delta$. The integrals of a Hamiltonian system with n degrees of freedom are given by $n-1$ coproducts of the Casimir

$$F_k(p,q) = \triangle^k(\tilde{\mathcal{C}}) \qquad k = 1, 2, \cdots, n-1 \,, \tag{52}$$

and arbitrary Hamiltonian

$$H(p,q) = \triangle^{n-1}(h(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)) \,. \tag{53}$$

An easy computation shows that $H, F_1, \cdots, F_{n-1}$ are in involution, and F_k are functionally independent by definition.

6.1. Modified structures

Let us introduce the deformed Poisson bracket

$$\{\tilde{x}, \tilde{y}\}_{\mathcal{F}} = \tilde{z} \,, \quad \{\tilde{z}, \tilde{x}\}_{\mathcal{F}} = \tilde{y} \,, \quad \{\tilde{y}, \tilde{z}\}_{\mathcal{F}} = f(\tilde{x}, \varepsilon) \,, \tag{54}$$

and the deformed coproduct

$$\begin{aligned} \triangle_\varepsilon(\tilde{x}) &= \tilde{x} \otimes 1 + 1 \otimes \tilde{x} \,, & (55)\\ \triangle_\varepsilon(\tilde{y}) &= \tilde{y} \otimes e^{\varepsilon \tilde{x}/2} + e^{-\varepsilon \tilde{x}/2} \otimes \tilde{y} \,, & (56)\\ \triangle_\varepsilon(\tilde{z}) &= \tilde{z} \otimes e^{\varepsilon \tilde{x}/2} + e^{-\varepsilon \tilde{x}/2} \otimes \tilde{z} \,, & (57) \end{aligned}$$

where $\lim_{\varepsilon \to 0}(f(\tilde{x}, \varepsilon)) = \alpha \tilde{x}_1 - \beta/2$, $x = \tilde{x}_1$, $y = \tilde{x}_2$, $z = \tilde{x}_3$, and ε is a constant. Thus the deformed Poisson tensor reads

$$\Lambda_\varepsilon = f(\tilde{x}, \varepsilon) \partial_{\tilde{y}} \wedge \partial_{\tilde{z}} + \tilde{y} \partial_{\tilde{z}} \wedge \partial_{\tilde{x}} + \tilde{z} \partial_{\tilde{x}} \wedge \partial_{\tilde{y}} \,, \tag{58}$$

and the Casimir is

$$\tilde{\mathcal{C}}_\varepsilon = g(\tilde{x}, \varepsilon) + \tilde{y}^2 + \tilde{z}^2 \,, \tag{59}$$

where $g(\tilde{x}, \varepsilon) = \int f(\tilde{x}, \varepsilon) d\tilde{x}$. Let us assume that $\triangle_\varepsilon$ is a Poisson mapping, then the following equalities must hold

$$\triangle_\varepsilon(\{\tilde{x}, \tilde{y}\}_{\mathcal{F}}) = \{\triangle_\varepsilon(\tilde{x}), \triangle_\varepsilon(\tilde{y})\}_{\mathcal{F}\otimes\mathcal{F}} \,, \tag{60}$$

$$\triangle_\varepsilon(\{\tilde{z}, \tilde{x}\}_{\mathcal{F}}) = \{\triangle_\varepsilon(\tilde{z}), \triangle_\varepsilon(\tilde{x})\}_{\mathcal{F}\otimes\mathcal{F}} \,, \tag{61}$$

$$\triangle_\varepsilon(\{\tilde{y}, \tilde{z}\}_{\mathcal{F}}) = \{\triangle_\varepsilon(\tilde{y}), \triangle_\varepsilon(\tilde{z})\}_{\mathcal{F}\otimes\mathcal{F}} \,. \tag{62}$$

It is easy to verify that the first two relations are always satisfied. The situation is different with the last relation

$$\triangle_\varepsilon(\{\tilde{y}, \tilde{z}\}_{\mathcal{F}}) = \{\tilde{y}, \tilde{z}\}_{\mathcal{F}} \otimes e^{\varepsilon\tilde{x}} + e^{-\varepsilon\tilde{x}} \otimes \{\tilde{y}, \tilde{z}\}_{\mathcal{F}} \,. \tag{63}$$

It is obvious that $\{\tilde{y}, \tilde{z}\}_{\mathcal{F}} \neq \alpha\tilde{x} - \beta/2$. A relation that satisfies all the requirements is

$$\{\tilde{y}, \tilde{z}\}_{\mathcal{F}} = \frac{\alpha}{\varepsilon} \sinh(\varepsilon\tilde{x}) - \frac{\beta}{2} e^{\varepsilon\tilde{x}} \,. \tag{64}$$

From the above relations we obtain

$$\tilde{x} = \frac{p}{\beta} \,, \tag{65}$$

$$\begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix} = \sqrt{1 + \frac{\beta(e^{\varepsilon p/\beta} - 1)}{\varepsilon} - 4\alpha \left[\frac{\sinh(\varepsilon p/2\beta)}{\varepsilon}\right]^2} \times \begin{pmatrix} \sin(\beta q) \\ \cos(\beta q) \end{pmatrix} . \tag{66}$$

Hence the deformed coproduct of the Casimir reads

$$\begin{aligned} \triangle_\varepsilon(\tilde{\mathcal{C}}) &= \frac{\beta}{\varepsilon} \left[e^{\varepsilon p_1/\beta} + e^{\varepsilon p_2/\beta}\right] - 4\alpha[\Theta^2(p_1) + \Theta^2(p_2)] + \beta[\Psi(p_1) + \Psi(p_2)] \\ &+ \sqrt{1 + \beta\Psi(p_1) - 4\alpha\Theta(p_1)}\, \sqrt{1 + \beta\Psi(p_2) - 4\alpha\Theta(p_2)}\, \cos(\beta(q_1 - q_2)) \,, \end{aligned} \tag{67}$$

where

$$\Psi(p)\varepsilon = \exp(\varepsilon p/\beta) - 1 \tag{68}$$

and

$$\Theta(p)\varepsilon = \sinh(\varepsilon p/2\beta) \,. \tag{69}$$

So we derived a family of completely integrable Hamiltonian system with two degrees of freedom, with the second integral given by (67) and the Hamiltonian defined by the deformed coproduct of an arbitrary smooth function $h = h(\tilde{x}, \tilde{y}, \tilde{z})$.

7. Concluding remarks

Some new families of integrable Hamiltonian systems have been presented. They have been obtained according to a integrable sequence

$$(R^3, \Lambda) \xrightarrow{\mathcal{C}} (\mathcal{L}_\delta, \tilde{\Lambda}) \xrightarrow{\triangle} \left(\mathcal{F} \otimes \mathcal{F}, \omega = \sum_{i=1}^{2} dp_i \wedge dq_i, \triangle(\tilde{\mathcal{C}})\right)$$

with the Poisson tensor $\Lambda = (\alpha x_1 - \beta/2)\partial_2 \wedge \partial_3 + x_2\partial_3 \wedge \partial_1 + x_3\partial_1 \wedge \partial_2$. The procedure to obtain integrable Hamiltonian systems has been generalized to any number of degrees of freedom by making use of the k-th coproduct: $\triangle^k : \mathcal{F} \to \mathcal{F}^{\otimes^{k+1}}$. We have considered modifications of the Poisson bialgebra $(\mathcal{F}, \triangle, \{\ ,\ \})$ by introducing the deformed coproduct $\triangle_\varepsilon$ and the deformed Poisson tensor Λ_ε. These modifications also provide new classes of completely integrable systems.

For case $\alpha = \beta = 0$ the deformed coproduct defines the quantum group $U_\varepsilon(e(2))$, whereas for $\alpha \neq 0, \beta = 0$ it leads to $U_\varepsilon(su(2)^*)$. The deformed Poisson–Lie structures with the deformed coproduct may be identified with nonstandard quantum deformations of these algebras.

We note that using the results of this paper one can easily generalize the Calogero system (*cf.* [15–17]) — linked with the Poisson structure [4] — to more complex integrable Hamiltonian systems.

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