RENYI ENTROPIES IN PARTICLE CASCADES

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Renyi entropies for particle distributions following from the general cascade models are discussed. The p-model and the β distribution introduced in earlier studies of cascades are discussed in some detail. Some phenomenological consequences are pointed out.

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1. Introduction

Recently, two of us proposed that Renyi entropies [1] may provide a useful tool for studies of correlations between particles created in high-energy collisions [2]. We have also studied these quantities in several models [3, 4]. In the present paper we continue this line of research, extending our discussion to models of particle production based on the multiplicative cascade mechanism, considered earlier by several authors [5–7].

2. The cascade models

We will be considering a multiplicative cascade of J steps with two branches at each vertex, *i.e.* $M = 2^J$ bins at the end of the cascade. Each bin is labelled by a set of J numbers taking values 0 or 1: $(00\ldots0)$,$(10\ldots0)$,$\ldots$, $(11\ldots1)$. Thus at each step, each bin is split into two and the density in the new bins is calculated by multiplying the density in the parent bin by the random numbers $q_{\ldots0}$ and $q_{\ldots1}$ distributed according to the probability

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$p(q_{\dots 0}, q_{\dots 1})$. This construction leads to the following probability distribution of the particle densities $e_{00\dots 0}, \dots, e_{11\dots 1}$

$$P(J; e_{0\dots 0}, \dots, e_{1\dots 1}) = \int \prod_{j=1}^{J} \prod_{k_1,\dots,k_{j-1}=0}^{1} dq_{k_1\dots k_{j-1}0} dq_{k_1\dots k_{j-1}1}$$
$$\times p(q_{k_1\dots k_{j-1}0}, q_{k_1\dots k_{j-1}1}) \prod_{k_1,\dots,k_J=0}^{1} \delta\left(e_{k_1\dots k_J} - \prod_{j=1}^{J} q_{k_1\dots k_j} \right) . \quad (1)$$

Without any loss of generality we can furthermore assume that — on the average — in each vertex the density is conserved:

$$\int dq_0 dq_1 p(q_0, q_1)(q_0 + q_1) = \langle q_0 \rangle + \langle q_1 \rangle = 1 \, . \quad (2)$$

The particle distribution is obtained as a Poisson transform of (1), *i.e.*,

$$P(J; n_{0\dots 0}, \dots, n_{1\dots 1}) = \int de_{0\dots 0} \dots de_{1\dots 1} P(J; e_{0\dots 0}, \dots, e_{1\dots 1})$$
$$\times \exp(-[e_{0\dots 0} + \dots + e_{1\dots 1}]\bar{N}) (\bar{N} e_{0\dots 0})^{n_{0\dots 0}} \dots (\bar{N} e_{1\dots 1})^{n_{1\dots 1}} \frac{1}{n_{0\dots 0}! \dots n_{1\dots 1}!}, \quad (3)$$

where $\bar{N}$ is the average number of particles. This procedure does not introduce any new correlations (apart from those which are already present in (1) [5]).

The δ-functions in (1) allow us to perform the integrations in (3) with the result

$$P(J; n_{0\dots 0}, \dots, n_{1\dots 1})$$
$$= \bar{N}^N \int \prod_{j=1}^{J} \prod_{k_1,\dots,k_{j-1}=0}^{1} dq_{k_1\dots k_{j-1}0} dq_{k_1\dots k_{j-1}1} p(q_{k_1\dots k_{j-1}0}, q_{k_1\dots k_{j-1}1})$$
$$\times \exp(-[q_0 q_{00} \dots q_{0\dots 0} + \dots + q_1 q_{11} \dots q_{1\dots 1}]\bar{N})$$
$$\times \frac{(q_0 q_{00} \dots q_{0\dots 0})^{n_{0\dots 0}} \dots (q_1 q_{11} \dots q_{1\dots 1})^{n_{1\dots 1}}}{n_{0\dots 0}! \dots n_{1\dots 1}!} , \quad (4)$$

which can be rewritten as

$$P(J; n_{0\dots 0}, \dots, n_{1\dots 1}) = W(N) P(J; N; n_{0\dots 0}, \dots, n_{1\dots 1}) \, , \quad (5)$$

where

$$N = n_{0\dots 0} + \dots + n_{1\dots 1} \quad (6)$$

is the total number of particles and $W(N)$ is the Poisson distribution

$$W(N) = e^{-\bar{N}} \frac{\bar{N}^N}{N!} , \tag{7}$$

and

$$\begin{aligned} & P(J;N;n_{0\ldots0},...,n_{1\ldots1}) \\ & = \int \prod_{j=1}^{J} \prod_{k_1,...,k_{j-1}=0}^{1} dq_{k_1\ldots k_{j-1}0} dq_{k_1\ldots k_{j-1}1} p(q_{k_1\ldots k_{j-1}0}, q_{k_1\ldots k_{j-1}1}) \\ & \times \exp(-[q_0 q_{00} \ldots q_{0\ldots0} + \ldots + q_1 q_{11} \ldots q_{1\ldots1} - 1]\bar{N}) \\ & \times \frac{N!(q_0 q_{00} \ldots q_{0\ldots0})^{n_{0\ldots0}} \ldots (q_1 q_{11} \ldots q_{1\ldots1})^{n_{1\ldots1}}}{n_{0\ldots0}! \ldots n_{1\ldots1}!} . \end{aligned} \tag{8}$$

From the definition of the coincidence probabilities

$$C_l(J) = \sum_{n_{0\ldots0},\ldots,n_{1\ldots1}} [P(J,N;n_{0\ldots0},\ldots,n_{1\ldots1})]^l , \tag{9}$$

one easily sees that

$$C_l(J) = \sum_N [W(N)]^l C_l(J,N) \tag{10}$$

so that it is enough to consider coincidence probabilities for a fixed total number of particles $C_l(J,N)$.

The actual shape of the distribution represented by (8) is determined by the vertex distribution $p(q_0,q_1)$. To illustrate that the models we consider do indeed span a rather broad class of distributions, it is useful to consider two limiting cases.

(i) At each vertex one branch takes the value $q = 0$ and the other one $q = 1$. Consequently, only one final bin is occupied. The probability that this will be any given bin equals $1/2^J$ because all bins are equivalent. Consequently, for the coincidence probabilities one obtains

$$C_l(J,N) = 2^{-J(l-1)} . \tag{11}$$

The Renyi entropies

$$H_l(J,N) \equiv -\frac{\log C_l(J,N)}{l-1} = J \log 2 \tag{12}$$

depend neither on l nor on N.

(ii) At each vertex the density splits equally into two branches. In this case $q = 1 - q = 1/2$ so that each bin contains the same particle density. Consequently, for fixed N, the particle distribution is the one of Bernoulli:

$$P(J;N;n_{00\ldots0},\ldots,n_{11\ldots1}) = M^{-N}\frac{N!}{n_{00\ldots0}!\ldots n_{11\ldots1}!}\,, \tag{13}$$

where we have denoted by M the total number of bins ($M = 2^J$).

For large N the coincidence probabilities can be obtained directly from the definition, by changing the sum into a (multidimensional) integral and using the saddle point method [4]. The result is

$$C_l(J,N) = \left(\sqrt{2\pi N}\right)^{(M-1)(1-l)}\left(\sqrt{M}\right)^{M(l-1)}\left(\sqrt{l}\right)^{1-M} \tag{14}$$

and thus

$$H_l(J,N) = \frac{M-1}{2}\log\left(\frac{2\pi N}{M}\right) - \frac{1}{2}\log M + \frac{M-1}{2}\frac{\log l}{l-1}\,. \tag{15}$$

In this case the dependence on the number of steps of the cascade is exponential. One notices also the linear dependence on the logarithm of the particle density.

3. Density conservation

A significant simplification of the distribution (4) is obtained when the density is conserved in each vertex[1]:

$$p(q_0,q_1) = \delta(q_0+q_1-1)f(q_0,q_1)\,. \tag{16}$$

The delta function simplifies the problem considerably, because it implies the equality

$$q_0q_{00}\ldots q_{0\ldots0} + \ldots + q_1q_{11}\ldots q_{1\ldots1} = 1 \tag{17}$$

and thus there are no fluctuations in the total particle density. In consequence, the integral in (4) factorises. One obtains

$$P(J;N;n_{0\ldots0},\ldots,n_{1\ldots1})$$
$$= \int\prod_{j=1}^{J}\prod_{k_1,\ldots,k_{j-1}=0}^{1} dq_{k_1\ldots k_{j-1}0}\,dq_{k_1\ldots k_{j-1}1}\,p(q_{k_1\ldots k_{j-1}0},q_{k_1\ldots k_{j-1}1})$$

[1] This may be considered as a generalisation of the so–called p-model [8].

$$\times (q_0 q_{00} \dots q_{0\dots0})^{n_{0\dots0}} \dots (q_1 q_{11} \dots q_{1\dots1})^{n_{1\dots1}} \frac{N!}{n_{0\dots0}! \dots n_{1\dots1}!}$$
$$= M(n_0,n_1)M(n_{00},n_{01})M(n_{10},n_{11}) \dots M(n_{1\dots10},n_{1\dots11}) \frac{N!}{n_{0\dots0}! \dots n_{1\dots1}!}\,, \tag{18}$$

where $n_{\dots}$ denotes the number of particles in a bin labelled by $\dots$. It is simply the sum of the numbers of particles in the final bins — after J steps — which originated from the given bin, *e.g.*,

$$n_0 = n_{00} + n_{01}; \quad n_{10} = n_{100} + n_{101}; \quad n_{101} = n_{1010} + n_{1011}\,. \tag{19}$$

$M(k,l)$ are the moments of the vertex distribution:

$$M(k,l) = \int dq_0 dq_q p(q_0,q_1) q_0^k q_1^l = \int dq f(q,1-q) q^k (1-q)^l\,. \tag{20}$$

The factorised form of (18) allows us to write down a general formula for the coincidence probabilities:

$$C_l(J,N) = \sum_{n_{0\dots0}+\dots+n_{1\dots1}=N} M(n_0,n_1)^l M(n_{00},n_{01})^l \dots$$
$$\times M(n_{10},n_{11})^l \dots M(n_{0\dots00},n_{0\dots01})^l \dots M(n_{1\dots10},n_{1\dots11})^l \left(\frac{N!}{n_{0\dots0}! \dots n_{1\dots1}!}\right)^l. \tag{21}$$

This formula still looks rather formidable, so we better make use of the recursive nature of the probabilities (18)

$$P(J+1;N;n_{00\dots0},\dots,n_{11\dots1})$$
$$= \frac{N!}{n_0! n_1!} M(n_0,n_1) P(J;n_0;n_{00\dots0},\dots,n_{11\dots1}) P(J;n_1;n_{10\dots0},\dots,n_{11\dots1})\,, \tag{22}$$

since it allows one to write down a recursive relation between the coincidence probabilities of cascades of different rank

$$C_l(J+1,N) = \sum_{n_{0\dots}+\dots+n_{1\dots}=N} P^l(J+1;N;n_{0\dots},\dots,n_{1\dots})$$
$$= \sum_{n_{0\dots}+\dots+n_{1\dots}=N} \left(\frac{N!}{n_0! n_1!}\right)^l [M(n_0,n_1)]^l P^l(J;n_0;n_{0\dots}\dots) P^l(J;n_1;n_{1\dots}\dots)$$
$$= \sum_{n_0+n_1=N} \left(\frac{N!}{n_0! n_1!}\right)^l [M(n_0,n_1)]^l C_l(J,n_0) C_l(J,n_1)\,, \tag{23}$$

with the initial condition

$$C_l(0,N) = 1\,. \tag{24}$$

This formula is more effective for practical applications than the general expression (21).

Introducing

$$\tilde{C}_l(J,N) = \frac{1}{(N!)^l} C_l(J,N)\,, \tag{25}$$

Eq. (23) can be rewritten in the simpler form

$$\tilde{C}_l(J+1,N) = \sum_{n_0+n_1=N} [M(n_0,n_1)]^l \tilde{C}_l(J,n_0)\tilde{C}_l(J,n_1)\,, \tag{26}$$

with

$$\tilde{C}_l(0,N) = \frac{1}{(N!)^l}\,. \tag{27}$$

4. The β distribution

To obtain more insight into the structure of the cascade models we have studied in some detail a special case of the vertex distribution, the so-called β distribution[2] for which

$$f(q,1-q) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} q^{a-1}(1-q)^{b-1}\,. \tag{28}$$

For symmetric distributions we have, of course, $b = a$. Using this, we can explicitly calculate the moments with the result

$$M(n_0,n_1) = \frac{\Gamma(a+n_0)}{\Gamma(a)} \frac{\Gamma(a+n_1)}{\Gamma(a)} \frac{\Gamma(2a)}{\Gamma(2a+n_0+n_1)}\,. \tag{29}$$

This allows us to simplify somewhat the recurrence relation (23). Introducing

$$\hat{C}(J,N) \equiv \left[\frac{\Gamma(a+N)}{N!\Gamma(a)}\right]^l C_l(J,N)\,, \tag{30}$$

we obtain

$$\hat{C}_l(J+1,N) = \left[\frac{\Gamma(a+N)\Gamma(2a)}{\Gamma(2a+N)\Gamma(a)}\right]^l \sum_{n_0+n_1=N} \hat{C}_l(J,n_0)\hat{C}_l(J,n_1)\,, \tag{31}$$

[2] To our knowledge, this distribution was first applied to multiparticle cascades in [9].

with the initial condition

$$\hat{C}_l(0,N) = \left[\frac{\Gamma(a+N)}{N!\Gamma(a)}\right]^l \ . \tag{32}$$

The recursive relation (31) with the initial condition (32) allows us to calculate the coincidence probabilities of any order for arbitrary J and N. Unfortunately, the explicit closed solution seems to be out of reach. Fortunately, these relations are well-suited for numerical work. In the next section we show the results of the numerical evaluation of C_2 in a broad region of the parameter a. As every multiplicative cascade the system is multifractal. The degree of fractality of the cascade is controlled by the parameter a. The fractal dimensions D_p is given by [10]

$$D_p = 1 - \frac{f_p}{p-1}\,, \tag{33}$$

where f_p are intermittency exponents [5]

$$f_p = \frac{\log[\langle q^p\rangle/\langle q\rangle^p]}{\log 2} = \frac{\log[2^p M(p,0)]}{\log 2}\,. \tag{34}$$

For $a \to 0$, $D_p \to 0$; for $a \to \infty$, $D_p \to 1$, *i.e.* the intermittency disappears.

5. Numerical analysis of the second Renyi entropy

Now we describe behaviour of coincidence probabilities $C_l(J,N)$ (we fix $l = 2$ in this section). We will study first the dependence on the number of particles for a fixed depth of the cascade. Figure 1 presents plots of $H_2(J,N)$

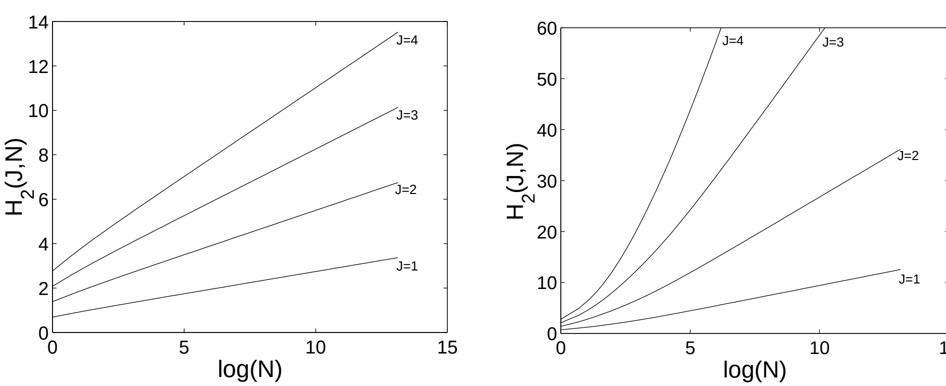

Fig. 1. $H_2(J,N)$ *vs.* $\log N$ for some values of J. Left subplot is for $a = 0.1$ while the right one is for $a = 5$.

vs $\log N$ for two values of parameter a — for the left one $a = 0.1$ and for the right $a = 5$. One sees that for large values of N the Renyi entropy reaches its asymptotic behaviour $\sim \log N$ (see Section 7 for details). The asymptotic regime is, however, reached much faster for small a than for large a. The second observation is that the Renyi entropy increases sharply with increasing a. This confirms the idea that it can be used as a measure of the "erraticity" [11] of the system.

The dependence on $J = \log_2 M$ for a given N is shown in figure 2. We see here a similar pattern as in Fig. 1 with the roles of J and $\log N$ exchanged.

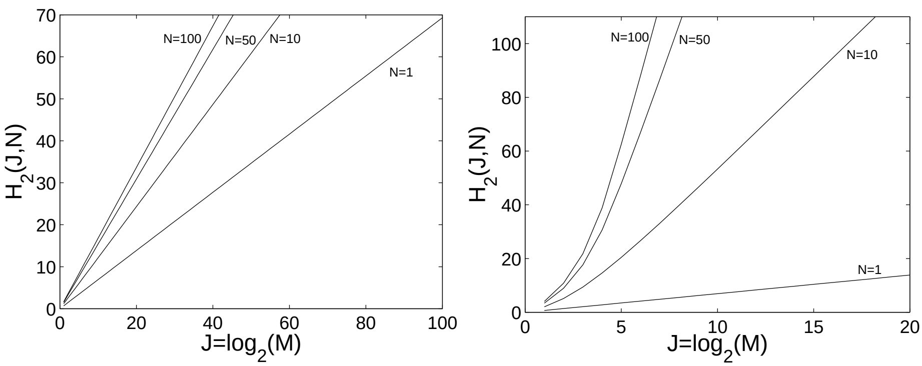

Fig. 2. Dependence of $H_2(J,N)$ on J for some values of N. Left subplot is for $a = 0.1$ while the right one is for $a = 5$.

In the next two sections we discuss the asymptotic behaviour in detail.

6. Asymptotic behaviour at large *J* and fixed *N*

This limit is of obvious interest since it is easily realized in experiment: one simply has to split a given phase-space region into a large number of bins, a procedure well-known from the studies of intermittency [5]. We study this limit employing the recurrence relation (31).

To this end we first observe that Eq. (31) can be — schematically — rewritten in the form

$$\hat{C}_l(J+1,N) = \Omega_N \left[\hat{C}_l(J,N) + H(J,N)\right] , \tag{35}$$

where

$$\Omega_N = 2\left[\frac{\Gamma(a+N)\Gamma(2a)}{\Gamma(2a+N)\Gamma(a)}\right]^l ,$$

$$H(J,N) = \frac{1}{2}\sum_{n=1}^{N-1} \hat{C}_l(J,n)\hat{C}_l(J,N-n) . \tag{36}$$

$\hat{C}_l(0,N)$ is given by (32), and $\hat{C}_l(J,0) = 1$.

To solve (35) for a given N we note that $H(J,N)$ contains only $\hat{C}_l(J,n)$ with $n < N$ and thus can be considered as a known function. With this remark the solution of (35) is easy to find:

$$\hat{C}_l(J,N) = [\Omega_N]^J\,\hat{C}_l(0,N) + \sum_{j=1}^{J}[\Omega_N]^j\,H(J-j,N)\ . \tag{37}$$

This allows us to construct the solution starting from $N=1$ and then step-by-step for higher N.

In Appendix A the first three steps ($N=1,2,3$) are explained in detail. Here we only summarise the main features of the result.

The solution for $\hat{C}_l(J,N)$ is a sum of terms which are proportional to $(\Omega_i\Omega_j\dots\Omega_k)^J$ where the sum of all indices is equal to N, but the asymptotic behaviour is determined only by two of them. If parameter a is smaller than that obtained as a solution of the equation $\Omega_N = \Omega_1^N$ then $\Omega_N > \Omega_1^N$ and the dominating term is of the form Ω_N^J. Thus

$$C_l(J,N) \quad\rightarrow\quad M^{-\nu_l(N)}\Phi_l(N) \tag{38}$$

with $M = 2^J$ and

$$\nu_l(N) = -\frac{\log\Omega_N}{\log 2}\ , \tag{39}$$

and Φ_l depends only on N. For larger values of a we have $\Omega_N < \Omega_1^N$ and the leading term is $(\Omega_1^N)^J$ so we obtain the formula (38) with

$$\nu_l(N) = -\frac{\log\Omega_1^N}{\log 2} = N(l-1)\ . \tag{40}$$

For the transition value $\Omega_N = \Omega_1^N$ the asymptotic behaviour cannot be described by a simple exponential law. The details for some values of N are given in the Appendix A.

In Fig. 3 the limiting value (for $J = \log_2 M \rightarrow \infty$) of the ratio H_2/J is plotted versus a for several values of N. One sees that it increases with increasing a until a plateau is reached. This shows explicitly that the Renyi entropy increases with increasing randomisation of the system.

It is interesting to note that the value of $\nu_l(N)$ is related to the intermittency exponent f_k of the cascade. Indeed, as shown in [5]

$$f_k = \frac{\log[\langle q^k\rangle/\langle q\rangle^k]}{\log 2} = k - 1 + \frac{1}{\log 2}\log\left[\frac{\Gamma(a+k)\Gamma(2a+1)}{\Gamma(a+1)\Gamma(2a+k)}\right] \tag{41}$$

and thus we obtain the relation

$$\nu_l(N) = Nl - 1 - lf_N \tag{42}$$

which is valid as long as $f_N > (N-1)/l$. This is another manifestation of the fact that the Renyi entropy is very sensitive to multiparticle correlations.

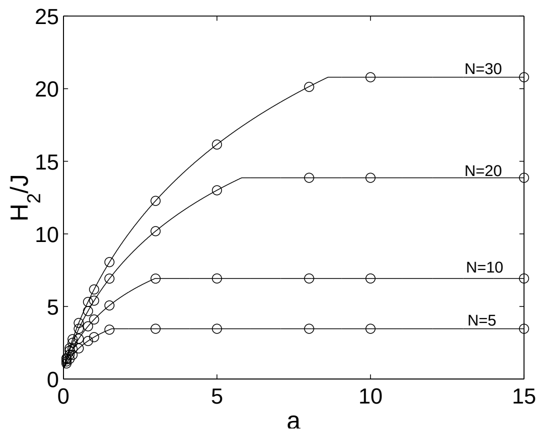

Fig. 3. Asymptotic dependence of Renyi entropy for large number of bins. Solid line represents theoretical predictions while values obtained from numerical calculations (J up to 5000) are represented by circles.

7. Asymptotic behaviour at large N, fixed J

This limit has more theoretical than practical significance, because it is not easily realized in an experiment. Indeed, when the number of bins is fixed, the number of particles can only be increased by increasing the bin size. But this means increasing the phase-space volume. This can be realized only if the region we consider is uniform enough, which is not easy to guarantee.

Derivation of the asymptotic form for $C_l(1,N)$ is presented in Appendix B. Using the same technique it is not too difficult to show by induction that the behaviour for arbitrary J is of the form $N^{-\tau}$. The value of τ depends on the value of a

$$\tau = [M(J)-1](l-1) \quad \text{for} \quad a > 2^{J-1}(l-1)/l\,, \tag{43}$$

$$\begin{aligned}\tau &= [M(j-1)-1](l-1) + (J-j+1)al\,,\\ &\quad \text{for} \quad 2^{j-1}(l-1)/l > a > 2^{j-2}(l-1)/l\,, \quad J \geq j \geq 2\,,\end{aligned} \tag{44}$$

$$\tau = Jal \quad \text{for} \quad a < (l-1)/l\,, \tag{45}$$

where $M(j) = 2^j$.

It is more difficult, however, to determine the coefficient in front of the power law. In Fig. 4 the asymptotic value (for $N \to \infty$) of the ratio $H_2/\log N$ is plotted *versus* a for several values of $M = 2^J$. One sees that it increases with increasing a until a plateau is reached. One can make two interesting observations.

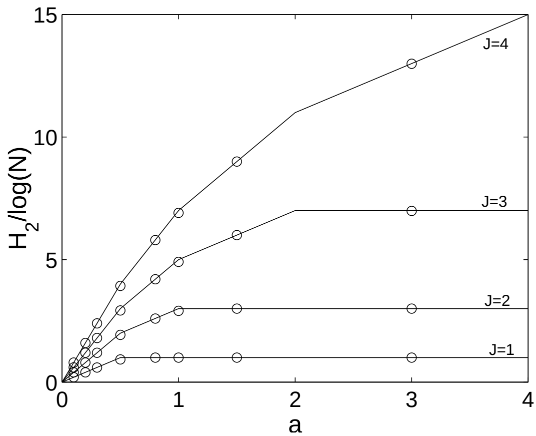

Fig. 4. Asymptotic dependence of Renyi entropy for large number of particles. Solid line represents theoretical predictions while values obtained from numerical calculations (N up to 500000) are represented by circles.

(a) Although $H_2/\log N$ is, for each M, a continuous function of a, its derivative is not. There are "phase transitions" at $a = 2^{j-2}$, $j = 1, \ldots, J$.

(b) The asymptotic value at large a differs by factor 2 from that obtained for Bernoulli distribution [2]. Thus the approach of the cascade model to the Bernoulli limit is *not uniform*: the order in which one takes the limit $a \to \infty$ is important.

8. Conclusions

In summary, we have investigated the Renyi entropy for systems of particles created by a multiplicative cascade mechanism. A general formula for binary cascades was written down and some special cases were studied in more detail. Our main conclusions can be summarized as follows:

(i) It was shown that when particle density is conserved at each vertex of the cascade, the corresponding probability distribution factorises. Consequently, a relatively simple (albeit non–linear) recurrence relation for coincidence probabilities can be written down. This recurrence relation, connecting coincidence probability for a cascade of the length J to those of length $J-1$, simplifies considerably the detailed studies of their properties.

(ii) The cascade of the β–distribution in the vertex

$$f(q) = \frac{\Gamma(2a)}{\Gamma^2(a)}[q(1-q)]^{a-1} \tag{46}$$

was considered in some detail. The asymptotic behaviour at large particle densities and/or at large lengths of the cascade was shown to be of the form

(a) for $J \to \infty$, N fixed

$$H_l(J,N) \sim \nu_l(N,a)J; \tag{47}$$

(b) for $N \to \infty$, J fixed

$$H_l(J,N) \sim \tau_l(N,a)\log N. \tag{48}$$

The functions $\nu_l(N,a)$ and $\tau_l(N,a)$ are described in Sections 6 and 7 where also some interesting discontinuities are pointed out.

(iii) In particular, although it may seem that for $a \to \infty$ the cascade mechanism should tend to the one described by the simple Bernoulli law, it turns out that this limit is not uniform. As a result, the limits $a \to \infty$ cannot be interchanged with the limits $J \to \infty$, N fixed and/or $N \to \infty$, J fixed.

(iv) An interesting relation between $\nu_l(N,a)$ and the intermittency exponents was found.

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Appendix A

Asymptotics at large J for $N = 1, 2, 3$

$N = 1$

$$\Omega_1 = 2^{1-l}, \quad \hat{C}_l(0,1) = a^l, \quad H(J,1) = 0 \tag{A.1}$$

so that

$$\hat{C}_l(J,1) = \Omega_1^J \hat{C}_l(0,1) \quad \to \quad C_l(J,1) = \Omega_1^J = M^{1-l} \tag{A.2}$$

$N = 2$

$$\Omega_2 = 2^{1-l}\left(\frac{a+1}{2a+1}\right)^l, \quad \hat{C}_l(0,2) = \left(\frac{a(a+1)}{2}\right)^l,$$
$$H(J,2) = \frac{1}{2}\hat{C}_l(J,1)^2 = \frac{1}{2}(\Omega_1^2)^J\hat{C}_l(0,1)^2 \tag{A.3}$$

and thus

$$\hat{C}_l(J,2) = \Omega_2^J\hat{C}_l(0,2) + \frac{1}{2}\hat{C}_l(0,1)^2\frac{(\Omega_1^2)^J - \Omega_2^J}{\Omega_1^2 - \Omega_2}\Omega_2 \tag{A.4}$$

and

$$\begin{aligned} C_l(J,2) \;=\; & \left(1 - \frac{1}{2}\frac{\hat{C}_l(0,1)^2}{\hat{C}_l(0,2)}\frac{\Omega_2}{\Omega_1^2 - \Omega_2}\right)M^{-\nu} \\ & + \frac{1}{2}\frac{\hat{C}_l(0,1)^2}{\hat{C}_l(0,2)}\frac{\Omega_2}{\Omega_1^2 - \Omega_2}M^{-2(l-1)} \end{aligned} \tag{A.5}$$

with $\nu = -\log\Omega_2/\log 2$. The asymptotic behaviour at large J depends crucially on the value of a. For small values of a (such that $\Omega_1^2 < \Omega_2$) the asymptotics is given by the first term in (A.5) since $l-1 < \nu < 2(l-1)$ whereas for larger values of a $(2(l-1) < \nu < 2l-1)$ the asymptotic behaviour is governed by the second term. When ν is equal to $2(l-1)$ $\left(a = \frac{2^{1/l}-2}{2-2^{1/l+1}}\right)$ then $C_l(J,2)$ is given by

$$C_l(J,2) = \left(1 + \frac{1}{2}J\frac{\hat{C}_l(0,1)^2}{\hat{C}_l(0,2)}\right)M^{-2(l-1)} \tag{A.6}$$

$N = 3$

$$\Omega_3 = 2^{1-l}\left[\frac{(a+2)(a+1)}{(2a+2)(2a+1)}\right]^l, \quad \hat{C}_l(0,3) = \left[\frac{a(a+1)(a+2)}{6}\right]^l. \tag{A.7}$$

In the following we will concentrate only on generic solution — that is for values of a that does not satisfy any of the equations: $\Omega_1^2 = \Omega_2$, $\Omega_1^3 = \Omega_3$, $\Omega_2\Omega_1 = \Omega_3$ — handling those cases would give us a lot of formulas with no new insight. $H(J,3)$ is given thus by

$$H(J,3) = \Omega_1^J\hat{C}_l(0,1)\left[\Omega_2^J\hat{C}_l(0,2) + \frac{1}{2}\hat{C}_l(0,1)^2\frac{(\Omega_1^2)^J - \Omega_2^J}{\Omega_1^2 - \Omega_2}\Omega_2\right] \tag{A.8}$$

and consequently

$$
\begin{aligned}
\hat{C}_l(J,3) \;=\; & \Omega_3^J \left\{ \hat{C}_l(0,3)\hat{C}_l(0,1)\hat{C}_l(0,2)\frac{\Omega_3}{\Omega_1\Omega_2-\Omega_3} \right. \\
& -\frac{1}{2}\hat{C}_l(0,1)^3\frac{\Omega_2}{\Omega_1^2-\Omega_2}\frac{\Omega_3}{\Omega_1^3-\Omega_3} \\
& \left. +\frac{1}{2}\hat{C}_l(0,1)^3\frac{\Omega_2}{\Omega_1^2-\Omega_2}\frac{\Omega_3}{\Omega_1\Omega_2-\Omega_3} \right\} \\
& +(\Omega_1\Omega_2)^J \left\{ \hat{C}_l(0,1)\hat{C}_l(0,2)\frac{\Omega_3}{\Omega_1\Omega_2-\Omega_3} \right. \\
& \left. -\frac{1}{2}\hat{C}_l(0,1)^3\frac{\Omega_2}{\Omega_1^2-\Omega_2}\frac{\Omega_3}{\Omega_1\Omega_2-\Omega_3} \right\} \\
& +(\Omega_1^3)^J\frac{1}{2}\hat{C}_l(0,1)^3\frac{\Omega_2}{\Omega_1^2-\Omega_2}\frac{\Omega_3}{\Omega_1^3-\Omega_3}\,. \qquad \text{(A.9)}
\end{aligned}
$$

Since one can show that $\Omega_1\Omega_2 \le \max(\Omega_3, \Omega_1^3)$ then the asymptotic behaviour will only depend either on Ω_3^J or $(\Omega_1^3)^J$ so for a smaller than the solution of $\Omega_1^3 = \Omega_3$, the asymptotic value will be given by the first term in (A.9) whereas for larger values of a the asymptotics is governed by the last term.

Let us generalise: from (36) and (37) one can see that $\hat{C}_l(J,N)$ is a sum of terms which all are of the form

$$(\Omega_i\Omega_j\ldots\Omega_k)^J B_l(N;i,j,\ldots,k)\,, \qquad \text{(A.10)}$$

where $i+j+\ldots+k=N$ and $B_l(N;i,j,\ldots,k)$ is a coefficient which does not depend on J (in generic case). It is not difficult to give recursive expression for coefficients $B_l(N+1;\ldots)$ in terms of $B_l(n;\ldots)$ where $n=1,2,\ldots,N$ with the initial condition $B_l(1;1)=\hat{C}_l(0,1)=a^l$. In order to explain it clearly we introduce now the following notation:

- φ_n will denote set of numbers $i_1,\ldots,i_k$ $(k=1,\ldots,n)$ such that their sum is equal to n,
- $\varphi_n\cup\varphi_m$ is a sum of two such sets (of course $\varphi_n\cup\varphi_m$ is a φ_{n+m}),
- if $\varphi_n=i,j,\ldots,k$ then $\Omega_{\varphi_n}=\Omega_i\Omega_j\ldots\Omega_k$.

Using this notation we have the following expressions for coefficients

$$
\begin{aligned}
& B(N;\varphi_N\neq N) \\
& = \frac{1}{2}\left(\sum_{n=1}^{N-1}\sum_{\varphi_n\cup\varphi_{N-n}=\varphi_N} B(n;\varphi_n)B(N-n;\varphi_{N-n})\right)\frac{\Omega_N}{\Omega_{\varphi_N}-\Omega_N}\,, \qquad \text{(A.11)}
\end{aligned}
$$

and

$$B(N;N) = \hat{C}(0,N) - \sum_{\varphi_N \neq N} B(N;\varphi_N)\,. \tag{A.12}$$

Since

$$\lim_{a\to\infty} \Omega_N \;=\; 2^{1-Nl}\,, \tag{A.13}$$

$$\lim_{a\to 0} \Omega_N \;=\; 2^{1-l} = \Omega_1 \tag{A.14}$$

one can see that for small values of a, term Ω_N^J will dominate whereas for large a, $(\Omega_1^N)^J$ will be dominating. We were not able to prove that for all values of a, N and l the asymptotic behaviour is governed either by the Ω_N^J or by $(\Omega_1^N)^J$ but the numerical results do indicate so. Example of this situation is presented in Fig. 5 where we used $N = 5$ and $l = 2$.

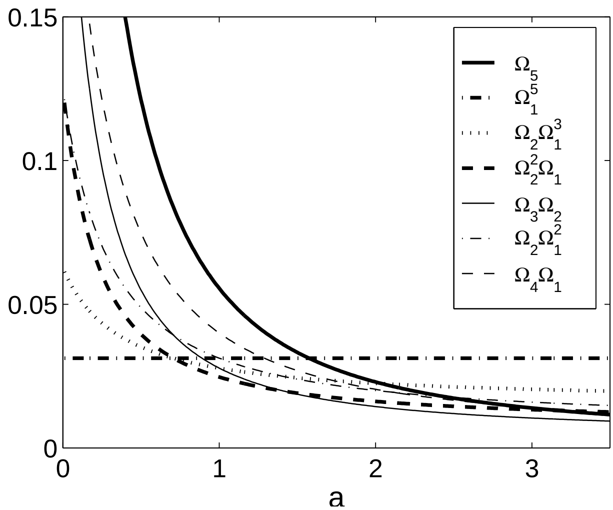

Fig. 5. Dependence of all $\Omega_i\Omega_j\ldots\Omega_k$ on a for $N = 5$ and $l = 2$.

Appendix B

Asymptotics at large N for $J = 1$

We again use Eq. (31) and taking into account its symmetry we write

$$\hat{C}_l(J+1,N) = 2\left[\frac{\Gamma(a+N)\Gamma(2a)}{\Gamma(2a+N)\Gamma(a)}\right]^l \sum_{n=0}^{N/2} \hat{C}_l(J,n)\hat{C}_l(J,N-n)\,. \tag{B.1}$$

At large N we can approximate this formula by

$$\hat{C}_l(J+1,N) = 2\left[\frac{\Gamma(2a)}{\Gamma(a)}\right]^l N^{-al}\sum_{n=0}^{N/2}\hat{C}_l(J,n)\hat{C}_l(J,N-n)\,. \tag{B.2}$$

As the first step we analyse $C_l(1,N)$. At large N the initial condition (32) gives

$$\hat{C}_l(0,N) = \left[\frac{N^{a-1}}{\Gamma(a)}\right]^l . \tag{B.3}$$

When substituted into (B.2) it gives

$$\hat{C}_l(1,N) = 2\left[\frac{\Gamma(2a)}{\Gamma^3(a)}\right]^l N^{-al}N^{l(a-1)}\sum_{n=0}^{N/2}\hat{C}_l(0,n)[(1-n/N)\Gamma(a)]^{l(a-1)}\,. \tag{B.4}$$

The behaviour of this expression at large N depends on the convergence of the series. If

$$l(a-1) > -1, \quad i.e, \quad a > (l-1)/l \tag{B.5}$$

the series is divergent, and we have

$$\begin{aligned}\sum_{n=0}^{N/2}\hat{C}_l(0,n)\left[(1-n/N)^{a-1}\Gamma(a)\right]^l &\to \frac{1}{2}N^{l(a-1)+1}\int_0^1 du\, u^{l(a-1)}(1-u)^{l(a-1)} \\ &= \frac{1}{2}N^{l(a-1)+1}\frac{\Gamma^2[l(a-1)+1]}{\Gamma[2l(a-1)+2]}\,,\end{aligned} \tag{B.6}$$

and thus denoting

$$Z_l(a) = \frac{\Gamma^l(2a)}{\Gamma^{2l}(a)}\,, \tag{B.7}$$

$$\hat{C}_l(1,N) = \frac{Z_l(a)\Gamma^2[l(a-1)+1]}{\Gamma^l(a)\Gamma[2l(a-1)+2]}N^{la-2l+1}\,. \tag{B.8}$$

From this we finally obtain

$$C_l(1,N) = \Gamma^l(a)N^{l(1-a)}\hat{C}_l(1,N) = \frac{Z_l(a)\Gamma^2[l(a-1)+1]}{\Gamma[2l(a-1)+2]}N^{1-l}. \tag{B.9}$$

Note that for $l=1$ we recover $C_1 = 1$. If

$$a < \frac{l-1}{l}\,, \tag{B.10}$$

the series in (B.4) is convergent and thus does not depend on N for large N. Denoting

$$\Phi_l(a) = 2\sum_{n=0}^{\infty}\left[\frac{\Gamma(N+a)}{N!\Gamma(a)}\right]^l , \tag{B.11}$$

we obtain

$$\hat{C}_l(1,N) = \frac{Z_l(a)\Phi_l(a)}{\Gamma^l(a)}N^{-la}N^{l(a-1)} = \frac{Z_l(a)\Phi_l(a)}{\Gamma^l(a)}N^{-l} \tag{B.12}$$

and thus

$$C_l(1,N) = Z_l(a)\Phi_l(a)N^{-al}. \tag{B.13}$$

Note that this is the result for $a < 1 - 1/l$ which is not possible for $l = 1$, so there is no contradiction.

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