THE δ-DEFORMATION OF THE FOCK SPACE

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A deformation of the Fock space based on the finite difference replacement for the derivative is introduced. The deformation parameter is related to the dimension of the finite analogue of the Fock space.

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1. Introduction

In recent years there has been a growing interest in discretizations of quantum mechanics based on the finite difference replacement for the derivative. This is motivated by the well-known speculations that below the Planck scale the conventional notions of space and time break down and the new discrete structures are likely to emerge. This has echoes in the arguments put forward in string theory and quantum gravity. We also mention the technical reasons for the application of discrete models. Let us only recall the lattice gauge theories. As a matter of fact the connection has been shown in Ref. [1] between ordinary quantum mechanics on a equidistant lattice, where the the role of the derivative is played by the forward or backward discrete derivative, and q-deformations utilizing the Jackson derivative, nevertheless no explicit form of the corresponding deformation of the Fock space has been provided in [1]. On the other hand, there are indications [2] that approaches based on the central difference operator are more adequate for discretization of quantum mechanics than those using asymmetric forward or backward discrete derivatives.

In this paper we introduce a deformation of the Fock space, such that the creation and annihilation operators are elements of the quotient field of the deformed Heisenberg algebra generated by the usual position operator and the central difference operator. The deformation parameter δ describing the fixed coordinate spacing is naturally related to the dimension of the finite-dimensional space which can be regarded as an analogue of the Fock space. In the formal limit $\delta \to 0$ we arrive at the infinite-dimensional space coinciding with the usual Fock space.

2. The δ-deformation of the Heisenberg algebra

As mentioned in the introduction there are indications that discretizations of quantum mechanics should involve the central difference operator such that

$$\Delta_\delta f(x) = \frac{f(x+\delta) - f(x-\delta)}{2\delta}\,. \tag{2.1}$$

In view of (2.1) the discrete counterpart of the momentum operator is given by

$$\hat{p}_\delta = -\mathrm{i}\Delta_\delta\,, \tag{2.2}$$

where we set $\hbar = 1$. Furthermore, it seems to us that the most natural candidate for the position operator in any discretized version of quantum mechanics is the standard one of the form

$$\hat{x} f(x) = x f(x)\,. \tag{2.3}$$

In order to close the algebra satisfied by the operators $\hat{p}_\delta$ and $\hat{x}$ we introduce the operator I_δ defined by

$$I_\delta f(x) = \frac{f(x+\delta) + f(x-\delta)}{2}\,. \tag{2.4}$$

It follows that

$$[\hat{x}, \hat{p}_\delta] = \mathrm{i}I_\delta\,, \qquad [\hat{x}, I_\delta] = -\mathrm{i}\delta^2\hat{p}_\delta\,, \qquad [\hat{p}_\delta, I_\delta] = 0\,. \tag{2.5}$$

Using (2.1)–(2.4) we also find easily the following Casimir operator for the algebra (2.5):

$$I_\delta^2 + \delta^2\hat{p}_\delta^2 = I\,. \tag{2.6}$$

We remark that (2.5) is a deformation of the $e(2)$ algebra corresponding to $\delta = 1$. Evidently,

$$\hat{p}_\delta = \tfrac{1}{\delta}\sin\delta\hat{p}\,, \qquad I_\delta = \cos\delta\hat{p}\,, \tag{2.7}$$

where $\hat{p} = -i\frac{d}{dx}$ is the usual momentum operator, so the contraction of the algebra (2.5) referring to $\delta \to 0$, is the usual Heisenberg algebra

$$[\hat{x}, \hat{p}] = iI, \qquad [\hat{x}, I] = 0, \qquad [\hat{p}, I] = 0\,. \tag{2.8}$$

We now discuss the representations of the algebra (2.5) in the Hilbert space $\mathcal{H} = L^2(\boldsymbol{R}, dx)$ specified by the scalar product

$$\langle f|g\rangle = \int\limits_{-\infty}^{\infty} dx f^*(x)g(x)\,. \tag{2.9}$$

Consider the operator U_δ defined by

$$U_\delta := I_\delta - i\delta\hat{p}_\delta \,. \tag{2.10}$$

Using the hermicity and boundedness of the operators I_δ and $\hat{p}_\delta$ following directly from (2.1), (2.2), (2.4) and (2.9), and utilizing (2.6) and (2.5) we find that U_δ is unitary. We point out that the unitarity condition satisfied by U_δ is simply an equivalent form of the Casimir (2.6). Making use of the relations

$$\begin{aligned} \hat{p}_\delta &= -\tfrac{i}{2\delta}(U_\delta - U_\delta^\dagger) \,, \\ I_\delta &= \tfrac{1}{2}(U_\delta + U_\delta^\dagger) \,, \end{aligned} \tag{2.11}$$

implied by (2.10), we arrive at the following equivalent form of the algebra (2.5):

$$[\hat{x}, U_\delta] = \delta U_\delta \,. \tag{2.12}$$

Consider now the abstract eigenvalue equation

$$\hat{x}|x\rangle = x|x\rangle \,, \tag{2.13}$$

where $x \in \boldsymbol{R}$, and $f(x)$ is related with $\langle x|f\rangle$ (see below). From equations (2.12) and (2.13) it follows that the action of the operators U_δ and $U_\delta^\dagger$ on the vectors $|x\rangle$ is of the form

$$\begin{aligned} U_\delta|x\rangle &= |x+\delta\rangle \,, \\ U_\delta^\dagger|x\rangle &= |x-\delta\rangle \,. \end{aligned} \tag{2.14}$$

Therefore

$$U_\delta^{\pm n}|x\rangle = |x \pm n\delta\rangle \,. \tag{2.15}$$

Thus, it turns out that we can generate the whole basis $\{|\lambda + n\delta\rangle\}$, where n is integer, of an irreducible representation of the algebra (2.12) and thus the algebra (2.5) from the unique vector $|\lambda\rangle$, where $\lambda \in [0,\delta)$. It is clear that λ labels irreducible representations of the algebra. On the other hand, the choice of λ determines in view of the relation

$$\hat{x}|\lambda + n\delta\rangle = (\lambda + n\delta)|\lambda + n\delta\rangle \,, \tag{2.16}$$

where n is integer, the position of the lattice on the real line. We point out that, in order to control the contraction $\delta \to 0$, we use the unnormalized vectors $|\lambda + n\delta\rangle$, satisfying $\langle \lambda + n\delta|\lambda + n'\delta\rangle = \frac{1}{\delta}\delta_{nn'}$. Further, it is also evident that the original Hilbert space $\mathcal{H} = L^2(\boldsymbol{R}, dx)$ can be written as a direct integral

$$\mathcal{H} = \int_{[0,\delta)}^{\oplus} d\lambda\, \mathcal{H}_\lambda\,, \tag{2.17}$$

where $\mathcal{H}_\lambda$ is the Hilbert space of functions defined on a lattice with spacings δ, with the scalar product

$$\langle f|g\rangle_\lambda = \sum_{n=-\infty}^{\infty} f^*(\lambda + n\delta)g(\lambda + n\delta)\delta\,, \tag{2.18}$$

where $f(\lambda + n\delta) = \langle \lambda + n\delta|f\rangle$.

We now specialize, without loss of generality, to the case of $\lambda = 0$. We have

$$\begin{aligned} \hat{x}|n\delta\rangle &= n\delta|n\delta\rangle\,, \\ \hat{p}_\delta|n\delta\rangle &= -\tfrac{i}{2\delta}(|(n+1)\delta\rangle - |(n-1)\delta\rangle)\,, \\ I_\delta|n\delta\rangle &= \tfrac{1}{2}(|(n+1)\delta\rangle + |(n-1)\delta\rangle)\,. \end{aligned} \tag{2.19}$$

The last two equations from (2.19) follow directly from (2.11) and

$$U_\delta|n\delta\rangle = |(n+1)\delta\rangle, \qquad U_\delta^\dagger|n\delta\rangle = |(n-1)\delta\rangle\,. \tag{2.20}$$

Clearly, the realization of the abstract Hilbert space of states is specified by the scalar product (2.18) with $\lambda = 0$, *i.e.*

$$\langle f|g\rangle = \sum_{n=-\infty}^{\infty} \langle f|n\delta\rangle\langle n\delta|g\rangle\delta = \sum_{n=-\infty}^{\infty} f^*(n\delta)g(n\delta)\delta\,, \tag{2.21}$$

where $f(n\delta) = \langle n\delta|f\rangle$.

We remark that the operator $\hat{x}$ is self-adjoint by standard arguments, with domain $\{f \in l^2(\delta\boldsymbol{Z})|xf \in l^2(\delta\boldsymbol{Z})\}$, where $\boldsymbol{Z}$ designates the set of integers and $l^2(\delta\boldsymbol{Z})$ is the space of square summable functions on the infinite lattice with spacing δ, with the scalar product (2.18), where $\lambda = 0$. The operators $\hat{p}_\delta$ and I_δ are bounded and symmetric so they are essentially self-adjoint on the whole $l^2(\delta\boldsymbol{Z})$.

We now study the representation generated by eigenvectors $|\varphi\rangle_\delta$, $-\frac{\pi}{\delta} \le \varphi \le \frac{\pi}{\delta}$, of the unitary operator U_δ such that

$$U_\delta|\varphi\rangle_\delta = e^{-i\delta\varphi}|\varphi\rangle_\delta\,. \tag{2.22}$$

The representation spanned by the vectors $|\varphi\rangle_\delta$ is applied in section 4 to the proof of a relationship between the actual treatment and some generalization of the harmonic oscillator. It follows immediately from (2.11) and (2.22) that

$$\begin{aligned} \hat{x}|\varphi\rangle_\delta &= -\mathrm{i}\frac{d}{d\varphi}|\varphi\rangle_\delta, \\ \hat{p}_\delta|\varphi\rangle_\delta &= \tfrac{1}{\delta}\sin(\delta\varphi)|\varphi\rangle_\delta\,, \\ I_\delta|\varphi\rangle_\delta &= \cos(\delta\varphi)|\varphi\rangle_\delta\,. \end{aligned} \tag{2.23}$$

The completeness of the vectors $|\varphi\rangle_\delta$ gives rise to the functional representation of vectors

$$\langle f|g\rangle = \frac{1}{2\pi}\int_{-\frac{\pi}{\delta}}^{\frac{\pi}{\delta}} f^*(\varphi)g(\varphi)d\varphi\,, \tag{2.24}$$

where $f(\varphi) = \langle\varphi|f\rangle$, and we have omitted for brevity the dependence of $f(\varphi)$ on δ.

Our purpose now is to analyze the contraction $\delta \to 0$ of the representations (2.21) and (2.24) introduced above. Taking into account (2.22) and (2.20) we find that the passage from the representation spanned by the vectors $|n\delta\rangle$ and that generated by the vectors $|\varphi\rangle_\delta$ can be described by the kernel

$$\langle n\delta|\varphi\rangle_\delta = e^{in\delta\varphi}\,. \tag{2.25}$$

Equations (2.24) and (2.25) taken together yield

$$\langle n\delta|n'\delta\rangle = \frac{1}{2\pi}\int_{-\frac{\pi}{\delta}}^{\frac{\pi}{\delta}} e^{i(n-n')\delta\varphi}d\varphi = \frac{\sin\pi(n-n')}{\pi(n-n')\delta}\,. \tag{2.26}$$

Therefore

$$\langle n\delta|n'\delta\rangle = \tfrac{1}{\delta}\delta_{nn'}\,, \tag{2.27}$$

whenever $\delta \neq 0$. On the other hand, defining the continuum limit as

$$n \to \infty, \qquad \delta \to 0, \qquad n\delta = \text{const} = x\,, \tag{2.28}$$

we find that (2.26) takes the form

$$\lim_{\substack{n,n'\to\infty,\ \delta\to 0\\ n\delta=x,\, n'\delta=x'}} \langle n\delta|n'\delta\rangle = \delta(x-x')\,. \tag{2.29}$$

Hence, we get

$$\lim_{\substack{n\to\infty,\ \delta\to 0\\ n\delta=x}} |n\delta\rangle = |x\rangle\,, \tag{2.30}$$

where $|x\rangle$, $x \in \boldsymbol{R}$, are the usual normalized eigenvectors of the position operator for a quantum mechanics on a real line. This observation is consistent with the fact that for $\delta \to 0$ the sum from (2.21) is simply the integral sum for the scalar product in $L^2(\boldsymbol{R}, dx)$. By (2.19) and (2.28) it is also evident that in the limit $\delta \to 0$ we arrive at the Heisenberg algebra (2.8). We have thus shown that the contraction referring to $\delta \to 0$ of the representation of the algebra (2.5) given by (2.21) and (2.19) coincides with the standard coordinate L^2 representation of the Heisenberg algebra (2.8). Analogously, we have

$$ {}_\delta\langle\varphi|\varphi'\rangle_\delta = \sum_{n=-\infty}^{\infty} e^{-in\delta(\varphi-\varphi')}\delta \,. \tag{2.31} $$

Therefore,

$$ \lim_{\delta\to 0} {}_\delta\langle\varphi|\varphi'\rangle_\delta = 2\pi\delta(\varphi-\varphi') \,, \tag{2.32} $$

and we can identify

$$ \lim_{\delta\to 0} |\varphi\rangle_\delta = \sqrt{2\pi}|p\rangle \,, \tag{2.33} $$

where $p = \varphi$, and $|p\rangle$, $p \in \boldsymbol{R}$, are the normalized eigenvectors of the momentum operator. Further, in view of (2.23) the case $\delta \to 0$ really corresponds to the Heisenberg algebra (2.8). So the representation specified by (2.24) coincides in the limit $\delta \to 0$ with the standard momentum representation. We conclude that the introduced deformation works both on the level of the algebra and the representation.

Finally, we recall that the operator I_δ can be regarded as an evolution operator for a free particle on a lattice on a real line [1]. Indeed, following [1] we set

$$ f(x, t_n) := I_\delta^n f(x) \,, \tag{2.34} $$

with the imaginary time $t_n := n\delta^2/i$. On introducing the Hamiltonian for a free particle with the unit mass of the form

$$ H_\delta := \frac{1}{\delta^2}(1 - I_\delta) \,, \tag{2.35} $$

we arrive at the Schrödinger equation satisfied by the time-dependent functions (2.34) such that

$$ i\tilde{\partial}_t f = H_\delta f \,, \tag{2.36} $$

where $\tilde{\partial}_t$ is the discrete time derivative defined by

$$ \tilde{\partial}_t f(x,t) := \frac{1}{\Delta t}[f(x, t + \Delta t) - f(x,t)] \,, \tag{2.37} $$

where $\Delta t := \delta^2/i$. We point out that in view of (2.35) and the second equation of (2.7) we really recover in the limit $\delta \to 0$ the Hamiltonian of a free particle on a real line $H = \hat{p}^2/2$.

3. The δ-deformation of the Heisenberg–Weyl algebra

In this section we study the δ-deformation of the Heisenberg–Weyl algebra satisfied by the Bose creation and annihilation operators. Let us introduce the following family of operators:

$$A(s)=\tfrac{1}{\sqrt{2}}[\hat{x}+i(1-\delta^2 s)\hat{p}_\delta I_\delta^{-1}]\,,\qquad A^\dagger(s)=\tfrac{1}{\sqrt{2}}[\hat{x}-i(1-\delta^2 s)\hat{p}_\delta I_\delta^{-1}]\,,\tag{3.1}$$

where $s=0,1,\ldots$. Clearly, these operators reduce to the standard Bose creation and annihilation operators in the limit $\delta\to 0$. We point out that in this limit $A(s)$ and $A^\dagger(s)$ do not depend on s. Notice that in view of hermicity of generators of the algebra (2.5) $A^\dagger(s)$ is really the Hermitian conjugate of $A(s)$. It should also be noted that in the representation spanned by the vectors $|\varphi\rangle_\delta$ the action of the operator I_δ^{-1} is simply the multiplication by $\sec\delta\varphi$. It is clear that, in opposition to the operator I_δ, the operator I_δ^{-1} is unbounded. Obviously, the Bose operators (3.1) are unbounded as well. We now seek the vectors $|s\rangle$ and functions $\alpha(s)$ and $\beta(s)$, satisfying

$$A(s)|s\rangle=\alpha(s)|s-1\rangle\,,\qquad A^\dagger(s)|s\rangle=\beta(s)|s+1\rangle\,,\qquad s=0,1,\ldots,\tag{3.2}$$

We stress that we have not designated for brevity the dependence of vectors $|s\rangle$ on the parameter δ *i.e.* $|s\rangle\equiv|s\rangle_\delta$. Referring back to (3.2), we are simply looking for the δ-deformation of vectors spanning the occupation number representation. Using the following form of the Casimir (2.6) which can be obtained with the help of (3.1):

$$A(s+1)A^\dagger(s)-A^\dagger(s-1)A(s)=(1-\delta^2 s)I\,,\tag{3.3}$$

where I is the unit operator, we get

$$\alpha(s+1)\beta(s)-\alpha(s)\beta(s-1)=1-\delta^2 s\,.\tag{3.4}$$

Hence, setting $\alpha(0)=0$ and solving the elementary recurrence (3.4) we obtain

$$\alpha(s)\beta(s-1)=s-\tfrac{\delta^2}{2}s(s-1)\,.\tag{3.5}$$

The following solution of (3.5) consistent with the limit values $\alpha(s)=\sqrt{s}$ and $\beta(s)=\sqrt{s+1}$, corresponding to $\delta=0$, when $|s\rangle$ span the usual occupation number representation can be guessed easily:

$$\alpha(s)=\sqrt{s-\tfrac{\delta^2}{2}s(s-1)}\,,\qquad \beta(s)=\sqrt{s+1-\tfrac{\delta^2}{2}s(s+1)}\,,\tag{3.6}$$

so we have

$$A(s)|s\rangle=\sqrt{s-\tfrac{\delta^2}{2}s(s-1)}|s-1\rangle,\quad A^\dagger(s)|s\rangle=\sqrt{s+1-\tfrac{\delta^2}{2}s(s+1)}|s+1\rangle\,.\tag{3.7}$$

Now, by virtue of

$$\langle s|A^\dagger(s)A(s)|s\rangle = [s - \tfrac{\delta^2}{2}s(s-1)]\langle s-1|s-1\rangle \geq 0\,, \tag{3.8}$$

we see that the sequence of s and thus $|s\rangle$ should truncate. The only possibility left is to set

$$\delta^2 = \frac{1}{s_{\rm max}}\,. \tag{3.9}$$

Indeed, by (3.1) we then have

$$A(s_{\rm max}) = A^\dagger(s_{\rm max}) = \tfrac{1}{\sqrt{2}}\hat{x}\,. \tag{3.10}$$

Using this and (3.7), we find

$$|s_{\rm max}+1\rangle = |s_{\rm max}-1\rangle\,, \tag{3.11}$$

where $|s_{\rm max}+1\rangle = A^\dagger(s_{\rm max})|s_{\rm max}\rangle$. We have thus shown that instead of δ we can use the parameter $s_{\rm max}$ exceeding by one the dimension of the system of vectors $\{|s\rangle\}_{0\leq s\leq s_{\rm max}}$. Such systems for $s_{\rm max}=1$, $s_{\rm max}=2$ and so on, can be interpreted as a finite-dimensional analogues of the usual infinite-dimensional Fock space. The latter evidently refers to the case with $s_{\rm max}=\infty$, when $\delta=0$.

We now discuss the algebra satisfied by the operators (3.1), that is the δ-deformation of the Heisenberg–Weyl algebra. Taking into account (3.7) we get

$$\begin{aligned} A(s') &= \left[1-\frac{\delta^2(s-s')}{2(\delta^2 s-1)}\right]A(s) + \frac{\delta^2(s-s')}{2(\delta^2 s-1)}A^\dagger(s),\\ A^\dagger(s') &= \frac{\delta^2(s-s')}{2(\delta^2 s-1)}A(s) + \left[1-\frac{\delta^2(s-s')}{2(\delta^2 s-1)}\right]A^\dagger(s)\,, \qquad s<s_{\rm max}\,. \end{aligned} \tag{3.12}$$

Making use of (2.5), (3.1), (3.12) and the following form of the Casimir (2.6), which can be easily derived with the help of (3.1):

$$\delta^2[A(s)-A^\dagger(s)]^2 = 2(1-\delta^2 s)(1-I_\delta^{-2})\,, \tag{3.13}$$

we arrive at the commutation relations such that

$$\begin{aligned} \left[A(s),A^\dagger(s')\right] &= \frac{[1-\delta^2}{2(s+s')]I_\delta^{-2}}\,,\\ [A(s),A(s')] &= [A^\dagger(s'),A^\dagger(s)] = \tfrac{\delta^2}{2}(s'-s)I_\delta^{-2}, \qquad s,s'\leq s_{\rm max}\,, \end{aligned}$$

$$\begin{aligned}
\left[A(s), I_\delta^{-2k}\right] &= [A^\dagger(s), I_\delta^{-2k}] = [A(s_{\max}), I_\delta^{-2k}] \\
&= k\frac{\delta^2}{1-\delta^2 s}B_k(s)\,, \quad s < s_{\max}\,, \\
[A(s), B_k(s')] &= [A^\dagger(s), B_k(s')] \\
&= 2k(1-\delta^2 s')I_\delta^{-2k} - (2k+1)(1-\delta^2 s')I_\delta^{-2(k+1)}\,, \\
\left[B_k(s), I_\delta^{-2l}\right] &= [I_\delta^{-2k}, I_\delta^{-2l}] \\
&= [B_k(s), B_l(s')] = 0\,, \quad s, s' \le s_{\max}\,, \quad k, l = 1, 2, \ldots\,,
\end{aligned} \tag{3.14}$$

where $B_k(s) = [A(s) - A^\dagger(s)]I_\delta^{-2k}$. We remark that due to the commutator of $A(s)$ and $B_k(s')$, the algebra (3.14) is infinite dimensional. It should also be noted that in view of the following relation:

$$A(s) = (1 - \tfrac{\delta^2 s}{2})A(0) + \tfrac{\delta^2 s}{2}A^\dagger(0)\,, \qquad 0 \le s \le s_{\max}\,, \tag{3.15}$$

which is an immediate consequence of (3.1), $A(s)$, $A^\dagger(s)$ and $B_k(s)$ can be regarded as a discrete curve in the algebra generated by $A(0)$, $A^\dagger(0)$, I_δ^{-2k} and $B_k(0)$. Of course, (3.14) reduce to the Heisenberg–Weyl algebra in the limit $\delta \to 0$, that is $s_{\max} \to \infty$.

4. The δ-deformation of the Fock space

We now discuss the δ-deformation of the Fock space expressed by (3.7) in a more detail. We first observe that the generation of the states $|s\rangle$, with $s \ge 1$, from the "vacuum vector" $|0\rangle$ can be described with the help of the second equation of (3.7) by

$$|s\rangle = \left(\prod_{s'=0}^{s-1} \frac{1}{\sqrt{s'+1-\frac{\delta^2}{2}s'(s'+1)}}\right) A^\dagger(s-1)\cdots A^\dagger(1)A^\dagger(0)|0\rangle\,, \quad 0 < s \le s_{\max}\,. \tag{4.1}$$

The vectors $|s\rangle$ are not orthonormal. In fact, using the first equation of (3.12) with $s' = s+1$, (3.7) and calculating the expectation value of the Casimir (3.13) in the state $|s\rangle$ with the use of the first equation of (3.14) for $s = s'$, we find

$$\langle 1|1\rangle = \frac{2}{2-\delta^2}\langle 0|0\rangle\,,$$

$$\langle s|s\rangle = \frac{\delta^2}{[2(\delta^2 s-1)-\delta^2][s-\frac{\delta^2}{2}s(s-1)]}\sum_{s'=0}^{s-2}(\delta^2 s'-1)\langle s'|s'\rangle$$
$$+\left(1+\frac{\delta^2[\delta^2(s-1)-1]}{[2(\delta^2 s-1)-\delta^2][s-\frac{\delta^2}{2}s(s-1)]}\right)\langle s-1|s-1\rangle,\quad 2\le s\le s_{\max}\,. \tag{4.2}$$

Furthermore, Eqs. (3.12) and (3.7) taken together yield

$$\begin{aligned}\sqrt{s-\tfrac{\delta^2}{2}s(s-1)}\langle s|s'\rangle \;=\; &\left[1-\frac{\delta^2(s'-s+1)}{2(\delta^2 s'-1)}\right]\\ &\times\sqrt{s'-\tfrac{\delta^2}{2}s'(s'-1)}\langle s-1|s'-1\rangle\\ &+\frac{\delta^2(s'-s+1)}{2(\delta^2 s'-1)}\sqrt{s'+1-\tfrac{\delta^2}{2}s'(s'+1)}\langle s-1|s'+1\rangle\,,\\ &0<s\le s_{\max},\quad 0\le s'<s_{\max}\,.\end{aligned} \tag{4.3}$$

The equations (4.2) and (4.3) form the closed system which enables to calculate the inner product $\langle s|s'\rangle$ for arbitrary $s,s'\le s_{\max}$. In particular, utilizing the relation

$$\langle s|s+1\rangle = 0,\qquad s\le s_{\max}\,, \tag{4.4}$$

implied by (4.3) and using recursively (4.3) we find that

$$\langle s|s'\rangle = 0,\qquad s,s'\le s_{\max}\,, \tag{4.5}$$

where s is even and s' is odd.

We finally discuss the concrete realization of the introduced δ-deformation of the abstract Fock space in the representation (2.24). On using (2.23) and (3.7) we arrive at the following system:

$$\begin{aligned}\left[\frac{d}{d\varphi}+(1-\delta^2 s)\frac{1}{\delta}\mathrm{tg}\delta\varphi\right]f_s(\varphi) \;&=\; -i\sqrt{2}\sqrt{s-\tfrac{\delta^2}{2}s(s-1)}\,f_{s-1}(\varphi)\,,\\ \left[\frac{d}{d\varphi}-(1-\delta^2 s)\frac{1}{\delta}\mathrm{tg}\delta\varphi\right]f_s(\varphi) \;&=\; -i\sqrt{2}\sqrt{s+1-\tfrac{\delta^2}{2}s(s+1)}\,f_{s+1}(p)\,,\end{aligned} \tag{4.6}$$

where $f_s(\varphi)=\langle\varphi|s\rangle$. We remark that the system (4.6) is the special case of the more general one

$$\begin{aligned}\left[\frac{d}{dx}+k(s,x)\right]y_s(x) &= \mu(s)y_{s-1}(x)\,,\\ \left[-\frac{d}{dx}+k(s,x)\right]y_s(x) &= \nu(s)y_{s+1}(x)\,.\end{aligned} \tag{4.7}$$

The system (4.7) was studied by Jannussis *et al* [3] in the context of the generalization of the Infeld–Hull method of factorization in the case of the harmonic oscillator. Analyzing the compatibility of the two second order differential equations implied by (4.7) they showed that besides the periodic solution there exists the following one:

$$k(s,x) = a\mathrm{ctg}(ax+\theta)\,s - \frac{b}{a}\mathrm{ctg}(ax+\theta) + \frac{c}{\sin(ax+\theta)}\,, \tag{4.8}$$

provided

$$\mu(s)\nu(s-1) = -a^2 s(s-1) + 2bs + \lambda\,, \tag{4.9}$$

where a, b, c, θ and λ are arbitrary constants. A look at (4.8), (4.9), (4.6) and (3.5) is enough to conclude that the actual treatment refers to the case with $a=\delta$, $b=1$, $c=0$, $\theta=\pi/2$ and $\lambda=0$. We point out that within the formalism introduced herein the second order equations implied by (4.6) are simply the realization of the abstract equations

$$\begin{aligned}A(s+1)A^\dagger(s)|s\rangle &= \beta(s)\alpha(s+1)|s\rangle\,,\\ A^\dagger(s-1)A(s)|s\rangle &= \alpha(s)\beta(s-1)|s\rangle\,.\end{aligned} \tag{4.10}$$

in the representation (2.24). The compatibility of the Eqs. (4.10) is ensured by the Casimir (3.3). In this sense the actual approach can be interpreted as an abstract form of the Infeld–Hull factorization method.

We now return to (4.6). Using (4.6) and the limit

$$\lim_{\delta\to 0}(\cos\delta\varphi)^{\frac{1}{\delta^2}} = e^{-\frac{\varphi^2}{2}}\,, \tag{4.11}$$

we find

$$f_0(\varphi) = \pi^{-\frac{1}{4}}(\cos\delta\varphi)^{\frac{1}{\delta^2}}\,. \tag{4.12}$$

Furthermore, utilizing (4.6) and

$$\frac{d}{d\varphi}(\cos\delta\varphi)^{\frac{1}{\delta^2}} = -\frac{\mathrm{tg}\delta\varphi}{\delta}(\cos\delta\varphi)^{\frac{1}{\delta^2}}\,, \qquad \frac{d}{d\varphi}\left(\frac{\mathrm{tg}\delta\varphi}{\delta}\right) = 1+\delta^2\left(\frac{\mathrm{tg}\delta\varphi}{\delta}\right)^2, \tag{4.13}$$

we get

$$f_s(\varphi) = \frac{\pi^{-\frac{1}{4}}(-i)^s}{(\sqrt{2})^s}\left(\prod_{s'=0}^{s-1}\frac{1}{\sqrt{s'+1-\frac{\delta^2}{2}s'(s'+1)}}\right)H_s^{(\delta)}\left(\frac{\mathrm{tg}\delta\varphi}{\delta}\right)(\cos\delta\varphi)^{\frac{1}{\delta^2}}\,, \tag{4.14}$$

where $1 \leq s \leq s_{\max}$, and $H_s^{(\delta)}(x)$ are the polynomials satisfying the recurrence

$$\begin{aligned} H_{s+1}^{(\delta)}(x) &= (2-\delta^2 s)xH_s^{(\delta)}(x) - (1+\delta^2x^2)H_s^{(\delta)\prime}(x)\,,\\ H_0^{(\delta)}(x) &= 1\,, \end{aligned} \tag{4.15}$$

where the prime designates the differentiation with respect to x. Of course, $H_s^{(\delta)}(x)$ are simply the δ-deformation of the usual Hermite polynomials refering to the limit $\delta \to 0$, *i.e.* $s_{\max} \to \infty$. The first few δ-deformed Hermite polynomials are of the form

$$\begin{aligned} H_0^{(\delta)}(x) &= 1\,,\\ H_1^{(\delta)}(x) &= 2x\,,\\ H_2^{(\delta)}(x) &= 4(1-\delta^2)x^2-2\,,\\ H_3^{(\delta)}(x) &= 8(1-\delta^2)(1-2\delta^2)x^3-12(1-\delta^2)x\,,\\ H_4^{(\delta)}(x) &= 16(1-\delta^2)(1-2\delta^2)(1-3\delta^2)x^4-48(1-\delta^2)(1-2\delta^2)x^2+12(1-\delta^2)\,. \end{aligned} \tag{4.16}$$

As with the standard Hermite polynomials the general formula on the δ-deformed ones can be derived such that

$$\begin{aligned} H_0^{(\delta)}(x) &= 1,\\ H_s^{(\delta)}(x) &= \sum_{j=0}^{[\frac{s}{2}]}(-1)^j\frac{s!}{j!(s-2j)!}2^{s-2j}\left[\prod_{s'=0}^{s-j-1}(1-\delta^2 s')\right]x^{s-2j},\ 1\leq s\leq s_{\max}\,, \end{aligned} \tag{4.17}$$

where $[y]$ is the biggest integer in y.

We finally write down the following formula on the matrix elements $\langle s|s'\rangle$ implied by (2.24) and (4.14):

$$\langle s|s'\rangle = \frac{1}{2\pi}\int_{-\frac{\pi}{\delta}}^{\frac{\pi}{\delta}} f_s^*(\varphi)f_{s'}(\varphi)d\varphi\,, \tag{4.18}$$

where $f_s(\varphi)$ is given by (4.12) and (4.14). The calculation of the integral from (4.18) for arbitrary s, s' seems to be more complicated than the solution of the recurrences (4.2) and (4.3). It should be noted however that (4.18) enables to calculate the squared norm of the "vacuum vector" $|0\rangle$ parametrizing solutions of (4.2) and (4.3). Namely, we find

$$\langle 0|0\rangle = \frac{1}{2\pi^{\frac{3}{2}}} \int\limits_{-\frac{\pi}{\delta}}^{\frac{\pi}{\delta}} (\cos\delta\varphi)^{\frac{2}{\delta^2}}\, d\varphi = \sqrt{\frac{s_{\max}}{\pi}}\, \frac{(2s_{\max}-1)!!}{(2s_{\max})!!} = \frac{\sqrt{s_{\max}}}{\pi}\, \frac{\Gamma(s_{\max}+\frac{1}{2})}{\Gamma(s_{\max}+1)}\,, \tag{4.19}$$

where $\delta^2 s_{\max} = 1$ and $\Gamma(x)$ is the gamma function.

5. Conclusion

We have introduced in this work the deformation of the Fock space based on the utilization of the central difference operator instead of the usual derivative. It should be mentioned that there exist alternative approaches for discretization of quantum mechanics relying on finite difference representations of the usual Heisenberg [4] or Heisenberg–Weyl algebra [5]. Nevertheless, the general problem with them is the interpretation of the nonequivalence of the obtained representations of the canonical commutation relations and the standard Schrödinger one. Some problems with the spectrum of operators within such approaches have been also reported [4]. We also recall the discretization of the harmonic oscillator introduced in [6] relying on the replacement of the Hermite polynomials with the Kravchuk polynomials in a discrete variable as well as the finite-dimensional counterpart of the Fock space spanned by the eigenvectors of the phase operator discussed in [7]. In analogy with the actual treatment in both approaches taken up in [6] and [7] the standard infinite-dimensional Fock space refers to the formal limit $N \to \infty$, where N is dimension of the finite-dimensional discrete version of the Fock space. Moreover, in the case with the discretization described in [6] one can recognize a counterpart of the parameter δ specified by (3.9) such that $\delta \simeq N^{-\frac{1}{2}}$. Nevertheless, besides of those similarities we have also serious differences. For example, in opposition to the operators (3.1) the generalizations of the Bose operators introduced in [6] do not depend on the index labelling the basis of the finite-dimensional analogue of the Fock space. On the other hand, the alternatives to the number states discussed in [7] form the orthonormal set. This is not the case for the states $|s\rangle$ described herein. Last but not least we point out that besides of quantum mechanics the results of this paper would be of importance in the theory of differential equations. We only recall the abstract form of the Infeld–Hull method of factorization described by the equations (4.10) and (3.3).

REFERENCES

[1] A. Dimakis, F. Müller-Hoissen, *Phys. Lett.* **B295**, 242 (1992).

[2] A.Z. Górski, J. Szmigielski, *J. Math. Phys.* **39**, 545 (1998).

[3] A. Jannussis, G. Karagannis, P. Kanagopoulos, G. Brodimos, *Lett. Nuovo Cim.* **38**, 155 (1983).

[4] A. Dimakis, F. Müller-Hoissen, T. Striker, *J. Phys.* **A29** 6861 (1996).

[5] Y.M. Smirnov, A.V. Turbiner, *Mod. Phys. Lett.* **A10**, 1795 (1995).

[6] N.M. Atakishiyev, S.K. Suslov, *Theor. Math. Phys.* **85**, 1055 (1991).

[7] D.T. Pegg, S.M. Barnett, *Europhys. Lett.* **6**, 483 (1988).